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Special Semigroup Classes Over Some Monoid Constructions and a New Example of a Finitely Presented Monoid with a Non-Finitely Generated Group of Units

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Abstract. In this paper, necessary and sufficient conditions are studied for Bruck-Reilly and gener- alized Bruck-Reilly ∗-extensions of direct product of k monoids to be regular, unit regular, completely regular and orthodox. Moreover, we give an example of a finitely presented monoid (generalized Bruck-Reilly ∗-extension of Bruck-Reilly extension of a free group with infinite rank), the group of units of which is not finitelygenerated.

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Keywords and Phrases: Generalized Bruck-Reilly ∗-extension, finite generation, finite presentability.

Bazı Monoid Yapıları Üzerinde Özel Yarıgrup Sınıfları ve Terslenebilir Elemanlarının Grubu Sonlu Üreteçli Olmayan Sonlu Sunumlu Yeni Bir Monoid Örneği

Özet. Bu makalede k tane monoidin direkt çarpımının Bruck-Reilly ve genelleştirilmiş Bruck-Reilly *- genişlemelerinin, regüler, terslenebilir regüler, tamamen regüler ve orthodox olabilmesi için gerek ve yeter koşullar çalışılmıştır. Ayrıca, biz terslenebilir elemanlarının grubu sonlu üreteçli olmayan sonlu sunumlu bir monoid (sonsuz ranklı bir serbest grubun Bruck-Reilly genişlemesinin genelleştirilmiş Bruck-Reilly *-genişlemesi) örneği verdik.

Anahtar Kelimeler: Genelleştirilmiş Bruck-Reilly *-genişlemesi, sonlu üreteçlilik, sonlu sunumluluk.

1. INTRODUCTION and PRELIMINARIES

The Bruck-Reilly extension, studied by Bruck [3], Reilly [14] and Munn [12], is con- sidered a fundamental construction in the theory of semigroups. Many classes of regular semigroups are characterized by Bruck-Reilly extensions; for instance, any bisimple regu- lar *w*-semigroup is isomorphic to a Reilly extension of a group [14] and any simple regular *w*-semigroup is isomorphic to a Bruck-Reilly extension of a finite chain of groups [10, 11]. Also a presentation for the *Bruck-Reilly extension* was given in [7]. Later on, in another important paper [1], the author obtained a new monoid, namely the *generalized Bruck- Reilly ∗-extension*, and presented the structure of the ∗-bisimple type *A* *w*-semigroup. Later on, in [15] the authors studied the structure theorem of the ∗-bisimple type *A* w^2- semigroups as the generalized Bruck-Reilly ∗-extension. Moreover, in a joint work [9], it has been recently defined a presentation for the generalized Bruck-Reilly ∗-extension and then obtained a Gröbner-Shirshov basis of this new construction.

In [13], the authors studied *regularity, unit regularity, completely regularity and ortho- dox properties* over Bruck-Reilly and generalized Bruck-Reilly ∗-extensions of monoids. In this paper, as a next step of these studies, we investigated these semigroup proper- ties over Bruck-Reilly and generalized Bruck-Reilly ∗-extensions of direct product of *k* monoids. Moreover, by using

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Bruck-Reilly and generalizaed Bruck-Reilly $*$-extensions, we give an example of a monoid defined by finitely many generators and defining relations, the group of units of which is finitely generated.

Now let us present the following fundamental material that will be needed in this paper. We refer the reader to [5, 6] for more detail knowledge.

A semigroup S is called

- *regular* if all its elements are regular. An element $x \in S$ is called regular if there exists $y \in S$ such that $xyx = x$.
- *unit- regular* if for each $x \in S$ there exists a unit y of S for which $x = xyx$.
- *completely regular* if for each $x \in S$ there exists $y \in V(x)$ ($V(x)$: the set of inverses of x) such that $xy = yx$.
- *idempotent* if for each $x \in S$ then $x \in E(S)$. ($E(S)$: the set of idempotent elements in S)
- *orthodox* if it is regular and the idempotent set forms a subsemigroup.

Groups are of course regular semigroups, but the class of regular semigroups is vastly more extensive than the class of groups (see [6]). Further, to have an inverse element can also be important in a semigroup. Therefore we call S is an *inverse semigroup* if every element has exactly one inverse. The well known examples of inverse and completely regular semigroups are groups. Besides bands are completely regular semigroups and semilattices are inverse semigroups.

2. BRUCK-REILLY EXTENSION OF MONOIDS

Let T be a monoid with an endomorphism θ defined on it such that $T\theta$ is in the $\mathcal{H}$-class of the identity 1_T of T. Also, let $\mathbb{N}^0$ denotes the set of non-negative integers. Hence the set $\mathbb{N}^0 \times T \times \mathbb{N}^0$ with the multiplication

$$(m.x.n)(m',x',n') = \left(m-n+t,(x\theta^{t-n})\left(x'\theta^{t-m'}\right),n'-m'+t\right), \qquad (1)$$

where $t = max(n,m')$ and θ^0 is the identity map on T, forms a monoid with identity $(0,1_T,0)$. Then this monoid is called the *Bruck-Reilly extension* of T determined by θ and denoted by $BR(T,\theta)$.

This construction is a generalization of constructions given in [3], [12] and [14]. The presentation of this construction is given below.

Theorem 2.1 *[7] Let T be a monoid defined by the presentation $\langle X \mid R\rangle$ and let*

$\theta : T \to T$ be an endomorphism. The monoid $BR(T,\theta)$ is then defined by the presentation

$$< X, b, c\ ;\ R, bc = 1, bx = (x\theta)b, xc = c(x\theta) >, \quad (2)$$

where $x \in X$.

The following properties of $BR(T,\theta)$ are easy to derive from the definition $BR(T,\theta)$:
(BR1) $T \cong \{0\} \times T \times \{0\}$
(BR2) The element b is right invertible but not left invertible; the element c is left invertible but not right invertible.
(BR3) $U\big(BR(T,\theta)\big) = \{0\} \times U(T) \times \{0\} \cong U(T)$. ($U\big(BR(T,\theta)\big)$ and $U(T)$ denotes the group of units in $BR(T,\theta)$ and T respectively. This is actually a generalization of the construction created by Bruck [3], Munn [12] and Reilly [14].

In the above references, the authors used $BR(T,\theta)$ to prove that every semigroup embeds in a simple monoid, and to characterize special classes of inverse semigroups. In [12], Munn showed that $BR(T,\theta)$ is an inverse semigroup if and only if T is inverse. So, the following result is a direct consequence of [12, Theorem 3.1].

Corollary 2.2 *Let T be an arbitrary monoid. Then* $BR(T,\theta)$ *is regular if and only if T is regular.*

Also in [12, Theorem 3.1], the author proved that $(m,x,n) \in E(BR(T,\theta))$ if and only if $m = n$ and $x \in E(T)$. Now let T_i $(1 \le i \le k)$ be monoids. We denote $T_1 \times T_2 \times \ldots \times T_k = \{(t_1, t_2, \ldots, t_k) : t_i \in T_i\}$ by $\times_{i=1}^{k} T_i$. Let φ be an endomorphism of $\times_{i=1}^{k} T_i$.

Theorem 2.3. $BR\big(\times_{i=1}^{k} T_i, \varphi\big)$ *is regular if and only if* $\times_{i=1}^{k} T_i$ *is regular.*

Proof. Let $BR\big(\times_{i=1}^{k} T_i, \varphi\big)$ be regular. Then for any $(m,(t_1,t_2,\ldots,t_k),n) \in BR\big(\times_{i=1}^{k} T_i, \varphi\big)$, there exists an element $(n,(t_1',t_2',\ldots,t_k'),m) \in BR\big(\times_{i=1}^{k} T_i, \varphi\big)$ such that

$(m,(t_1,t_2,\ldots,t_k),n) = (m,(t_1,t_2,\ldots,t_k),n)(n,(t_1',t_2',\ldots,t_k'),m)(m,(t_1,t_2,\ldots,t_k),n)$. By using (1), we have $(t_1,t_2,\ldots,t_k) = (t_1,t_2,\ldots,t_k)(t_1',t_2',\ldots,t_k')(t_1,t_2,\ldots,t_k)$. Hence $\times_{i=1}^{k} T_i$ is regular.

Conversely, let us suppose that $\times_{i=1}^{k} T_i$ is regular. Then for any $(t_1,t_2,\ldots,t_k) \in \times_{i=1}^{k} T_i$, there exists an element $(t_1',t_2',\ldots,t_k') \in \times_{i=1}^{k} T_i$ such that $(t_1,t_2,\ldots,t_k) = (t_1,t_2,\ldots,t_k)(t_1',t_2',\ldots,t_k')(t_1,t_2,\ldots,t_k)$. Now we need to show that for any $(m,(t_1,t_2,\ldots,t_k),n) \in BR\big(\times_{i=1}^{k} T_i, \varphi\big)$, there exists an element $(p,(h_1,h_2,\ldots,h_k),q) \in BR\big(\times_{i=1}^{k} T_i, \varphi\big)$ such that

$(m,(t_1,t_2,\ldots,t_k),n) = (m,(t_1,t_2,\ldots,t_k),n)(p,(h_1,h_2,\ldots,h_k),q)(m,(t_1,t_2,\ldots,t_k),n)$.
For $p = n, q = m$ and $(h_1,h_2,\ldots,h_k) = (t_1',t_2',\ldots,t_k')$, the following equality holds:
$(m,(t_1,t_2,\ldots,t_k),n) = (m,(t_1,t_2,\ldots,t_k),n)(n,(t_1',t_2',\ldots,t_k'),m)(m,(t_1,t_2,\ldots,t_k),n)$. Consequently, $BR\big(\times_{i=1}^{k} T_i, \varphi\big)$ is regular.

Theorem 2.4 $BR\big(\times_{i=1}^{k} T_i, \varphi\big)$ *is unit regular if and only if* $\times_{i=1}^{k} T_i$ *is unit regular.*

Proof. Let $BR\big(\times_{i=1}^{k} T_i, \varphi\big)$ be unit regular. Then for any $(m,(t_1,t_2,\ldots,t_k),n) \in BR\big(\times_{i=1}^{k} T_i, \varphi\big)$ there exists an element $(n,(t_1',t_2',\ldots,t_k'),m) \in G$ (where G is the group of units of $BR\big(\times_{i=1}^{k} T_i, \varphi\big)$) such that

$(m,(t_1,t_2,\dots,t_k),n) = (m,(t_1,t_2,\dots,t_k),n)(n,(t_1',t_2',\dots,t_k'),m)(m,(t_1,t_2,\dots,t_k),n)$. By using (1), we have $(t_1,t_2,\dots,t_k) = (t_1,t_2,\dots,t_k)(t_1',t_2',\dots,t_k')(t_1,t_2,\dots,t_k)$. Hence $\times_{i=1}^{k} T_i$ is unit regular.

Conversely, let us suppose that $\times_{i=1}^{k} T_i$ is unit regular. Then for any $(t_1,t_2,\dots,t_k) \in \times_{i=1}^{k} T_i$, there exists an element $(t_1',t_2',\dots,t_k') \in G_{\times_{i=1}^{k} T_i}$ (where $G_{\times_{i=1}^{k} T_i}$ is the group of units of $\times_{i=1}^{k} T_i$) such that $(t_1,t_2,\dots,t_k) = (t_1,t_2,\dots,t_k)(t_1',t_2',\dots,t_k')(t_1,t_2,\dots,t_k)$. Now we need to show that for any $(m,(t_1,t_2,\dots,t_k),n) \in BR(\times_{i=1}^{k} T_i,\varphi)$, there exists an element $(p,(h_1,h_2,\dots,h_k),q) \in G$ such that

$$(m,(t_1,t_2,\dots,t_k),n) = (m,(t_1,t_2,\dots,t_k),n)(p,(h_1,h_2,\dots,h_k),q)(m,(t_1,t_2,\dots,t_k),n).$$

For $p=n, q=m$ and $(h_1,h_2,\dots,h_k) = (t_1',t_2',\dots,t_k')$, the following equality holds:
$(m,(t_1,t_2,\dots,t_k),n) = (m,(t_1,t_2,\dots,t_k),n)(n,(t_1',t_2',\dots,t_k'),m)(m,(t_1,t_2,\dots,t_k),n)$. Consequently, $BR(\times_{i=1}^{k} T_i,\varphi)$ is unit regular.

Lemma 2.5 Let $(m,(t_1,t_2,\dots,t_k),n) \in BR(\times_{i=1}^{k} T_i,\varphi)$. If $(m,(t_1,t_2,\dots,t_k),n)$ is completely regular then $m=n$.

Proof. Let $(m,(t_1,t_2,\dots,t_k),n) \in BR(\times_{i=1}^{k} T_i,\varphi)$ be completely regular. Then there exists an element $(n,(t_1',t_2',\dots,t_k'),m) \in V((m,(t_1,t_2,\dots,t_k),n))$ with $(t_1',t_2',\dots,t_k')$ is an inverse of $(t_1,t_2,\dots,t_k)$ (by [12]) such that
$(m,(t_1,t_2,\dots,t_k),n)(n,(t_1',t_2',\dots,t_k'),m) = (n,(t_1',t_2',\dots,t_k'),m)(m,(t_1,t_2,\dots,t_k),n)$. From this equality we have $m=n$ and $(t_1,t_2,\dots,t_k)(t_1',t_2',\dots,t_k') = (t_1',t_2',\dots,t_k')(t_1,t_2,\dots,t_k)$.

Theorem 2.6 $S = \{(m,(t_1,t_2,\dots,t_k),m) : (t_1,t_2,\dots,t_k) \in \times_{i=1}^{k} T_i, m \in \mathbb{N}^0\} \leq BR(\times_{i=1}^{k} T_i,\varphi)$ *is completely regular if and only if* $\times_{i=1}^{k} T_i$ *is completely regular.*

Proof. Let $S \leq BR(\times_{i=1}^{k} T_i,\varphi)$ be completely regular. Then for any $(m,(t_1,t_2,\dots,t_k),n) \in BR(\times_{i=1}^{k} T_i,\varphi)$, there exists an element $(n,(t_1',t_2',\dots,t_k'),m) \in V((m,(t_1,t_2,\dots,t_k),n))$ with $(t_1',t_2',\dots,t_k')$ is an inverse of $(t_1,t_2,\dots,t_k)$ such that
$(m,(t_1,t_2,\dots,t_k),n)(n,(t_1',t_2',\dots,t_k'),m) = (n,(t_1',t_2',\dots,t_k'),m)(m,(t_1,t_2,\dots,t_k),n)$. From this equality we have $(t_1,t_2,\dots,t_k)(t_1',t_2',\dots,t_k') = (t_1',t_2',\dots,t_k')(t_1,t_2,\dots,t_k)$. Hence $\times_{i=1}^{k} T_i$ is completely regular.

Conversely, let us suppose that $\times_{i=1}^{k} T_i$ is completely regular. Then for any $(t_1,t_2,\dots,t_k) \in \times_{i=1}^{k} T_i$, there exists an element $(h_1,h_2,\dots,h_k) \in V((t_1,t_2,\dots,t_k))$ such that
$(t_1,t_2,\dots,t_k)(h_1,h_2,\dots,h_k) = (h_1,h_2,\dots,h_k)(t_1,t_2,\dots,t_k)$. Now we need show that for any $(m,(t_1,t_2,\dots,t_k),m) \in BR(\times_{i=1}^{k} T_i,\varphi)$, there exists an element $(m,(h_1,h_2,\dots,h_k),m) \in V((m,(t_1,t_2,\dots,t_k),m))$ such that $(m,(t_1,t_2,\dots,t_k),m)(m,(h_1,h_2,\dots,h_k),m) = (m,(h_1,h_2,\dots,h_k),m)(m,(t_1,t_2,\dots,t_k),m)$. The last equality is clear from (1). Consequently $S \leq BR(\times_{i=1}^{k} T_i,\varphi)$ is completely regular.

Theorem 2.7 $BR(\times_{i=1}^{k} T_i,\varphi)$ *is orthodox if and only if* $\times_{i=1}^{k} T_i$ *is orthodox.*

Proof. Let $BR(\times_{i=1}^{k} T_i,\varphi)$ be orthodox. Then for any $(m,(t_1,t_2,\dots,t_k),n) \in BR(\times_{i=1}^{k} T_i,\varphi)$, there exists an element $(n,(t_1',t_2',\dots,t_k'),m) \in BR(\times_{i=1}^{k} T_i,\varphi)$ with $(t_1,t_2,\dots,t_k) = (t_1,t_2,\dots,t_k)(t_1',t_2',\dots,t_k')(t_1,t_2,\dots,t_k)$ such that

$(m,(t_1,t_2,\dots,t_k),n) = (m,(t_1,t_2,\dots,t_k),n)(n,(t'_1,t'_2,\dots,t'_k),m)(m,(t_1,t_2,\dots,t_k),n)$. On the other hand,

$$E\left(BR\left(\times_{i=1}^{k} T_i,\varphi\right)\right) = \left\{(m,(e_1,e_2,\dots,e_k),m):\ (e_1,e_2,\dots,e_k) \in \times_{i=1}^{k} T_i, (e_1,e_2,\dots,e_k)^2 = (e_1,e_2,\dots,e_k), m \in \mathbb{N}^0\right\}$$

is a subsemigroup of $BR\left(\times_{i=1}^{k} T_i,\varphi\right)$. By commutativity $E(\times_{i=1}^{k} T_i)$ is a subsemigroup of $\times_{i=1}^{k} T_i$. Moreover, by Theorem 2.3, we know that $BR\left(\times_{i=1}^{k} T_i,\varphi\right)$ is regular if and only if $\times_{i=1}^{k} T_i$ is regular. Consequently if $\times_{i=1}^{k} T_i$ is orthodox.

Conversely, let $\times_{i=1}^{k} T_i$ be orthodox. Then $\times_{i=1}^{k} T_i$ is regular and $E(\times_{i=1}^{k} T_i)$ is a subsemigroup of $\times_{i=1}^{k} T_i$. Now for any $(m,(e_1,e_2,\dots,e_k),m),(n,(f_1,f_2,\dots,f_k),n) \in E\left(BR\left(\times_{i=1}^{k} T_i,\varphi\right)\right)$ we have

$$(m,(e_1,e_2,\dots,e_k),m)(n,(f_1,f_2,\dots,f_k),n) = (\max(m,n),\left((e_1,e_2,\dots,e_k)\theta^{\max(m,n)-m}\right)(f_1,f_2,\dots,f_k)\theta^{\max(m,n)-n},\max(m,n)).$$

Then $E\left(BR\left(\times_{i=1}^{k} T_i,\varphi\right)\right)$ is a subsemigroup of $BR\left(\times_{i=1}^{k} T_i,\varphi\right)$. Moreover, by Theorem 2.3, we know that $\times_{i=1}^{k} T_i$ is regular if and only if $BR\left(\times_{i=1}^{k} T_i,\varphi\right)$ is regular. Consequently, $BR\left(\times_{i=1}^{k} T_i,\varphi\right)$ is orthodox.

3. GENERALIZED BRUCK-REILLY *-EXTENSION of MONOIDS

Suppose that T is an arbitrary monoid having H_1^* and H_1 as the $\mathcal{H}^*$- and $\mathcal{H}$- classes, respectively. Let us also suppose that each of H_1^* and H_1 contains the identity element 1_T of T. Moreover, let us assume that β and γ are morphisms from T into H_1^* and, for an element u in H_1, let λ_u be the inner automorphism of H_1^* defined by $x \mapsto uxu^{-1}$ such that $\gamma\lambda_u = \beta\gamma$.
Now one can consider the set $S = \mathbb{N}^0 \times \mathbb{N}^0 \times T \times \mathbb{N}^0 \times \mathbb{N}^0$ into a semigroup with a multiplication

$$(m,n,x,p,q)(m',n',x',p',q') = \begin{cases} \left(m, n-p+d, \left(x\beta^{d-p}\right)\left(x'\beta^{d-n'}\right), p'-n'+d, q'\right), & if\ q=m' \\ \left(m,n,x\left(\left(\left(u^{-n'}(x'\gamma)u^{p'}\right)\gamma^{q-m'-1}\right)\beta^{p}\right), p, q'-m'+q\right), & if\ q>m' \\ \left(m-q+m', n', \left(\left(\left(u^{-n}(x\gamma)u^{p}\right)\gamma^{m'-q-1}\right)\beta^{n'}\right)x', p', q'\right), & if\ q<m' \end{cases}$$

where $d = \max(p,n')$ and β^0,γ^0 are interpreted as the identity map of T, and also u^0 is interpreted as the identity 1_T of T. In [15], Shung and Wang showed that S is a monoid with the identity $(0,0,1_T,0,0)$ of T. In fact this new monoid $S = \mathbb{N}^0 \times \mathbb{N}^0 \times T \times \mathbb{N}^0 \times \mathbb{N}^0$ is denoted by $GBR^*(T;\beta,\gamma;u)$ and called *generalized Bruck-Reilly *-extension* of T determined by the morphisms β,γ and the element u.

In [9], the authors found the presentation of this new monoid construction and obtained the normal form of elements by using Gröbner-Shirshov basis method.

Theorem 3.1 *[9] Let T be a monoid defined by the presentation $\langle X \mid R\rangle$ and let β,γ be morphisms from T into H_1^*. Then the monoid $GBR^*(T;\beta,\gamma;u)$ is defined by the presentation*

$$\langle X,y,z,b,c \mid R, yz=1, bc=1, yx=(x\gamma)y, xz=z(x\gamma), bx=(x\beta)b, xc=c(x\beta), yb=uy, yc=u^{-1}y, bz=zu, cz=$$

where $x \in X$.

The following properties of $GBR^*(T;\beta,\gamma;u)$ are easy to derive from the definition of $GBR^*(T;\beta,\gamma;u)$:
(GBR1) $T \cong \{0\} \times \{0\} \times T \times \{0\} \times \{0\}$
(GBR2) $U(GBR^*(T;\beta,\gamma;u)) = \{0\} \times \{0\} \times U(T) \times \{0\} \times \{0\} \cong U(T)$. ($U(GBR^*(T;\beta,\gamma;u))$ and $U(T)$ denotes the group of units in $GBR^*(T;\beta,\gamma;u)$ and T, respectively.)

In [15], it has been proved the following two lemmas:

Lemma 3.2 If $(m,n,x,p,q) \in GBR^*(T;\beta,\gamma;u)$ then $(m,n,x,p,q) \in E(GBR^*(T;\beta,\gamma;u))$ if and only if $m = q, n = p$ and $x \in E(T)$

Lemma 3.3 If $(m,n,x,p,q) \in GBR^*(T;\beta,\gamma;u)$ then (m,n,x,p,q) has an inverse$(m',n',x',p',q') \in S$ if and only if $m' = q, n' = p, p' = n, q' = m$ and x' is an inverse of x in T.

Then we have an immediate consequence of Lemma 3.3 as in the following.

Corollary 3.4 Let T be a monoid. Then $GBR^*(T;\beta,\gamma;u)$ is regular if and only if T is regular.

Now we generalize this result to generalized Bruck-Reilly *-extension of direct product of k monoids.

Theorem 3.5 $GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$ *is regular if and only if* $\times_{i=1}^{k} T_i$ *is regular.*

Proof. Let $GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$ be regular. Then for any $(m,n,(t_1,t_2,\dots,t_k),p,q) \in GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$, there exists an element $(q,p,(t_1',t_2',\dots,t_k'),n,m) \in GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$ such that

$$(m,n,(t_1,t_2,\dots,t_k),p,q)(q,p,(t_1',t_2',\dots,t_k'),n,m)(m,n,(t_1,t_2,\dots,t_k),p,q) = (m,n,(t_1,t_2,\dots,t_k),p,q).$$

By the multiplication of generalized Bruck-Reilly *-extension, we have

$(t_1,t_2,\dots,t_k) = (t_1,t_2,\dots,t_k)(t_1',t_2',\dots,t_k')(t_1,t_2,\dots,t_k)$. Consequently, $\times_{i=1}^{k} T_i$ is regular.

Conversely, let us suppose that $\times_{i=1}^{k} T_i$ is regular. Then for any $(t_1,t_2,\dots,t_k) \in \times_{i=1}^{k} T_i$, there exists an element $(t_1',t_2',\dots,t_k') \in \times_{i=1}^{k} T_i$ such that $(t_1,t_2,\dots,t_k) = (t_1,t_2,\dots,t_k)(t_1',t_2',\dots,t_k')(t_1,t_2,\dots,t_k)$. Now we need to show that for any $(m,n,(t_1,t_2,\dots,t_k),p,q) \in GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$, there exists an element $(m',n',(t_1',t_2',\dots,t_k'),p',q') \in GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$ such that

$$(m,n,(t_1,t_2,\dots,t_k),p,q) = (m,n,(t_1,t_2,\dots,t_k),p,q)(m',n',(t_1',t_2',\dots,t_k'),p',q')(m,n,(t_1,t_2,\dots,t_k),p,q).$$

For $m' = q,\ n' = p,\ p' = n,\ q' = m$, this equality holds. So $GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$ is regular.

Theorem 3.6 $GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$ *is unit regular if and only if* $\times_{i=1}^{k} T_i$ *is unit regular.*

Proof. Let $GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$ be unit regular. Then for any $(m, n, (t_1, t_2, \dots, t_k), p, q) \in GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$, there exists an element $(q, p, (t_1', t_2', \dots, t_k'), n, m) \in G$ (where G is the group of units of $GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$) such that

$$(m, n, (t_1, t_2, \dots, t_k), p, q)(q, p, (t_1', t_2', \dots, t_k'), n, m)(m, n, (t_1, t_2, \dots, t_k), p, q) = (m, n, (t_1, t_2, \dots, t_k), p, q).$$

By using the multiplication of generalized Bruck-Reilly *-extension, we have $(t_1, t_2, \dots, t_k) = (t_1, t_2, \dots, t_k)(t_1', t_2', \dots, t_k')(t_1, t_2, \dots, t_k)$. Therefore, $\times_{i=1}^{k} T_i$ is unit regular.

Conversely, let us suppose that $\times_{i=1}^{k} T_i$ is unit regular. Then for any $(t_1, t_2, \dots, t_k) \in \times_{i=1}^{k} T_i$, there exists an element $(t_1', t_2', \dots, t_k') \in G_{\times_{i=1}^{k} T_i}$ (where $G_{\times_{i=1}^{k} T_i}$ is the group of units of $\times_{i=1}^{k} T_i$) such that $(t_1, t_2, \dots, t_k) = (t_1, t_2, \dots, t_k)(t_1', t_2', \dots, t_k')(t_1, t_2, \dots, t_k)$. Now we need to show that for any $(m, n, (t_1, t_2, \dots, t_k), p, q) \in GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$, there exists an element $(m', n', (t_1', t_2', \dots, t_k'), p', q') \in GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$ such that

$$(m, n, (t_1, t_2, \dots, t_k), p, q) = (m, n, (t_1, t_2, \dots, t_k), p, q)(m', n', (t_1', t_2', \dots, t_k'), p', q')(m, n, (t_1, t_2, \dots, t_k), p, q).$$

For $m' = q,\ n' = p,\ p' = n,\ q' = m$ it is seen that this equality holds. Hence $GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$ is unit regular.

Lemma 3.7 *Let $(m, n, (t_1, t_2, \dots, t_k), p, q) \in GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$. If $(m, n, (t_1, t_2, \dots, t_k), p, q)$ is completely regular then $m = q$ and $n = p$.*

Proof. Let $(m, n, (t_1, t_2, \dots, t_k), p, q) \in GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$ be completely regular. Then there exists an element $(q, p, (t_1', t_2', \dots, t_k'), n, m) \in V((m, n, (t_1, t_2, \dots, t_k), p, q))$ with $(t_1', t_2', \dots, t_k')$ is an inverse of $(t_1, t_2, \dots, t_k)$ (by [12]) such that $(q, p, (t_1', t_2', \dots, t_k'), n, m) = (q, p, (t_1', t_2', \dots, t_k'), n, m)(m, n, (t_1, t_2, \dots, t_k), p, q)$. By the multiplication of generalized Bruck-Reilly *-extension, we have $m = q, n = p$ and $(t_1, t_2, \dots, t_k)(t_1', t_2', \dots, t_k') = (t_1', t_2', \dots, t_k')(t_1, t_2, \dots, t_k)$.

Theorem 3.8 $S = \{(m, n, (t_1, t_2, \dots, t_k), n, m) : (t_1, t_2, \dots, t_k) \in \times_{i=1}^{k} T_i, m \in \mathbb{N}\} \leq GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$ *is completely regular if and only if* $\times_{i=1}^{k} T_i$ *is completely regular.*

Proof. Let $S \leq GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$ be completely regular. Then for any $(m, n, (t_1, t_2, \dots, t_k), n, m) \in GBR^*(\times_{i=1}^{k} T_i; \beta, \gamma; u)$, there exists an element $(m, n, (t_1', t_2', \dots, t_k'), n, m) \in V((m, n, (t_1, t_2, \dots, t_k), n, m))$ with $(t_1', t_2', \dots, t_k')$ is an inverse of $(t_1, t_2, \dots, t_k)$ such that

$$(m, n, (t_1, t_2, \dots, t_k), n, m)(m, n, (t_1', t_2', \dots, t_k'), n, m) = (m, n, (t_1', t_2', \dots, t_k'), n, m)(m, n, (t_1, t_2, \dots, t_k), n, m)$$

By the multiplication of generalized Bruck-Reilly *-extension, we have $(t_1, t_2, \dots, t_k)(t_1', t_2', \dots, t_k') = (t_1', t_2', \dots, t_k')(t_1, t_2, \dots, t_k)$. Hence $\times_{i=1}^{k} T_i$ is completely regular.

Conversely, let us suppose that $\times_{i=1}^{k} T_i$ is completely regular. Then for any $(t_1, t_2, \dots, t_k) \in \times_{i=1}^{k} T_i$, there exists an element $(t_1', t_2', \dots, t_k')) \in V((t_1, t_2, \dots, t_k))$ such that

$(t_1,t_2,\dots,t_k)(t_1',t_2',\dots,t_k')=(t_1',t_2',\dots,t_k')(t_1,t_2,\dots,t_k)$. Now we need to show that for any $(m,n,(t_1,t_2,\dots,t_k),n,m)\in GBR^*(\times_{i=1}^k T_i;\beta,\gamma;u)$, there exists an element $(m,n,(t_1',t_2',\dots,t_k'),n,m)\in V((m,n,(t_1,t_2,\dots,t_k),n,m))$ such that

$$(m,n,(t_1,t_2,\dots,t_k),n,m)(m,n,(t_1',t_2',\dots,t_k'),n,m)=(m,n,(t_1',t_2',\dots,t_k'),n,m)(m,n,(t_1,t_2,\dots,t_k),n,m).$$

It is an inverse of $(t_1,t_2,\dots,t_k)$ it is a routine matter to show the lasst equality. Consequently, $S\le GBR^*(\times_{i=1}^k T_i;\beta,\gamma;u)$ is completely regular.

Theorem 3.9 $GBR^*(\times_{i=1}^k T_i;\beta,\gamma;u)$ is orthodox if and only if $\times_{i=1}^k T_i$ is orthodox.

Proof. Let $GBR^*(\times_{i=1}^k T_i;\beta,\gamma;u)$ be orthodox. Then for any $(m,n,(t_1,t_2,\dots,t_k),p,q)\in GBR^*(\times_{i=1}^k T_i;\beta,\gamma;u)$, there exists an element $(q,p,(h_1,h_2,\dots,h_k),n,m)\in GBR^*(\times_{i=1}^k T_i;\beta,\gamma;u)$ with $(t_1,t_2,\dots,t_k)=(t_1,t_2,\dots,t_k)(h_1,h_2,\dots,h_k)(t_1,t_2,\dots,t_k)$ such that

$$(m,n,(t_1,t_2,\dots,t_k),p,q)=(m,n,(t_1,t_2,\dots,t_k),p,q)(q,p,(h_1,h_2,\dots,h_k),n,m)(m,n,(t_1,t_2,\dots,t_k),p,q).$$

On the other hand,

$$E\left(GBR^*\left(\times_{i=1}^k T_i;\beta,\gamma;u\right)\right)=\left\{(m,n,(e_1,e_2,\dots,e_k),n,m):\ (e_1,e_2,\dots,e_k)\in\times_{i=1}^k T_i, m,n\in\mathbb{N}^0\right\}$$

is a subsemigroup of $GBR^*(\times_{i=1}^k T_i;\beta,\gamma;u)$. Hence $E(\times_{i=1}^k T_i)$ is a subsemigroup of $\times_{i=1}^k T_i$. Moreover, by Theorem 3.5, we know that $GBR^*(\times_{i=1}^k T_i;\beta,\gamma;u)$ is regular if and only if $\times_{i=1}^k T_i$ is regular. Consequently $\times_{i=1}^k T_i$ is orthodox.

Conversely, let $\times_{i=1}^k T_i$ be orthodox. Then $\times_{i=1}^k T_i$ is regular and $E(\times_{i=1}^k T_i)$ is a subsemigroup of $\times_{i=1}^k T_i$. We need to show that for any $(m,n,(e_1,e_2,\dots,e_k),n,m),(m',n',(f_1,f_2,\dots,f_k),n',m')\in E\left(GBR^*\left(\times_{i=1}^k T_i;\beta,\gamma;u\right)\right)$,
$(m,n,(e_1,e_2,\dots,e_k),n,m)(m',n',(f_1,f_2,\dots,f_k),n',m')\in E\left(GBR^*\left(\times_{i=1}^k T_i;\beta,\gamma;u\right)\right)$. Now by considering the multiplication $(m,n,(e_1,e_2,\dots,e_k),n,m)(m',n',(f_1,f_2,\dots,f_k),n',m')$, we have the following cases:

Case (i): If $m=m'$, then we get

$$(m,n,(e_1,e_2,\dots,e_k),n,m)(m',n',(f_1,f_2,\dots,f_k),n',m')=\left(m,d',(e_1,e_2,\dots,e_k)\beta^{d'-n}\right)(f_1,f_2,\dots,f_k)\beta^{d'-n'},d',m')$$

where $d'=\max(n,n')$. Since $(e_1,e_2,\dots,e_k),(f_1,f_2,\dots,f_k)\in E(\times_{i=1}^k T_i)$, we deduce that $(f_1,f_2,\dots,f_k)\beta^{d'-n'},(e_1,e_2,\dots,e_k)\beta^{d'-n}\in E(\times_{i=1}^k T_i)$ i.e.
$\left((f_1,f_2,\dots,f_k)\beta^{d'-n'}\right)\left((e_1,e_2,\dots,e_k)\beta^{d'-n}\right)=\left((e_1,e_2,\dots,e_k)\beta^{d'-n}\right)\left((f_1,f_2,\dots,f_k)\beta^{d'-n'}\right)$.

Case (ii): If $m<m'$ or $m>m'$, then we get

$(m,n,(e_1,e_2,\dots,e_k),n,m)(m',n',(f_1,f_2,\dots,f_k),n',m')$
$$= (m',n',\left(\left(\left(u^{-n}((e_1,e_2,\dots,e_k)\gamma)u^{n}\right)\gamma^{m'-m-1}\right)\beta^{n'}\right)(f_1,f_2,\dots,f_k),n',m')$$
$(m,n,(e_1,e_2,\dots,e_k),n,m)(m',n',(f_1,f_2,\dots,f_k),n',m') =$
$(m,n,(\left(\left(u^{-n'}((f_1,f_2,\dots,f_k)\gamma)u^{n'}\right)\gamma^{m-m'-1}\right)\beta^{n}(f_1,f_2,\dots,f_k),n,m)$

respectively. Since $\left(\left(u^{-n'}((f_1,f_2,\dots,f_k)\gamma)u^{n'}\right)\gamma^{m-m'-1}\right)\beta^{n} \in E(\times_{i=1}^{k} T_i)$, we obtain $E(GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$ is a subsemigroup of $GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$. Moreover, by Theorem 3.5, we know that $GBR^*(\times_{i=1}^{k} T_i;\beta,\gamma;u)$ is regular if and only if $\times_{i=1}^{k} T_i$ is regular. Hence the result.

4. A FINITELY PRESENTED MONOID WITH A NON-FINITELY GENERATED GROUP OF UNITS

The relationship between properties of a monoid M and its group of units $U(M)$ has been studying for many years. In this sense Adjan [2] showed the properties of finite presentability and solvable word problem for the special monoids and the group of units. Then Zhang [16] showed that the conjugacy problem for a special monoid was reducible to the conjugacy problem for its group of units. After that the same author in [17], proved that the group of units of every special monoid is finitely preseneted. In [4], Carvalho and Ruskuc gave a new example of a finitely presented monoid with a non- finitely generated group of units by considering double Bruck-Reilly extension of a free group with infinite rank. Then in [8], Karpuz showed another example takes the form of Bruck-Reilly extension of generalized Bruck-Reilly *-extension of free group with infinite rank. In this section, by Theorem 4.1, we give another example which takes the form of a generalized Bruck-Reilly *-extension of Bruck-Reilly extension of free group with infinite rank, FG_∞, defined by the following presentation

$$< a_0,a_1,\dots,a_0^{-1},a_1^{-1},\dots;a_i^{-\epsilon},a_i^{\epsilon}=1\ (\epsilon=\pm1, i\geq 0) >.$$

Theorem 4.1 *Let T denotes the monoid given as generalized Bruck-Reilly *-extension of Bruck-Reilly extension of FG_∞. The group of units of T defined by the finite presentation given in (16) is not finitely generated.*

Proof. Let θ be an endomorphism defined by $\theta: FG_\infty \to FG_\infty$, $a_i^{\epsilon} \mapsto a_{i+1}^{\epsilon}$. Hence we obtain the following presentation

$$< a_i^{\epsilon},b,c\ ;\ a_i^{-\epsilon}a_i^{\epsilon}=1,\ \ bc=1,\ \ ba_i^{\epsilon}=a_{i+1}^{\epsilon}b,\ \ a_i^{\epsilon}c=ca_{i+1}^{\epsilon}\ \ (\epsilon=\pm1, i\geq 0) >, \quad (3)$$

For $BR(FG_\infty,\theta)$. By using the relations $bc=1$ and $ba_i^{\epsilon}=a_{i+1}^{\epsilon}b$ we deduce that $a_{i+1}^{\epsilon}=ba_i^{\epsilon}c$. For $i=0$, we have $a_1^{\epsilon}=ba_0^{\epsilon}c$. For $i=1$, we get $a_2^{\epsilon}=ba_1^{\epsilon}c=b^2a_0^{\epsilon}c^2$. Then by inductive argument for $\epsilon=\pm1$ and $i\geq 0$ we obtain

$$a_i^{\epsilon}=b^ia_0^{\epsilon}c^i. \quad (4)$$

By using (4), we can eliminate all the generators $a_i^{\epsilon}(\epsilon=\pm1, i\geq 0)$ from the presentation (3). For simplicity, we use a^{ϵ} instead of a_0^{ϵ}, then we have the following finitely generated (but not finitely presented) presentation for $BR(FG_\infty,\theta)$:

$< a, a^{-1}, b, c\ ; b^i a^{-\epsilon} c^i b^i a^{\epsilon} c^i = 1, bc = 1, b^{i+1}a^{\epsilon}c^i = b^{i+1}a^{\epsilon}c^{i+1}b,\ b^i a^{\epsilon} c^{i+1} = cb^{i+1}a^{\epsilon}c^{i+1}\ (\epsilon = \pm 1, i \geq 0) >$ (5)

Now let us consider generalized Bruck-Reilly *-extension of $BR(FG_\infty,\theta)$ defined by (5) under the homomorphisms $\beta,\gamma: BR(FG_\infty,\theta) \to H_1^*$ (where H_1^* is the H^*-class which contains the identity of FG_∞) such that

$$\beta: b \mapsto b, c \mapsto c, a^{\epsilon} \mapsto ba^{\epsilon}c \text{ and } \gamma: b \mapsto b, c \mapsto c, a^{\epsilon} \mapsto ba^{\epsilon}c\ (\epsilon = \pm 1). \qquad (6)$$

Now we show that β and γ define homomorphism by checking β and γ map the relations given in (6) into relations that are valid in $BR(FG_\infty,\theta)$:

$$(b^i a^{-\epsilon} c^i b^i a^{\epsilon} c^i)\beta = b^i.ba^{-\epsilon}cc^i b^i ba^{\epsilon}cc^i = 1 = 1\beta,$$
$$\left(b^{i+1}a^{\epsilon}c^i\right)\beta = b^{i+1}.ba^{\epsilon}c.c^i = b^{i+2}a^{\epsilon}c^{i+1} = b^{i+2}a^{\epsilon}c^{i+2}b = (b^{i+1}a^{\epsilon}c^{i+1}b)\beta,$$
$$(b^i a^{\epsilon} c^i)\beta = b^i ba^{\epsilon}cc^{i+1} = b^{i+1}a^{\epsilon}c^{i+2} = cb^{i+2}a^{\epsilon}c^{i+2} = (cb^{i+2}a^{\epsilon}c^{i+2})\beta.$$

This can be shown similarly for γ. Hence we get the monoid $GBR^*(BR(FG_\infty,\theta),\beta,\gamma;u)$, say T, and the following presentation:

$$
\begin{aligned}
< a, a^{-1}, b, c, y, z, \bar{b}, \bar{c}\ ; \quad & b^i a^{-\epsilon} c^i b^i a^{\epsilon} c^i = 1, && (7)\\
& bc = 1, && (8)\\
& b^{i+1}a^{\epsilon}c^i = b^{i+1}a^{\epsilon}c^{i+1}b, \quad b^i a^{\epsilon} c^{i+1} = cb^{i+1}a^{\epsilon}c^{i+1}, && (9)\\
& \bar{b}\bar{c} = 1, \quad yz = 1, && (10)\\
& \bar{b}a^{\epsilon} = (a^{\epsilon}\beta)\bar{b},\ \ a^{\epsilon}\bar{c} = \bar{c}(a^{\epsilon}\beta), && (11)\\
& ya^{\epsilon} = (a^{\epsilon}\gamma)y,\ \ a^{\epsilon}z = (a^{\epsilon}\gamma), && (12)\\
& yb = uy,\ \ bz = zu,\ \ uyc = y,\ \ czu = z, && (13)\\
& y\bar{b} = uy,\ \ \bar{b}z = zu,\ \ uy\bar{c} = y,\ \ \bar{c}zu = z,\ \ \bar{b}b = b\bar{b},\ \ \bar{b}c = c\bar{b}, &&\\
& \bar{b}b = (b\beta)\bar{b},\ \ c\bar{c} = \bar{c}c,\ \ b\bar{c} = \bar{c}b, && (14)\\
& yb = by,\ \ yc = cy,\ \ bz = zb,\ \ cz = zc >. && (15)
\end{aligned}
$$

Now we consider a relation (7) and multiply it by $\bar{b}$ from the left and by $\bar{c}$ from the right, and by using the relations given in (10)-(15) we obtain

$$
\begin{aligned}
\bar{b}b^i a^{-\epsilon} c^i b^i a^{\epsilon} c^i \bar{c} = \bar{b}\bar{c} &\Longrightarrow\ b^i.ba^{-\epsilon}c.c^i b^i.ba^{\epsilon}c.c^i \bar{b}\bar{c} = 1\\
&\Longrightarrow\ b^{i+1}a^{-\epsilon}c^{i+1}b^{i+1}a^{\epsilon}c^{i+1} = 1.
\end{aligned}
$$

It can be easily seen that all relations in (7) are consequences of $a^{-\epsilon}a^{\epsilon} = 1$ and (10)-(15). Similarly, all relations (9) are consequences of the relations (9) for $i = 1$ and (10)-(15). Hence we conclude that our monoid is defined by the following presentation

$$
\begin{aligned}
< a, a^{-1}, b, c, y, z, \bar{b}, \bar{c}\ ; \quad & aa^{-1} = a^{-1}a = bc = \bar{b}\bar{c} = yz = 1,\\
& ba^{\epsilon} = ba^{\epsilon}cb,\ \ a^{\epsilon}c = cba^{\epsilon}c,\\
& b^{i+1}a^{\epsilon}c^i = b^{i+1}a^{\epsilon}c^{i+1}b,\ \ b^i a^{\epsilon} c^{i+1} = cb^{i+1}a^{\epsilon}c^{i+1}\\
& \bar{b}a^{\epsilon} = ba^{\epsilon}c\bar{b},\ \ a^{\epsilon}\bar{c} = \bar{c}ba^{\epsilon}c, && (16)
\end{aligned}
$$

$$ya^{\epsilon} = ba^{\epsilon}cy, \quad a^{\epsilon}z = zba^{\epsilon}c,$$
$$yb = uy, \quad bz = zu, \quad uyc = y, \quad czu = z,$$
$$y\bar{b} = uy, \quad \bar{b}z = zu, \quad uy\bar{c} = y, \quad \bar{c}zu = z,$$
$$\bar{b}b = b\bar{b}, \quad \bar{b}c = c\bar{b}, \quad c\bar{c} = \bar{c}c, \quad b\bar{c} = \bar{c}b,$$
$$yb = by, \quad yc = cy, \quad bz = zb, \quad cz = zc >,$$

which is finitely presented. By using properties (BR3) and (GBR2), we obtain the following:

$$U(T) = U\big(GBR(BR(FG_{\infty},\theta),\beta,\gamma;u)\big) \cong U\big(BR(FG_{\infty},\theta)\big) \cong \{0\} \times U(FG_{\infty}) \times \{0\} \cong U(FG_{\infty}) = FG_{\infty}$$

And so the group of units of T is not finitely generated.

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