Research Article

Pseudo-Riemannian Submanifolds with 3-planar Geodesics

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Abstract

In the present paper, we study pseudo-Riemannian submanifolds which have 3-planar geodesic normal sections. Further, we consider W-curves (helices) on pseudo-Riemannian submanifolds. Finally, we give necessary and sufficient condition for a normal section to be a W-curve on pseudo-Riemannian submanifolds.

Keywords: Pseudo-Riemannian submanifold, geodesic normal section, W-curve, planar geodesic.

3-Düzlemsel Geodezikli Yarı-Riemann Altmanifoldlar

Öz

Bu çalışmada, 3-düzlemsel geodezik normal kesitlere sahip yarı-Riemann almanifoldlar ele alınmıştır. Daha sonra, yarı-Riemann altmanifoldları üzerindeki W-eğrileri (helisler) incelenmiştir. Son olarak, yarı-Riemann altmanifoldları üzerindeki normal kesitlerin W-eğrisi olması için gerek ve yeter şartlar elde edilmiştir.

Anahtar Kelimeler: Yarı-Riemann altmanifold, geodezik normal kesit, W-eğri, düzlemsel geodezik

Introduction

Let $M^n$ be an n-dimensional Riemannian manifold. A regular curve $\gamma$ in $M^n$ is called a helix if its first and second curvatures are constant and the third curvature is zero. It has been shown that every helix in a Riemannian submanifold $M^n$ is also a helix in the ambient space [1]. For the pseudo-Riemannian manifold $M^p_r$, helices are defined almost the same way as the Riemannian case. The helices are characterized in Lorentzian submanifold $M^p_r \subset N^m_s$ [2].

A submanifold $M^p_r \subset N^m_s$ is said to have planar geodesics if the image of each geodesic of $M^p_r$ lies in a 2-plane of $N^m_s$ [3]. In the Riemannian case such submanifolds were studied in [4], [5], [6], and [7]. Recently, Kim studied minimal surfaces of pseudo-Euclidean spaces with

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geodesic normal sections [8].

In the present study, we give some results toward a characterization of 3-planar geodesic immersions $f : M^n_r \to N^m_s$ from an $n$-dimensional, connected pseudo-Riemannian manifold $M^n_r$ into an $m$-dimensional pseudo-Riemannian manifold $N^m_s$. Further, we consider $W$-curves (helices) on pseudo-Riemannian submanifolds. Finally, we give necessary and sufficient condition for a normal section to be a $W$-curve on pseudo-Riemannian submanifold $M^n_r$.

**Basic Concepts**

Let $M^n_r \subset N^m_s$ be a submanifold in an $m$-dimensional pseudo-Riemannian manifold $N^m_s$ of index $s$. Let $\nabla$ and $\tilde{\nabla}$ denote the covariant derivatives of $M^n_r$ and $N^m_s$ respectively. Then, for $X, Y \in T_p(M^n_r)$ the second fundamental form $h$ of $M^n_r$ is defined by

$$h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y.$$  \hfill (1)

For a normal vector field $\xi \in N(M^n_r)$ we put

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$  \hfill (2)

where $A_\xi$ is the shape operator and $D$ is the normal connection of $M^n_r$.

The covariant derivatives of $h$ is given by

$$(\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z)$$
$$- h(\tilde{\nabla}_X Y, Z) - h(Y, \nabla_X Z),$$  \hfill (3)

where $X, Y, Z \in T_p(M^n_r)$ and $\nabla$ is the Vander Waerden-Bortolotti connection [9]. Then the Codazzi equation

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) = (\nabla_Z h)(X, Y).$$  \hfill (4)

holds. If $\nabla h = 0$, then $h$ is said to be parallel [10].

The mean curvature vector field $H$ of $M^n_r$ is defined by

$$H = \frac{1}{n} \sum \langle e_i, e_i \rangle h(e_i, e_i), \ i = 1, ..., n,$$  \hfill (5)

where $\{e_1, e_2, ..., e_n\}$ is an orthonormal frame field of $M^n_r$. Consequently, $H$ is called parallel when $DH = 0$ holds.

If the second fundamental form $h$ satisfies

$$g(X, Y)H = h(X, Y),$$  \hfill (6)

for any $X, Y \in T_p(M^n_r)$, then $M^n_r$ is called a totally umbilical. A totally umbilical submanifold with parallel mean curvature vector fields is said to be an extrinsic sphere [11].

**Helices in a Pseudo-Riemannian Manifold**

Let $\gamma$ be a regular curve in a pseudo-Riemannian manifold $M^n_r$. We denote $\gamma'(s) = X$, when $\langle X, X \rangle = \varepsilon = \pm 1$ $\gamma$ is called a unit speed curve. The curve $\gamma$ is called a Frenet curve of rank $d$ ($0 \leq d \leq n$), if its derivatives $\gamma'(s), \gamma''(s), ..., \gamma^{(d)}(s)$ are linearly independent and $\gamma'(s), \gamma''(s), ..., \gamma^{(d+1)}(s)$ are no longer linearly independent for all $s \in I$ [7]. To each Frenet curve of order $d$ we can associate an orthonormal $d$ frame $\{V_1, V_2, ..., V_d\}$ along $\gamma$, called the Frenet
frame, and \( k_1, k_2, ..., k_{d-1} \) are curvature functions of \( \gamma \).

We have the following result.

**Proposition 1:** Let \( \gamma : I \rightarrow M^n_r \) be a non-null smooth curve of osculating order \( d \) in of \( M^n_r \), and \( \{V_1 = X, V_2, ..., V_d\} \) its Frenet frame. Then the following Frenet equations are hold:

\[
V_1' = \nabla_X X = \varepsilon_2 k_1 V_2, \tag{7}
\]

\[
V_2' = \nabla_X V_2 = -\varepsilon_1 k_1 V_1 + \varepsilon_2 k_2 V_3, \tag{8}
\]

\[
\vdots
\]

\[
V_{d-1}' = \nabla_X V_{d-1} = -\varepsilon_{d-2} k_{d-2} V_{d-2} + \varepsilon_{d-1} k_{d-1} V_d, \tag{9}
\]

\[
V_d' = \nabla_X V_d = -\varepsilon_{d-1} k_d V_{d-1}, \tag{10}
\]

where \( \varepsilon_i = \langle V_i, V_i \rangle = \pm 1, 1 \leq i \leq d - 1 \) and \( k_i \) are curvature functions of \( \gamma \).

**Definition 2:** A smooth curve \( \gamma \) of rank \( d \) on \( M^n_r \) is called a W-curve of rank \( d \) if its curvatures \( k_1, k_2, ..., k_{d-1} \) are all constant and \( k_d = 0 \) [7].

**Proposition 3:** Let \( \gamma \) be a non-null W-curve of rank 2 in \( M^n_r \). Then the third derivative \( \gamma''' \) of \( \gamma \) is a scalar multiple of \( \gamma' \). In this case necessarily

\[
\gamma'''(s) = -\varepsilon_1 \varepsilon_2 k_1^2 \gamma'(s). \tag{11}
\]

**Proof:** By the use of (7) we have

\[
\gamma'(s) = \varepsilon_2 k_1 V_2(s). \]

Furthermore, differentiating this equation with respect to \( s \) and using (8) we obtain

\[
\gamma''(s) = -\varepsilon_1 \varepsilon_2 k_2^2 X + \varepsilon_1 k_1 V_2(s) + \varepsilon_2 k_2 V_3(s) \tag{12}
\]

Since \( \gamma \) is a W-curve of rank 2 then by definition \( k_1 \) is constant and \( k_2 = 0 \) we get the result.

**Proposition 4:** Let \( \gamma \) be a non-null W-curve of \( M^n_r \). If \( \gamma \) is of osculating order 3 then

\[
\gamma'''(s) = -\varepsilon_2 (\varepsilon_1 k_1^2 + \varepsilon_3 k_2^2) \gamma''(s) \tag{13}
\]

holds.

**Proof:** Differentiating (12) and using the fact that \( k_1, k_2 \) are constant and \( k_3 = 0 \) we get the result.

**Planar Geodesic Immersions**

Let \( M^n_r \subset N^m_s \) be a submanifold in an \( m \)-dimensional pseudo-Riemannian manifold \( N^m_s \) of index \( s \). For \( p \in M^n_r \) and \( X \in T_p(M^n_r) \) the vector \( X \) and the normal space \( N^m_p(M^n_r) \) determine a \((m-n+1)\)-dimensional totally geodesic submanifold \( \Gamma \) of \( N^n_r \). The intersection of \( M^n_r \) with \( \Gamma \) gives rise a curve \( \gamma \) (in a neighborhood of \( p \)) called the normal section of \( M^n_r \) at \( p \) in the direction of \( X \) [12]. The submanifold \( M^n_r \) is said to have \( d \)-planar normal sections if for each normal section \( \gamma \) the higher order derivatives \( \gamma'(s), \gamma''(s), ..., \gamma^{(d)}(s), \gamma^{(d+1)}(s), 1 \leq d \leq m-n+1 \) are linearly dependent as vectors in \( \Gamma \) [12].

The submanifold \( M^n_r \) is said to have \( d \)-planar geodesic normal sections if each normal section of \( M^n_r \) is a geodesic of \( M^n_r \). The immersion in pseudo-Euclidean space with 2-planar geodesic normal section have been studied in [3]. See also [4].

**Example 5:** [3] Pseudo-Riemannian sphere
\[ S_s(c) = \left\{ p \in \mathbb{R}^n : \langle p - a, p - a \rangle = \frac{1}{c} \right\}, c > 0, \quad (14) \]

and pseudo-Riemannian hyperbolic space
\[ H_s(c) = \left\{ p \in \mathbb{R}^n : \langle p - a, p - a \rangle = \frac{1}{c} \right\}, c < 0, \quad (15) \]

both have 2-planar geodesic normal sections.

We have the following result.

**Proposition 6:** Let \( \gamma \) be a non-null geodesic normal section of \( M^n_r \subset N^m_s \). If \( \gamma'(s) = X(s) \), then we have
\[
\gamma''(s) = h(X, X), \quad (16)
\]
\[
\gamma'''(s) = -A_{h(x,x)}X + (\nabla_X h)(X, X), \quad (17)
\]
\[
\gamma^{(n)}(s) = -\nabla_X (A_{h(x,x)}X) - h(A_{h(x,x)}X, X) - A_{(\nabla_X h)(X, X)}X + (\nabla_X \nabla_X h)(X, X), \quad (18)
\]

**Definition 7:** The submanifold \( M^n_r \) (or the isometric immersion \( f \)) is said to be pseudo-isotropic at \( p \) if
\[
L = \langle h(X, X), h(X, X) \rangle, \]

is independent of the choice of unit vector \( X \) tangent to \( M^n_r \) at \( p \). In particular if \( L \) is independent of the points then \( M^n_r \) is said to be constant pseudo-isotropic.

The submanifold \( M^n_r \) is pseudo-isotropic if and only if
\[
\langle h(X, X), h(X, Y) \rangle = 0,
\]
for any orthonormal vectors \( X \) and \( Y \) [3].

The following results are well-known.

**Theorem 8:** [3] If a submanifold \( M^n_r \subset E^m_s \) has 2-planar geodesic normal sections, then it is a submanifold with zero mean curvature in a hypersphere \( S^{m-1}_{s-1} \) or \( H^{m-1}_{s-1} \) if and only if \( L \) is a non-zero constant.

**Theorem 9:** [8] The surface \( M^2_r \subset E^m_s \) with 2-planar geodesic normal sections is constant pseudo-isotropic.

**Theorem 10:** [13] Let \( M^n_r \) be a pseudo-Riemannian submanifold of index \( r \) of a pseudo-Euclidean space \( E^m_s \) of index \( s \) with geodesics normal sections. Then
\[
\langle (\nabla_X h)(X, X), (\nabla_X h)(X, X) \rangle \quad (19)
\]
is constant on their tangent bundle \( UM \) of \( M^n_r \).

**Theorem 11:** [13] Let \( M^2_r \) be a minimal surface of \( E^5_s \) with geodesics normal sections. Then we have

i) \( M^2_r \) has parallel second fundamental form and 0-pseudo isotropic (i.e. \( L=0 \)),

ii) \( M^2_r \) has 2-planar geodesic normal sections,

iii) \( M^2_r \) is flat.

**Main Results**

Submanifolds \( M^n \) in \( E^{n+d} \) with 3-planar normal sections have been studied by S.J. Li for the case \( M^n \) is isotropic [14] and sphered [15]. See also [16] for the case \( M^n \) is a product manifold in \( E^{n+d} \). In [17] the authors consider submanifolds in a real space form \( N^{n+d}(c) \) with 3-planar geodesic normal sections.
We proved the following results.

**Proposition 12:** Let $M^n_r \subset N^n_s$ be a submanifold with 3-planar geodesic normal sections then $M^n_r$ is constant pseudo-isotropic.

**Proof:** Similar to the proof of Lemma 4.1 in [18].

**Proposition 13:** Let $M^n_r \subset N^n_s$ be a submanifold with 3-planar geodesic normal sections then we have

\[(\nabla_X h)(X, X) = \varepsilon_2(Xk_1)V_2 + \varepsilon_2\varepsilon_3k_2k_3V_3, \tag{20}\]

and

\[A_{h(X, X)}X = \varepsilon_1\varepsilon_2k_1^2X, \tag{21}\]

hold.

**Proof:** Let $\gamma$ be a normal section of $M^n_r$ at point $p = \gamma(s)$ in the direction of $X$. Further, we suppose that $k_1(s)$ is positive. Then $k_1$ is also smooth and there exists a unit vector field $V_2$ along $\gamma$ normal to $M^n_r$ such that

\[h(X, X) = \langle V_2, V_2 \rangle k_1V_2. \tag{22}\]

Since $\tilde{\nabla}_X V_2$ is also tangent to $M^n_r$, there exists a vector field $V_3$ normal to $M^n_r$ and mutually orthogonal to $X$ and $V_2$ such that

\[\tilde{\nabla}_X V_2 = -\langle X, X \rangle k_1X + \langle V_3, V_2 \rangle k_2V_3. \tag{23}\]

Differentiating (22) covariantly and using (23) we get

\[(\tilde{\nabla}_X h)(X, X) = -\varepsilon_1\varepsilon_2k_1^2X
+ \varepsilon_2(Xk_1)V_2 + \varepsilon_2\varepsilon_3k_2k_3V_3, \tag{24}\]

where $\langle V_i, V_j \rangle = \varepsilon_i = \pm 1$. Comparing (24) with (17) we get the result.

**Proposition 14:** Let $\gamma$ be a normal section of $M^n_r$ at point $p = \gamma(s)$ in the direction of $X$. $\gamma$ is a non-null W-curve of rank 2 in $M^n_r$ if and only if

\[\nabla_X\nabla_X X + g(\nabla_X X, \nabla_X X)g(X, X)X = 0 \tag{25}\]

holds.

**Proof:** Since $\gamma'(s) = X(s)$, $\gamma''(s) = \nabla_X \nabla_X X$ and

\[g(X, X) = \varepsilon_1, \ g(\nabla_X X, \nabla_X X) = \varepsilon_2k_1^2. \]

So, by the use of the equality $\gamma''(s) = \varepsilon_2k_1V_2(s)$ we get the result.

**Theorem 15:** Let $M^n_r$ be a totally umbilical submanifold of $N^n_s$ with parallel mean curvature vector field. If the normal section $\gamma$ is a W-curve of osculating order 2. Then $\gamma$ is also a W-curve of $N^n_s$ with the same order.

**Proof:** Suppose $\gamma$ is a W-curve of rank 2 in $M^n_r$ then it satisfies the equality (25). Further, by the use of (1) we get

\[\gamma'' = \tilde{\nabla}_X X = \nabla_X X + h(X, X). \tag{26}\]

Since $M^n_r$ is totally umbilical then $g(X, X)H = h(X, X)$. So, the equation (26) reduces to

\[\gamma'' = \tilde{\nabla}_X X = \nabla_X X + g(X, X)H. \tag{27}\]

Differentiating the equation (27) with respect to $X$, we obtain
\[ \gamma^* = \ddot{\gamma}_X \dot{\gamma}_X = \gamma_X \dot{\gamma}_X + g(\gamma_X, \dot{\gamma}_X)H \]

Further, taking use of \( DH = 0 \) and (26)-(28) get

\[ \ddot{\gamma}_X \dot{\gamma}_X + g(\ddot{\gamma}_X, \dot{\gamma}_X)g(X, X)X = \gamma_X \ddot{\gamma}_X + g(\ddot{\gamma}_X, \dot{\gamma}_X)g(X, X)X \]

So, by previous proposition \( \gamma \) is a W-curve of rank 2 in \( N^m_s \).

References


