Some Nordhaus - Gaddum Type Relations On

Strong Efficient Dominating Sets

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Abstract – Let \( G = (V, E) \) be a simple graph with \( p \) vertices and \( q \) edges. A subset \( S \) of \( V(G) \) is called a strong (weak) efficient dominating set of \( G \) if for every \( v \in V(G) \), \( |N_s[v]\cap S| = 1 \) (\( |N_w[v]\cap S| = 1 \)). The minimum cardinality of a strong (weak) efficient dominating set \( G \) is called strong (weak) efficient domination number of \( G \) and is denoted by \( \gamma_{se}(G) \), \( \gamma_{we}(G) \). A graph \( G \) is strong efficient if there exists a strong efficient dominating set of \( G \). In this paper, the authors introduced a new parameter called the number of strong efficient dominating sets of a graph \( G \) denoted by \( \# \gamma_{se}(G) \) and studied some Nordhaus- Gaddum type relations on strong efficient domination number of a graph and its derived graph. The relation between the number of strong efficient dominating sets of a graph and its derived graph is also studied.

Keywords - Strong efficient dominating sets, Strong efficient domination number and number of strong efficient dominating sets

1. Introduction

Throughout this paper, only finite, undirected and simple graphs are considered. Let \( G = (V, E) \) be a graph with \( p \) vertices and \( q \) edges. The degree of any vertex \( u \) in \( G \) is the number of edges incident with \( u \) and is denoted by \( \text{deg} u \). The minimum and maximum degree of a vertex is denoted by \( \delta(G) \) and \( \Delta(G) \) respectively. A vertex of degree 0 in \( G \) is called an isolated vertex and a vertex of degree 1 in \( G \) is called a pendant vertex. A subset \( S \) of \( V(G) \) of a graph \( G \) is called a dominating set of \( G \) if every vertex in \( V(G) \) \( \setminus S \) is adjacent to a vertex \( S \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \). Sampathkumar and Pushpalatha introduced the concepts of strong and weak domination in graphs [6]. A subset \( S \) of \( V(G) \) is called a strong dominating set of \( G \) if for every \( v \in V(G) \) \( S \) there exists a \( u \in S \) such that \( u \) and \( v \) are adjacent and \( \text{deg} u \geq \text{deg} v \). A subset \( S \) of \( V(G) \) is called an efficient dominating set of \( G \) if for every \( v \in V(G) \), \( |N[v]\cap S| = 1 \)[1]. The concept of strong (weak) efficient domination in graphs was introduced by Meena, Subramanian and Swaminathan [4]. A subset \( S \) of \( V(G) \) is called a strong (weak) efficient dominating set of \( G \) if for every \( v \in V(G) \), \( |N_s[v]\cap S| = 1 \) (\( |N_w[v]\cap S| = 1 \)). \( N_s(v) = \{ u \in V(G) : uv \in E(G), \text{deg}(u) \geq \text{deg}(v) \} \). The minimum cardinality of a strong (weak) efficient dominating set is called strong (weak) efficient domination number and is
denoted by $\gamma_{se}(G)$. A graph $G$ is strong efficient if there exists a strong efficient dominating set of $G$. In this paper, the authors introduced a new parameter called the number of strong efficient dominating sets of a graph $G$ denoted by $\# \gamma_{se}(G)$ and studied some Nordhaus-Gaddum type relations on strong efficient domination number of a graph and its derived graph. They also studied the Nordhaus-Gaddum type relations on the number of strong efficient dominating sets of a graph and its derived graph. For all graph theoretic terminologies and notations, Harary [2] is followed.

2. Basic Definitions and Results

The following basic definitions and results are necessary for the present study.

Definition 2.1: A graph $G$ with vertex set $V(G) = \{v_1, v_2, v_3, ..., v_n\}$ for $n \geq 3$ and edge set $E(G) = \{v_i v_j | 1 \leq i \leq n\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_1 v_n\}$ is called a wheel graph of length $n$ and is denoted by $W_n$. The vertex $v$ is called the axial or central vertex of the wheel graph.

Definition 2.2: A gear graph $G_n$ is obtained from the wheel graph $W_n$ by adding a vertex between every pair of adjacent vertices in the cycle.

Definition 2.3: The Bistar $D_{m,n}$ is the graph obtained from $K_2$ by joining $m$ pendant edges to one end vertex of $K_2$ and $n$ pendant edges to the other end of $K_2$. The edge $K_2$ is called the central edge of $D_{m,n}$ and the vertices of $K_2$ are called the central vertices of $D_{m,n}$.

Definition 2.4: The H-graph of a path $P_n$ is the graph obtained from two copies of $P_n$ with vertices $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$ by joining the vertices $v_{n+1}$ and $u_{n+1}$ if $n$ is odd and the vertices $v_{\frac{n}{2}}$ and $u_{\frac{n}{2}+1}$ if $n$ is even.

Definition 2.5: The complement $\overline{G}$ of a graph $G$ has $V(G)$ as its vertex set and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$.

Definition 2.6 [7]: A vertex switching $G_v$ of a graph $G$ is obtained by taking a vertex $v$ of $G$, removing all edges incident to $v$ and adding edges joining $v$ to every vertex which are not adjacent to $v$ in $G$.

Result 2.7 [4]: $\gamma_{se}(K_{1,n}) = 1, n \in N$

Result 2.8 [4]: For any path $P_m, \gamma_{se}(P_m) = \begin{cases} n & \text{if } m = 3n, n \in N \\ n+1 & \text{if } m = 3n+1, n \in N \\ n+2 & \text{if } m = 3n+2, n \in N \end{cases}$

Result 2.9 [4]: $\gamma_{se}(C_{3n}) = n, n \in N$

Result 2.10 [4]: $\gamma_{se}(K_n) = 1, n \in N$

Result 2.11 [4]: $\gamma_{se}(D_{r,s}) = r + 1$ where $r \leq s$
Result 2.12 [5]: A graph $G$ does not admit a strong efficient dominating set if the distance between any two maximum degree vertices is exactly two.

Result 2.13 [4]: Any strong efficient dominating set is independent.

Result 2.14: If $G_1$ and $G_2$ are strong efficient graphs, then $G_1 \cup G_2$ is strong efficient.

3. Main Results

In this section, the authors studied the strong efficient domination number of some graphs and their derived graphs and also derived some Nordhaus- Gaddum type relations between them. It was found that there are several strong efficient dominating sets for a given graph. This motivated the authors to define a new parameter called the number of strong efficient dominating sets denoted by $\#_{se}(G)$. Using this parameter, the authors studied the Nordhaus- Gaddum type relations on the number of strong efficient dominating sets of a graph and its derived graph.

Definition 3.1: Let $G$ be a graph with a strong efficient dominating number $\gamma_{se}(G)$. The number of distinct strong efficient dominating sets of a graph $G$ is denoted by $\#_{se}(G)$.

Theorem 3.2: $\gamma_{se}(K_{1,n}) + \gamma_{se}(\overline{K_{1,n}}) = 3$ for all $n \geq 1$ and $\#_{se}(K_{1,n}) + \#_{se}(\overline{K_{1,n}}) = n+1$ for all $n \geq 1$

For: $K_{1,n}$ is the graph $K_n \cup K_1$. Therefore $\gamma_{se}(\overline{K_{1,n}}) = 2$ and $\#_{se}(\overline{K_{1,n}}) = n$. Hence the theorem.

Theorem 3.3: $\overline{P_n}$ is strong efficient if and only if $n \leq 4$. Also $\#_{se}(P_n) + \#_{se}(\overline{P_n}) = \{3 \text{ when } n = 2 \text{ or } 3 \} \quad 4 \text{ when } n = 4$

Proof: Let $\overline{P_n}$ be strong efficient. Suppose $n > 4$. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $\overline{P_n}$. Let $S$ be a strong efficient dominating set of $\overline{P_n}$. Let $\text{deg}(v_1) = \text{deg}(v_n) = n-2$ and $\text{deg}(v_i) = n-3$ for $2 \leq i \leq n-1$. Since $S$ is independent, $v_1$ and $v_n$ are adjacent, either $v_1 \in S$ or $v_n \in S$. Suppose $v_1 \in S$. Then $v_1$ strongly dominates all $v_i$ where $3 \leq i \leq n$ and $v_2$ must be an element of $S$. Then $|N_S[v_i] \cap S| \geq 2$ for $4 \leq i \leq n-1$, a contradiction. The proof is similar if $v_n \in S$. Further, when $n = 2$, $\overline{P_2}$ is $2K_1$, and when $n = 3$, $\overline{P_3}$ is $K_2 \cup K_1$, which are obviously strong efficient with $\gamma_{se}(\overline{P_2}) = 2$, $\#_{se}(\overline{P_2}) = 1$, $\gamma_{se}(\overline{P_3}) = 2$ and $\#_{se}(\overline{P_3}) = 2$. In $\overline{P_4}$, $\{v_1, v_2\}$ and $\{v_3, v_4\}$ are the strong efficient dominating sets. Hence $\gamma_{se}(\overline{P_4}) = 2$ and $\#_{se}(\overline{P_4}) = 2$.

Thus $\#_{se}(P_n) + \#_{se}(\overline{P_n}) = \{3 \text{ when } n = 2 \text{ or } 3 \} \quad 4 \text{ when } n = 4$

Theorem 3.4: $\overline{C_n}$ is strong efficient if and only if $n \leq 4$

Proof: Let $\overline{C_n}$ be strong efficient. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of $\overline{C_n}$. Suppose $n > 4$. Then $\overline{C_n}$ is a regular graph of degree $n-3$. Let $S$ be a strong efficient dominating set of $\overline{C_n}$. Let $v_i$ be an element of $S$. $v_i$ strongly dominates all the vertices other than $v_{i-1}$ and $v_{i+1}$. Since $S$ is independent and $v_{i-1}$ and $v_{i+1}$ are adjacent, either $v_{i-1} \in S$ or $v_{i+1} \in S$. Suppose $v_{i-1} \in S$. Then $|N_S(v_i) \cap S| \geq 2$, $i + 2 \leq j$, a contradiction. The proof is similar if $v_{i+1} \in S$. Further, suppose $n = 3$. Then $\overline{C_3}$ is $3K_1$ which has a unique strong efficient...
dominating set with strong efficient domination number 3. On the other hand, suppose n = 4. Then $\overline{C_4}$ is $2K_2$ which is also strong efficient with strong efficient domination number 2 and $\#\gamma_{se}(\overline{C_4}) = 4$. Therefore $\#\gamma_{se}(\overline{C_n}) + \#\gamma_{se}(\overline{C_n}) = 4$ when n = 3

**Result 3.5:** There exists a graph G for which $\gamma_{se}(G) + \gamma_{se}(\overline{G}) = \#\gamma_{se}(G) + \#\gamma_{se}(\overline{G})$

**Example:** Let $G = K_n$.

For, $\overline{K_n}$ is $nK_1$ which has a unique strong efficient dominating set with strong efficient domination number n. So $\gamma_{se}(K_n) + \gamma_{se}(\overline{K_n}) = n + 1$ and $\#\gamma_{se}(K_n) + \#\gamma_{se}(\overline{K_n}) = n + 1$ for all $n \geq 1$.

**Theorem 3.6:** The bistar $D_{r,s}$ is strong efficient with $\gamma_{se}(D_{r,s}) + \gamma_{se}(\overline{D_{r,s}}) = r + 3$ and $\#\gamma_{se}(D_{r,s}) + \#\gamma_{se}(\overline{D_{r,s}}) = r + s + 1$ for all $r, s \geq 1$ and where $r < s$

**Proof:** Let $V(D_{r,s}) = \{u_1, u_2, v_1 / 1 \leq i \leq r + s\}$ and $E(D_{r,s}) = \{u_1 v_i, u_2 v_j : r + 1 \leq i \leq r + s, 1 \leq j \leq r\} \cup \{v_i v_j / 1 \leq i \neq j \leq r + s\}$

In $D_{r,s}$, $u_1$ is adjacent with $v_{i+1}, v_{i+2}, ..., v_{i+s}, u_2$ is adjacent with $v_1, v_2, ..., v_r$, $v_i$ is adjacent with all the vertices other than $u_j$; $1 \leq i \leq r$ and $v_j$ is adjacent with all the vertices other than $u_j$; $r + 1 \leq j \leq r + s$. $u_1$ and $u_2$ are nonadjacent. Deg($u_1$) = $s$, Deg($u_2$) = $r = \delta(D_{r,s})$, deg($v_i$) = $r + s = \Delta(D_{r,s})$, $1 \leq i \leq r + s$. $\{v_1, u_1\}, \{v_2, u_1\}, ..., \{v_r, u_1\}, \{v_{r+1}, u_2\}, \{v_{r+2}, u_2\}, ..., \{v_{r+s}, u_2\}$ are the strong efficient dominating sets of $D_{r,s}$. So $\gamma_{se}(D_{r,s}) = 2$ whereas $\#\gamma_{se}(D_{r,s}) = r + s$. Hence the theorem.

**Corollary 3.7:** $\gamma_{se}(D_{r,r}) + \gamma_{se}(D_{r,r}) = r + 3$ and $\#\gamma_{se}(D_{r,r}) + \#\gamma_{se}(D_{r,r}) = 2r + 1$

**Theorem 3.8:** $\overline{W_n}$ is strong efficient if and only if $n \leq 4$

**Proof:** Suppose $n \geq 5$. Let $\overline{W_n}$ be strong efficient. Let S be a strong efficient dominating set of $\overline{W_n}$. Let $v, v_1, v_2, ..., v_n$ be the vertices of $\overline{W_n}$. $v$ is isolated in $\overline{W_n}$. $v_1$ is adjacent with all the vertices other than $v_{i-1}$ and $v_{i+1}$. As in the proof of theorem 2.4, a contradiction arises. Hence $n < 5$. Conversely, suppose $n = 3$. Then $\overline{W_3}$ is $4K_1$ which has a unique strong efficient dominating set. Thus $\gamma_{se}(\overline{W_3}) = 4$ and $\#\gamma_{se}(\overline{W_3}) = 1$. Suppose $n = 4$. Then $\overline{W_4}$ is $K_1 \cup 2K_2$ which is also strong efficient in which $\{v, v_1, v_2\}, \{v, v_1, v_4\}, \{v, v_3, v_2\}$ and $\{v, v_3, v_4\}$ are strong efficient dominating sets. Thus $\gamma_{se}(\overline{W_4}) = 3$ and $\#\gamma_{se}(\overline{W_4}) = 4$.

**Result 3.9:** Complement of a strong efficient graph need not be strong efficient.

**Example:** Consider the Gear graph $G_n$. Let $V(G_n) = \{v, v_1, v_2, v_3, ..., v_{2n}\}$. The vertex $v$ strongly dominates the vertices $v_1, v_3, ..., v_{2n-1}$. Hence $\{v, v_1, v_2, v_4, v_6, ..., v_{2n}\}$ is the unique strong efficient dominating set of $G_n$. Therefore $G_n$ is strong efficient. In $\overline{G_n}$, v is adjacent with $v_{2i}$ where $1 \leq i \leq n$. Deg($v$) = $n$. Each $v_{2i-1}$ is adjacent with all the vertices other than $v_{2j-2}, v_{2j}$ and v where $2 \leq j \leq n$. $v_1$ is adjacent with all the vertices other than $v_{2n}, v_2$ and v. $v_{2j}$ is adjacent with all the vertices other than $v_{2j-1}$ and $v_{2j+1}$. Deg($v_{2j-1}$) = $2n - 3; 1 \leq j \leq n$ and $\text{deg}(v_{2j}) = 2n - 2; 1 \leq i \leq n$. Suppose $\overline{G_n}$ is strong efficient. Let S be a strong efficient dominating set of $\overline{G_n}$. Since degree of any $v_{2i}$
is maximum for some i, \( v_{2i} \in S \) for some i. \( v_{2i} \) strongly dominates all the vertices other than \( v_{2i-1} \) and \( v_{2i+1} \). Since \( v_{2i-1} \) and \( v_{2i+1} \) are adjacent and S is independent, either \( v_{2i-1} \in S \) or \( v_{2i+1} \in S \). If \( v_{2i-1} \in S \), then \( v_{2i+1} \) is strongly dominated by both \( v_{2i-1} \) and \( v_{2i+1} \). \( |N_s[v_{2i+1}] \cap S| \geq 2 \), a contradiction. Proof is similar if \( v_{2i+1} \in S \). Hence \( \overline{G_n} \) is not strong efficient.

**Theorem 3.10:** The complement of the H-graph \( \overline{H_n} \) is strong efficient if and only if \( n \leq 4 \).

**Proof:** Suppose \( \overline{H_n} \) is strong efficient. Let \( v_1, v_2, v_3, ..., v_n, u_1, u_2, u_3, ..., u_n \) be the vertices of \( \overline{H_n} \). Suppose \( n > 4 \). Since the vertices \( v_1, v_n, u_1 \) and \( u_n \) are pendant vertices in \( H_n \), their degree is maximum in \( \overline{H_n} \) which is \( 2n-2 \). Let S be a strong efficient dominating set of \( \overline{H_n} \). Since the maximum degree vertices are mutually adjacent with each other S contains exactly one of them. Without loss of generality, let \( v_1 \in S \). \( v_1 \) strongly dominates all the vertices other than \( v_2 \). Therefore \( v_2 \in S \). But \( u_2 \) is adjacent with both \( v_1 \) and \( v_2 \) in \( \overline{H_n} \) and \( \deg(v_2) = \deg(u_2) \). \( |N_s[u_2] \cap S| = 2 > 1 \). Therefore \( n \leq 4 \). Conversely suppose \( n = 3 \), \( \overline{H_3} \) is given in the figure 1.

![Figure 1](image1.png) ![Figure 2](image2.png)

\( \{v_1, v_2\}, \{v_2, v_3\}, \{u_1, u_2\} \) and \( \{u_2, u_3\} \) are strong efficient dominating sets of \( \overline{H_3} \). Therefore \( \gamma_{se}(\overline{H_3}) = 2 \) and \# \( \gamma_{se}(\overline{H_3}) = 4 \). Suppose \( n = 4 \). \( \overline{H_4} \) is given in figure 2. \( \{v_1, v_2\} \) and \( \{u_3, u_4\} \) are strong efficient dominating sets of \( \overline{H_4} \). Therefore \( \gamma_{se}(\overline{H_4}) = 2 \) and \# \( \gamma_{se}(\overline{H_4}) = 2 \).

**Theorem 3.11:** The graph \( P_{m[v_i]} \) where \( v_i \) is an end vertex of the path \( P_m \) and \( m \geq 2 \) is strong efficient with

\[
\gamma_{se}(P_m) + \gamma_{se}[P_{m[v_i]}] = \begin{cases} 
3 \text{ when } m = 2 \\
2 \text{ when } m = 3 \\
n+2 \text{ when } m = 3n, n \geq 2, n \in \mathbb{N} \\
n+3 \text{ when } m = 3n+1, n \in \mathbb{N} \\
n+4 \text{ when } m = 3n+2, n \in \mathbb{N}
\end{cases}
\]

\[
\#\gamma_{se}(P_m) + \#\gamma_{se}[P_{m[v_i]}] = \begin{cases} 
3 \text{ when } m = 2 \\
2 \text{ when } m = 3n \text{ and } m = 3n + 2, n \in \mathbb{N} \\
3 \text{ when } m = 3n + 1, n \in \mathbb{N}
\end{cases}
\]

**Proof:** Let \( v_1, v_2, v_3, ..., v_m \) be the vertices of the path \( P_m \). Let \( i = 1 \). Let \( P_m[v_i] \) be the graph obtained by switching the end vertex \( v_i \) of the path \( P_m \).
Case (i): Suppose $m = 2$. Then $P_{2[v_1]}$ is $2K_1$ which is strong efficient with a unique strong efficient dominating set $\{v_1, v_2\}$. $\gamma_{se}(P_2) = 2$ and $\gamma_{se}[P_{2[v_1]}] = 3$ and $\#\gamma_{se}(P_2) + \#\gamma_{se}[P_{2[v_1]}] = 3$.

Case (ii): Suppose $m = 3$. Then $P_{3[v_1]}$ is $P_3$ which is strong efficient with a unique strong efficient dominating set $\{v_3\}$. In this case, $\gamma_{se}[P_{3[v_1]}] = 1$ and $\#\gamma_{se}[P_{3[v_1]}] = 1$. $\gamma_{se}(P_3) + \gamma_{se}[P_{3[v_1]}] = 2$.

Case (iii): Suppose $m \geq 4$. In $P_{m[v_1]}$, $v_1$ is adjacent with all the vertices other than $v_2$. Deg $(v_1) = m - 2 = \Delta(P_{m[v_1]}), \deg(v_2) = 1, \deg(v_i) = 3; 3 \leq i \leq m - 1$ and $\deg(v_m) = 2$. Now $v_1$ strongly dominates all the vertices of $P_m$ other than $v_2$. Therefore $\{v_1, v_3\}$ is the unique strong efficient dominating set of $P_{m[v_1]}$. Therefore $\gamma_{se}[P_{m[v_1]}] = 2$ and $\#\gamma_{se}[P_{m[v_1]}] = 1$. Proof is similar if $i = m$. Hence the result.

Theorem 3.12: The graph $P_{m[v_1, v_m]}$ where $v_1$ and $v_m$ are the end vertices of the path $P_m$ is strong efficient if and only if $m \neq 4$. Moreover

$$\gamma_{se}(P_m) + \gamma_{se}[P_{m[v_1, v_m]}] = \begin{cases} \ 3 & \text{when } m = 2 \text{ or } 2 \\ 4 & \text{when } m = 5 \text{ or } 6 \\ n+2 & \text{when } m = 3n \text{ and } n > 2 \\ n+3 & \text{when } m = 3n+1, n > 1 \\ n+4 & \text{when } m = 3n+2 \text{ and } n > 1 \end{cases}$$

$$\#\gamma_{se}(P_m) + \#\gamma_{se}[P_{m[v_1, v_m]}] = \begin{cases} \ 3 & \text{when } m = 2 \text{ or } 3 \text{ or } m = 3n \text{ or } m = 3n+2, n > 1 \\ 2 & \text{when } m = 5 \\ 4 & \text{when } m = 3n+1, n > 1 \\ 6 & \text{when } m = 6 \end{cases}$$

Proof: Let $v_1$, $v_2$, $v_3$, ..., $v_m$ be the vertices of the path $P_m$.

Case (i): Suppose $m = 2$. The graph $P_{2[v_1, v_m]}$ is $2K_1$ which has the unique strong efficient dominating set $\{v_1, v_2\}$. $\gamma_{se}(P_2) + \gamma_{se}[P_{2[v_1, v_m]}] = 3$ and $\#\gamma_{se}(P_2) + \#\gamma_{se}[P_{2[v_1, v_m]}] = 3$.

Case (ii): Suppose $m = 3$. The graph $P_{3[v_1, v_m]}$ is $K_2 \cup K_1$ which is obviously strong efficient with $\gamma_{se}[P_{3[v_1, v_m]}] = 2$ and $\#\gamma_{se}[P_{3[v_1, v_m]}] = 2$.

Case (iii): Suppose $m = 5$. In $P_{5[v_1, v_m]}$, $v_3$ is adjacent with all the vertices. Deg $(v_3) = \Delta(P_{5[v_1, v_m]})$. Therefore $v_3$ strongly dominates all the vertices and $\{v_3\}$ is the unique strong efficient dominating set. Hence $\gamma_{se}[P_{5[v_1, v_m]}] = 1$ and $\#\gamma_{se}[P_{5[v_1, v_m]}] = 1$.

Case (iv): Suppose $m = 6$. In the graph $P_{6[v_1, v_m]}$, $v_1$ and $v_4$ are adjacent with all the vertices other than $v_2$. $v_2$, $v_3$ and $v_6$ are adjacent with all the vertices other than $v_2$. Deg $(v_1) = \deg(v_1) = \deg(v_3) = \deg(v_4) = \deg(v_6) = 4 = \Delta(P_{6[v_1, v_m]}), \deg(v_2) = \deg(v_5) = 2 = \delta(P_{6[v_1, v_m]}), v_2$ and $v_5$ are non adjacent. Therefore $\{v_4, v_6\}, \{v_3, v_5\}, \{v_1, v_2\}$ and $\{v_3, v_5\}$ are strong efficient dominating sets of $P_{6[v_1, v_m]}$. Therefore $\gamma_{se}[P_{6[v_1, v_m]}] = 2$ and $\#\gamma_{se}[P_{6[v_1, v_m]}] = 4$.

Case (v): Suppose $m > 6$. In the graph $P_{m[v_1, v_m]}$, $v_1$ is adjacent with all the vertices other than $v_2$. Similarly $v_n$ is adjacent with all the vertices other than $v_{n-1}$. Deg $(v_1) = \deg(v_1) = n - 2 = \Delta(P_{m[v_1, v_m]}), \deg(v_2) = \deg(v_{n-1}) = 2$ and $\deg(v_i) = n - 3; 3 \leq i \leq n - 2$.
\(v_1\) strongly dominates all the vertices other than \(v_2\). Therefore \(\{v_1, v_2\}\) is a strong efficient dominating set of \(P_m[v_1, v_m]\) where \(m > 6\). Similarly \(\{v_{m-1}, v_m\}\) is also a strong efficient dominating set of \(P_m[v_1, v_m]\). So \(y_{se}(P_m[v_1, v_m]) = 2\) and \(\#y_{se}(P_m[v_1, v_m]) = 2\). Hence the result.

Conversely suppose \(m = 4\). Then \(P_4[v_1, v_2]\) is the cycle \(C_4\) which is not strong efficient.

**Theorem 3.13:** The graph \(P_m[v_1, v_m]\) is strong efficient if and only if \(m \leq 6\). Also

\[
\begin{align*}
\gamma_{se}(P_m) + \gamma_{se}(P_m[v_1, v_m]) &= 3 \text{ when } m \leq 4 \\
\gamma_{se}(P_m) + \gamma_{se}(P_m[v_1, v_m]) &= 4 \text{ when } m = 5, 6 \\
\#y_{se}(P_m) + \#y_{se}(P_m[v_1, v_m]) &= 3 \text{ when } m \leq 4 \\
\#y_{se}(P_m) + \#y_{se}(P_m[v_1, v_m]) &= 2 \text{ when } m = 5, 6
\end{align*}
\]

**Proof:** Let \(P_m[v_1, v_m]\) be strong efficient. Let \(v_1, v_2, v_3, \ldots, v_m\) be the vertices of the path \(P_m\). Suppose \(m \geq 7\). Now \(v_2\) is adjacent with all the vertices other than \(v_2\) and \(v_3\) is adjacent with the vertices other than \(v_1\) and \(v_2\). \(D(v_1) = m - 2\) and \(D(v_2) = m - 3\), \(D(v_3) = 2\), \(D(v_4) = 3\), \(D(v_5) = 4\); \(4 \leq i \leq n - 1\). Let \(S\) be a strong efficient dominating set of \(P_m[v_1, v_2]\) where \(m \geq 7\). \(v_1\) strongly dominates all the vertices other than \(v_2\). So \(v_1 \in S\). \(v_1\) and \(v_2\) are non adjacent. This is a contradiction since \(|N_S(v_j) \cap S| = 2 > 1\) for \(4 \leq j \leq n - 1\). Hence \(P_m[v_1, v_m]\) is not strong efficient when \(m \geq 7\).

Further, suppose \(m = 2\). \(P_2[v_1, v_2]\) is 2-\(K_1\) is strong efficient with a unique strong efficient dominating set \(\{v_1, v_2\}\). Suppose \(m = 3\). \(P_3[v_1, v_3]\) is \(K_2 \cup K_1\) which is strong efficient with strong efficient dominating sets \(\{v_1, v_2\}\) and \(\{v_3, v_2\}\). Suppose \(m = 4\) or \(5\). In both \(P_4[v_1, v_2]\) and \(P_5[v_1, v_2]\), the vertex \(v_4\) is adjacent with all the vertices. Therefore \(\{v_4\}\) is the unique strong efficient dominating set. Suppose \(m = 6\). In \(P_6[v_1, v_2]\), \(D(v_1) = D(v_4) = D(v_5) = 4\), \(D(v_6) = 3\) and \(D(v_3) = 2\). \(v_6\) is adjacent with all the vertices other than \(v_3\). Hence \(\{v_5, v_2\}\) is the unique strong efficient dominating set. Hence the result.

**Theorem 3.14:** The graph \(P_m[v_1, v_m]\), \(m \geq 3\) is strong efficient. Also

\[
\begin{align*}
\gamma_{se}(P_m) + \gamma_{se}(P_m[v_1, v_m]) &= \begin{cases} 
n + 2 & \text{when } m = 3n, n \in N \\
n + 3 & \text{when } m = 3n + 1, n \in N \\
n + 4 & \text{when } m = 3n + 2, n \in N
\end{cases} \\
\#y_{se}(P_m) + \#y_{se}(P_m[v_1, v_m]) &= \begin{cases} 
3 & \text{when } m = 3 \text{ or } 6 \text{ or } m = 3n + 1, n \in N \\
2 & \text{when } m = 3n, n \geq 3 \text{ or } m = 3n + 2, n \in N
\end{cases}
\end{align*}
\]

**Proof:** Case (i): Suppose \(m = 3\). Then \(P_3[v_1, v_3]\) is \(K_2 \cup K_1\) which is strong efficient with strong efficient dominating sets \(\{v_1, v_2\}\) and \(\{v_2, v_3\}\).

Case (ii): Let \(m = 6\). Then in \(P_6[v_1, v_2]\), \(v_1\) and \(v_2\) are the maximum degree vertices which are mutually adjacent and adjacent with all the vertices other than \(v_2\). Hence \(\{v_1, v_2\}\) and \(\{v_2, v_5\}\) are strong efficient dominating sets of \(P_6[v_1, v_2]\).

Case (iii): Let \(m \neq 3\) or \(6\). Then in \(P_m[v_1, v_2]\), \(v_1\) is adjacent with all the vertices other than \(v_2\) and \(D(v_1) = n - 2 = \Delta(P_m[v_1, v_2])\). \(v_2\) is isolated. Therefore \(\{v_1, v_2\}\) is the unique strong efficient dominating set of \(P_m[v_1, v_2]\). Hence the result.
Theorem 3.15: The graph $C_{3n[v_1]}$ is strong efficient with $\gamma_{se}(C_{3n}) + \gamma_{se}[C_{3n[v_1]}] = n + 3$
and $\#\gamma_{se}(C_{3n}) + \#\gamma_{se}[C_{3n[v_1]}] = \begin{cases} 5 & \text{for } n = 2 \\ 4 & \text{for } n \neq 2 \end{cases}$

Proof: Let $v_1, v_2, v_3, \ldots, v_{3n}$ be the vertices of the cycle $C_{3n}$.

Case (i): Suppose $n = 1$. $C_{3[v_1]}$ is $K_1 \cup K_2$ which is strong efficient with the strong efficient dominating sets $\{v_1, v_2\}$ and $\{v_1, v_3\}$.

Case (ii): Suppose $n = 2$. In $C_{6[v_1]}$, $v_1$ strongly dominates all the vertices of $C_6$ except $v_2$ and $v_3$. Clearly $\{v_1, v_2, v_6\}$ and $\{v_4, v_2, v_6\}$ are strong efficient dominating sets.

Case (iii): Suppose $n \geq 3$. In $C_{3n[v_1]}$, $v_1$ is adjacent with all the vertices other than $v_2$ and $v_{3n}$. $\text{Deg}(v_1) = 3n - 3 = \Delta(C_{3n[v_1]}), \text{Deg}(v_i) = 3, 3 \leq i \leq 3n - 1, \text{Deg}(v_2) = \text{Deg}(v_{3n}) = 1$. Hence $\{v_1, v_2, v_{3n}\}$ is the unique strong efficient dominating set of $C_{3n[v_1]}$. Hence the result.

Proposition 3.16: The graph $K_{1,n[v_1]}$ where $v$ is the central vertex of the star $K_{1,n}$ is strong efficient. Also $\gamma_{se}(K_{1,n}) + \gamma_{se}[K_{1,n[v]}] = n + 2$ and $\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[K_{1,n[v]}] = 2, n \geq 1$

Proof: Let $v, v_1, v_2, \ldots, v_n$ be the vertices of the star $K_{1,n}$. $K_{1,n[v_1]}$ is the graph $(n+1)K_1$. So $\gamma_{se}[K_{1,n[v]}] = n + 1$ and $\#\gamma_{se}[K_{1,n[v]}] = 1$. Therefore $\gamma_{se}(K_{1,n}) + \gamma_{se}[K_{1,n[v]}] = n + 2$ and $\#\gamma_{se}(K_{1,n}) + \#\gamma_{se}[K_{1,n[v]}] = 2, n \geq 1$

Proposition 3.17: The graph $K_{1,n[v_1]}$ where $v$ is any pendant vertex of the star $K_{1,n}$ is strong efficient if and only if $n = 1, 2$

Proof: Let $v, v_1, v_2, \ldots, v_n$ be the vertices of the star $K_{1,n}$. Let $K_{1,n[v_1]}$ is the graph obtained by switching the pendant vertex $v_1$ of the star $K_{1,n}$. When $n = 1$, the graph $K_{1,1[v_1]}$ is $2K_1$ which has the unique strong efficient dominating set $\{v, v_1\}$. When $n = 2$, the graph $K_{1,2[v_1]}$ is the path $P_3$ which has the unique strong efficient dominating set $\{v_2\}$. Conversely suppose $n \geq 3$. Let $S$ be a strong efficient dominating set of $K_{1,n[v_1]}$.

Case (i): Suppose $n = 3$. Then the graph $K_{1,3[v_1]}$ is the cycle $C_4$ which is necessarily strong efficient.

Case (ii): Suppose $n \geq 4$. $K_{1,n[v_1]}$ is the graph in which $v$ and $v_1$ are adjacent with $v_2, v_3, \ldots, v_n$. $\text{Deg}(v) = \text{Deg}(v_1) = n - 1, \text{Deg}(v_i) = 2$ for $2 \leq i \leq n$. Let $v_i \in S, v$ and $v_1$ are non adjacent so that $\{v_1, v\} \subseteq S$ for every strong efficient dominating set $S$. $|N_s[v] \cap S| = 2 > 1$, a contradiction. From both cases (i) and (ii), $n = 1, 2$.

Theorem 3.18: The gear graph $G_n$ is strong efficient for all $n \geq 3$.

Proof: Let $v, v_1, v_2, \ldots, v_{2n}$ be the vertices of the gear graph $G_n$. The vertex $v$ is adjacent with $v_{2i-1}; 1 \leq i \leq n$. $\text{Deg}(v) = n = \Delta(G_n), \text{Deg}(v_{2i-1}) = 3, 1 \leq i \leq n, \text{Deg}(v_{2i}) = 2, 1 \leq i \leq n$. $v$ strongly dominates all the vertices $v_{2i-1}; 1 \leq i \leq n$. The vertices $v_{2i}$ and $v_{2i-1}$ are mutually non adjacent with each other. Therefore $\{v, v_2, v_4, \ldots, v_{2n}\}$ is the unique strong efficient dominating set of $G_n$. Thus $\gamma_{se}(G_n) = n + 1$ and $\#\gamma_{se}(G_n) = 1$. 

Theorem 3.19: The graph $G_{n[v_{2i-1}]}$, $1 \leq i \leq n$ is strong efficient if and only if $n = 4$.

Proof: Let $v$, $v_1, v_2, \ldots, v_{2n}$ be the vertices of the graph $G_n$. Suppose $n = 4$. Let $i = 1$. Let $v, v_1, v_2, \ldots, v_8$ be the vertices of $G_{4[v_{2i-1}]}$. $v_1$ is adjacent with all the vertices other than $v, v_2$ and $v_8$. $\text{Deg}(v_1) = 5 = \Delta(G_{4[v_{2i-1}]})$. Therefore $v_1$ strongly dominates all the vertices other than $v, v_2$ and $v_8$ and these vertices are mutually non adjacent with each other. $\text{Deg}(v_2) = \text{Deg}(v_8) = 1$. $\text{Deg}(v) = 3$. $v$ is adjacent with $v_4, v_5$ and $v_6$. $\text{Deg}(v_4) = \text{Deg}(v_5) = \text{Deg}(v_7) = \text{Deg}(v_8) = 4$. $v_2$ is adjacent with $v_3$ and $v_8$ is adjacent with $v_7$. Hence $\{v, v_1, v_2, v_8\}$ is the unique strong efficient dominating set of $G_{4[v_{2i-1}]}$. Proof is similar if $2 \leq i \leq 4$. Thus $G_{4[v_{2i-1}]}$ is strong efficient. Hence $\gamma_{se}[G_{4[v_{1}]}] = 4$ and $\#\gamma_{se}[G_{4[v_{1}]}] = 1$. Conversely, let $G_{n[v_{2i-1}]}$, $1 \leq i \leq n$ be strong efficient. Suppose $n \neq 4$.

Case (i): Let $n = 3$. In $G_{3[v_{2i-1}]}$, $\text{deg}(v_3) = \text{deg}(v_5) = 4 = \Delta(G_{3[v_{2i-1}]}$) and $d(v_3, v_5) = 2$. Hence by the result 1.12, $G_{3[v_{2i-1}]}$ is not strong efficient.

Case (ii): Suppose $n \geq 5$. Let $i = 1$. Let $S$ be a strong efficient dominating set of $G_{n[v_{2i-1}]}$. In $G_{n[v_{2i-1}]}$, the vertex $v_1$ is adjacent with all the vertices other than $v_2, v_{2n}$ and $v$. $\text{Deg}(v_1) = 2n - 3 = \Delta(G_{n[v_{2i-1}]}). \text{deg}(v_{2i-1}) = 4; 2 \leq i \leq n, \text{deg}(v_{2i}) = 3; 2 \leq i \leq n - 1, \text{deg}(v) = n - 1$. The vertex $v_1$ is the unique maximum degree vertex and it strongly dominates all the vertices other than $v_2, v_{2n}$ and $v$. Therefore $v_1 \in S$. The vertices $v_2, v_{2n}$ and $v$ are mutually non adjacent with each other. Hence they belong to $S$. But $|V[S]| = 3 > 1$, for every $i \neq j$, a contradiction. Hence $G_{n[v_{2i-1}]}$ is not strong efficient. Proof is similar for other values of $i$.

Theorem 3.20: The graph $G_{n[v_{2i-1}]}$, $1 \leq i \leq n$, is strong efficient. Further $\gamma_{se}(G_n) + \gamma_{se}[G_{n[v_{2i-1}]}] = n + 4$ and $\#\gamma_{se}(G_n) + \#\gamma_{se}[G_{n[v_{2i-1}]}] = \begin{cases} 3 \text{when } n = 3 \\ 2 \text{ when } n \geq 4 \end{cases}$

Proof: Let $v, v_1, v_2, \ldots, v_{2n}$ be the vertices of the graph $G_n$. Let $i = 1$.

Case (i): Suppose $n = 3$. In $G_{3[v_{2i-1}]}$, $v_2$ and $v_5$ are mutually adjacent and are adjacent with all the vertices other than $v_1$ and $v_3$. $\text{Deg}(v_2) = \text{deg}(v_5) = 4 = \Delta(G_{3[v_{2i-1}]}). \text{deg}(v_1) = \text{deg}(v_3) = 2$. $v_2$ and $v_3$ strongly dominates all the vertices other than $v_1$ and $v_3$. Therefore $\{v_2, v_1, v_3\}$ and $\{v_2, v_1, v_3\}$ are strong efficient dominating sets of $G_{3[v_{2i-1}]}$. Hence $\gamma_{se}[G_{3[v_{2i-1}]}] = 3$ and $\#\gamma_{se}[G_{3[v_{2i-1}]}] = 2$.

Case (ii): Suppose $n \geq 4$. In $G_{n[v_{2i-1}]}$, $v_2$ is adjacent with all the vertices other than $v_1$ and $v_3$. $\text{Deg}(v_2) = 2(n - 1) = \Delta(G_{n[v_{2i-1}]}). v_2$ is the unique maximum degree vertex. $\text{deg}(v_1) = \text{deg}(v_3) = 2 = \Delta(G_{n[v_{2i-1}]}). v_2$ strongly dominates all the vertices other than $v_1$ and $v_3$ which are non adjacent. Hence $\{v_2, v_1, v_3\}$ is the unique strong efficient dominating set of $G_{n[v_{2i-1}]}$. Proof is similar for other values of $i$. Hence the result.

Theorem 3.21: The graph $G_{n[v]}$, where $v$ is the central vertex in a gear graph $G_n$, is strong efficient. Further $\gamma_{se}(G_n) + \gamma_{se}[G_{n[v]}] = 2n + 2$ and $\#\gamma_{se}(G_n) + \#\gamma_{se}[G_{n[v]}] = 2$.
Proof: Let \( v, v_1, v_2, \ldots, v_{2n} \) be the vertices of the graph \( G_n \). In \( G_n[v] \), \( v \) is adjacent with all the vertices \( v_{2i}; 1 \leq i \leq n \) and non adjacent with all the vertices \( v_{2i-1}; 1 \leq i \leq n \). Therefore \( G_n[v] \) is strong efficient. Hence the result.

Theorem 3.22: The graph obtained by switching any one of the central vertices of the bistar \( D_{r,s} \), \( r,s \geq 2 \) is not strong efficient.

Proof: Let \( u,v,v_1, v_2, v_3, \ldots, v_{r+s} \) be the vertices of the bistar \( D_{r,s} \), \( r,s \geq 2 \). \( v_1, v_2, \ldots, v_s \) are the pendant vertices adjacent with \( u \) and \( v_{r+1}, \ldots, v_{r+s} \) are the pendant vertices adjacent with \( v \). In \( D_{r,s[u]} \), both \( u \) and \( v \) are adjacent with the vertices \( v_{r+1}, \ldots, v_{r+s} \). But \( u \) and \( v \) are non adjacent. \( \text{Deg}(u) = \text{Deg}(v) = \Delta(D_{r,s[u]}) \) and \( d(u,v) = 2 \). Therefore by result 2.12, \( D_{r,s[u]} \) is not strong efficient.

Corollary 3.23: \( D_{1,s[u]} \), \( s \geq 2 \) where \( v \) is defined in theorem 3.22 is strong efficient.

Proof: Let \( v_1 \) be the pendant vertex adjacent with the central vertex \( u \) and \( v_2, v_3, \ldots, v_{s+1} \) be the pendant vertices adjacent with \( v \). \( D_{1,s[u]} \) is the graph \( P_3 \cup sK_1 \) which is strong efficient. \( \{v_1, v_2, \ldots, v_s, v_{s+1}\} \) is the unique strong efficient dominating set of \( D_{1,s[u]} \).

Corollary 3.24: \( D_{1,s[u]} \), \( s \geq 2 \) where \( u \) is defined in theorem 3.22 is not strong efficient.

Proof: In \( D_{1,s[u]} \), \( u \) and \( v \) are adjacent with the vertices \( v_2, v_3, \ldots, v_s, v_{s+1} \). Also \( u \) and \( v \) are non adjacent. \( \text{Deg}(u) = \text{Deg}(v) = s = \Delta(D_{1,s[u]}) \). Since \( d(u,v) = 2 \), by result 2.12, the graph \( D_{1,s[u]} \) is not strong efficient.

Theorem 3.25: Let \( D_{r,s[u,v]} \) be the graph obtained by switching both the central vertices \( u \) and \( v \) of the bistar \( D_{r,s} \). Then

\[
\gamma_{se}(D_{r,s}) + \gamma_{se}(D_{r,s[u,v]}) = r + 3 \text{ when } r \leq s
\]

\[
\#\gamma_{se}(D_{r,s}) + \#\gamma_{se}(D_{r,s[u,v]}) = 2 \text{ when } r < s
\]

\[
= 3 \text{ when } r = s
\]

Proof: Let \( u, v, v_1, v_2, \ldots, v_r, v_{r+1}, \ldots, v_{r+s} \) be the vertices of the bistar \( D_{r,s} \). \( v_1, v_2, \ldots, v_r \) are the pendant vertices adjacent with \( u \) and \( v_{r+1}, \ldots, v_{r+s} \) are the pendant vertices adjacent with \( v \). The graph \( D_{r,s[u,v]} \) is \( K_{1,r} \cup K_{1,s} \) which is strong efficient. \( \{u,v\} \) is the unique strong efficient dominating set of \( D_{r,s[u,v]} \). Hence the result.

Theorem 3.26: The graph \( D_{r,s[v_i]} \) where \( 1 \leq i \leq r + s \) obtained by switching a pendant vertex of the bistar \( D_{r,s} \) is strong efficient if and only if \( r = 1 \) and \( i = 1 \) or \( s = 1 \) and \( i = r + 1 \) or both \( r, s = 1 \).

Proof: Case(i): Let \( r, s \geq 2 \) and \( r < s \). Suppose \( D_{r,s[v_i]} ; 1 \leq i \leq r + s \) be strong efficient. Let \( S \) be a strong efficient dominating set of \( D_{r,s[v_i]} \).

Subcase (a): Suppose \( 1 \leq i \leq r \). In \( D_{r,s[v_i]} \), \( v_i \) is adjacent with all the vertices other than \( u \). \( \text{Deg}(v_i) = r + s = \Delta(D_{r,s[v_i]}), \text{Deg}(v) = s + 2, \text{Deg}(u) = r, \text{Deg}(v_j) = 2 \) for \( j \neq i \). \( v_i \) strongly
dominates all the vertices other than \( u \). Hence \( v, u \in S \). \(|N_S[v_k] \cap S| = 2 > 1, k \neq i, 1 \leq k \leq r \). This is a contradiction.

**Subcase i(b):** Suppose \( r+1 \leq i \leq r+s \). In \( D_{r,s}[v_i] \), \( v_i \) is adjacent with all the vertices other than \( v \). \( \text{Deg}(v_i) = r+s = \Delta(D_{r,s}[v_i]), \text{deg}(v) = s, \text{deg}(u) = r+2 \) and \( \text{deg}(v_j) = 2 \) for \( j \neq i \). \( v_i \) strongly dominates all the vertices other than \( v \). Hence \( v_i, v \in S \). \(|N_S[v_k] \cap S| = 2 > 1, k \neq i, r+1 \leq k \leq r+s \). This is a contradiction.

**Case(ii):** Suppose \( r, s > 2 \) and \( r = s \). Proof is similar to that of subcase i(a).

**Case(iii):** Suppose \( r, s = 2 \) and \( 1 \leq i \leq 2 \). In \( D_{1,2}[v_i] \), \( v_i \) is adjacent with all the vertices other than \( u \). \( \text{Deg}(v_i) = \text{deg}(v) = 4 = \Delta(D_{1,2}[v_i]), \text{deg}(u) = 2, \text{deg}(v_j) = 2 \) for \( j \neq i \). Also \( d(u, v) = 2 \). Hence by result 1.12, the graph \( D_{1,2}[v_i] \) is not strong efficient. Proof is similar if \( 3 \leq i \leq 4 \).

**Case(iv):** Suppose \( r = 1 \) and \( i \geq 2 \). In \( D_{1,s}[v_i] \), \( v_i \) is adjacent with all the vertices other than \( v \). \( \text{Deg}(v_i) = s+1 = \Delta(D_{1,s}[v_i]), \text{deg}(v) = s, \text{deg}(v_j) = 2 \) for \( j \neq i \). The vertex \( v_i \) strongly dominates all the vertices other than \( v \). Hence \( v_i, v \in S \). \(|N_S[v_k] \cap S| = 2 > 1, k \neq i, 2 \leq k \leq s+1 \). This is a contradiction.

**Case(v):** Suppose \( s = 1 \) and \( 1 \leq i \leq r \). Proof is similar to that of case(iv). Conversely

**Case(i):** Let \( r = 1 \) and \( i = 1 \). In \( D_{1,s}[v_i] \), the vertex \( v_1 \) is adjacent with all the vertices other than \( u \) and \( v \) is the full degree vertex. \( \{v\} \) is the unique strong efficient dominating set of \( D_{1,s}[v_i] \).

**Case(ii):** Let \( s = 1 \) and \( r = s+1 \). In \( D_{1,s}[v_{r-1}] \), the vertex \( v_{r-1} \) is adjacent with all the vertices other than \( v \) and \( u \) is the full degree vertex. \( \{u\} \) is the unique strong efficient dominating set of \( D_{1,s}[v_{r-1}] \).

**Case(iii):** Let \( r = s = 1 \). Proof is similar to that of case(i) and case(ii). Hence from all the above cases, the graph \( D_{r,s}[v_i] \) is strong efficient.

**Theorem 3.27:** The graph \( H_{n[u_i]} \) where \( u_i \) is the pendant vertex of the \( H \)- graph \( H_n \) is strong efficient if and only if \( n \neq 3 \).

**Proof:** Let \( u_i, v_j \) where \( 1 \leq i \leq n \) be the vertices of the graph \( H_n \). Suppose \( n \geq 4 \). In \( H_{n[u_i]} \), the vertex \( u_1 \) is adjacent with all the vertices other than \( u_2 \). \( u_1 \) strongly dominates all the vertices all the vertices other than \( u_2 \). \( \text{Deg}(u_1) = 2n-2 = \Delta(H_{n[u_i]}), \text{deg}(u_2) = 1 \). Hence \( \{u_1, u_2\} \) is the unique strong efficient dominating set. Similarly \( H_{n[u_n]} \), \( H_{n[v_1]} \) and \( H_{n[v_n]} \) are strong efficient.

Conversely, let \( H_{3[u_i]} \) be strong efficient. Let \( S \) be a strong efficient dominating set. In \( H_{3[u_i]} \), \( u_1 \) is adjacent with all the vertices other than \( u_2 \) and \( v_2 \) is adjacent with all the vertices other than \( u_3 \). \( \text{Deg}(u_1) = \text{deg}(v_2) = 4 = \Delta(H_{3[u_i]} \) and \( u_1, v_2 \) are adjacent. Therefore \( S \) contains either \( u_2 \) or \( v_2 \). \( \text{Deg}(v_1) = \text{deg}(v_3) = \text{deg}(u_2) = \text{deg}(u_3) = 2 = \delta(H_{3[u_i]} \). If \( u_1 \in S \), then \( u_2 \in S \) and \(|N_S[u_3] \cap S| = 2 > 1 \). This is a contradiction. If
\( \nu_2 \in S \), then \( u_3 \in S \) and \( |N_S[u_2] \cap S| = 2 > 1 \). This is also a contradiction. Proof is similar for the graphs \( H_3[u_3] \), \( H_3[\nu_3] \) and \( H_3[p_3] \). Hence the graph \( H_n[u_i] \) where \( \nu_i \) is the pendant vertex of the \( H \)- graph \( H_n \) is strong efficient if and only if \( n \neq 3 \).

**Theorem 3.28:** (i) \( H_n[u_{n+1 \over 2}] \), \( n \geq 3 \) and \( n \) is odd is strong efficient if and only if \( n \neq 3 \).

(ii) \( H_n[u_{n+1 \over 2}] \), \( n \geq 4 \) and \( n \) is even is strong efficient if and only if \( n \neq 4 \).

**Proof:** Let \( u_i, \nu_i \) where \( 1 \leq i \leq n \) be the vertices of the graph \( H_n \).

**Case (i):** Suppose \( n \neq 3 \) and \( n \) be odd. In \( H_n[u_{n+1 \over 2}] \) is adjacent with all the vertices other than \( u_{n-1 \over 2}, u_{n+3 \over 2} \) and \( \nu_{n+1 \over 2} \). Deg \( u_{n+1 \over 2} = 2n-4 = \Delta (H_n[u_{n+1 \over 2}]) \), deg \( u_{n-1 \over 2} = 1 = \delta (H_n[u_{n+1 \over 2}]) \) and deg \( \nu_{n+1 \over 2} = 2 = \deg (\nu_1) = \deg (\nu_n) = \deg (u_1) = \deg (u_n) \). deg \( u_k \) = deg \( \nu_k \) = 3 for \( k \neq {n+1 \over 2}, 1 \) and \( n \). Also \( u_{n+1 \over 2}, u_{n-1 \over 2}, u_{n+3 \over 2} \) and \( \nu_{n+1 \over 2} \) are mutually non adjacent. Therefore \( \{u_{n+1 \over 2}, u_{n-1 \over 2}, u_{n+3 \over 2}, \nu_{n+1 \over 2}\} \) is the unique strong efficient dominating set of \( H_n[u_{n+1 \over 2}] \). Hence \( H_n[u_{n+1 \over 2}] \) is strong efficient.

Conversely suppose \( n = 3 \). \( H_3[u_a] \) is the graph \( C_4 \cup 2K_1 \). Since \( C_4 \) is not strong efficient, \( H_3[u_a] \) is not strong efficient.

**Case (ii):** Suppose \( n \neq 4 \) and \( n \) be even. In \( H_n[u_{n+1 \over 2}] \), the vertex \( u_{n+1 \over 2} \) is adjacent with all the vertices other than \( u_{n-1 \over 2}, u_{n+3 \over 2} \) and \( \nu_{n+1 \over 2} \). Also Deg \( u_{n+1 \over 2} = 2n-4 = \Delta (H_n[u_{n+1 \over 2}]) \), deg \( u_{n-1 \over 2} = 1 = \delta (H_n[u_{n+1 \over 2}]) \) and deg \( \nu_{n+1 \over 2} = 2 = \deg (\nu_1) = \deg (\nu_n) = \deg (u_1) = \deg (u_n) \). All the other vertices are of degree 3. The vertex \( u_{n+1 \over 2} \) strongly dominates all the vertices other than \( u_{n-1 \over 2}, u_{n+3 \over 2} \) and \( \nu_{n+1 \over 2} \). Therefore \( \{u_{n+1 \over 2}, u_{n-1 \over 2}, u_{n+3 \over 2}, \nu_{n+1 \over 2}\} \) is the unique strong efficient dominating set of \( H_n[u_{n+1 \over 2}] \). Therefore \( H_n[u_{n+1 \over 2}] \) is strong efficient.

Conversely let \( n = 4 \). In \( H_4[u_{n+1 \over 2}] \), deg \( u_2 \) = 4 = \( \Delta (H_4[u_{n+1 \over 2}]) \). The vertex \( u_2 \) strongly dominates all the vertices other than \( u_1, u_3, \) and \( \nu_3 \). The vertex \( u_1 \) is an isolate. Deg \( u_1 = 1 = \delta (u_1) \), deg \( \nu_3 = 2 = \deg (\nu_1) = \deg (\nu_2) = 3 \). Suppose \( H_4[u_{n+1 \over 2}] \) is strong efficient. Let \( S \) be a strong efficient dominating set of \( H_4[u_{n+1 \over 2}] \). Hence \( u_2 \in S, u_1, \nu_3 \). \( u_3 \) and \( u_2 \) are mutually
non adjacent. Hence they belong to $S$. $|N_5[v_3] \cap S| = |\{u_2, v_3\}| = 2 > 1$. This is a contradiction. Hence the graph $H_{4[u_2]}$ is not strong efficient. Hence the theorem.

4. Conclusion

In this paper, the authors studied some Nordhaus- Gaddum type relations on strong efficient domination number of a graph and its derived graph. They introduced the concept of number of strong efficient dominating sets and studied the relation between the number of strong efficient dominating sets of a graph and its derived graph. Similar studies can be made on this type for various derived graphs.

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