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Some Fixed Point Theorems for Ordered Contractions in Partial b-Metric Spaces

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Abstract

In this paper, some fixed point theorems in a partial b-metric space endowed with a partial order are proved. The results of this paper generalize and extend the Banach contraction principle and some other known results in partial b-metric spaces endowed with a partial order. Some examples are given which illustrate the cases when new results can be applied while old ones cannot.

Keywords

Partial order
Partial b-metric space
b-metric space
Contraction
Fixed point

1. INTRODUCTION

Bakhtin [8] and Czerwik [29] introduced b-metric spaces as a generalization of metric spaces. In these spaces, the triangular inequality of the usual metric function was replaced by a more general inequality consisting a constant $s \geq 1$ such that for $s = 1$ we obtain the usual metric as a special case. They also obtained the generalized version of Banach contraction principle in such spaces. After this work, several interesting generalization in b-metric spaces have been obtained (see [1],[17],[21],[23],[24],[26],[32] and the references therein). Matthews [31] introduced the notion of partial metric spaces as a part of the study of denotational semantics of dataflow network. In these spaces, the usual metric was generalized by introducing the nonzero self-distance of points of space. Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verification.

On the other hand, Ran and Reurings [4] and Nieto and Lopez [13],[14] obtained the existence of fixed points of a self- mapping of a metric space equipped with a partial order. The fixed point results in spaces equipped with a partial order can be applied in proving existence and uniqueness of solutions for matrix equations as well as for boundary value problems of ordinary differential equations, integral equations, fuzzy equations, of problems in L-spaces etc. (see [2],[4],[7],[9],[10],[12],[13],[14],[15],[16][25]). The results of Ran and Reurings [4] and Nieto and Lopez [13],[14] were generalized by several authors (see, e.g., [2],[5],[6],[9],[11],[12],[18],[28],[33],[34]).

Shukla [32] generalized b-metric and partial metric spaces by introducing the notion of partial b-metric spaces and proved the Banach contraction principle in such spaces. Some generalizations and fixed point results on partial b-metric spaces can be found in [3],[19],[20],[35]. In this paper, we prove a generalization of Banach contraction principle in partial b-metric space endowed with a partial order. Our results generalize the results of Bakhtin [8], Czerwik [29], Ran and Reurings [4], Nieto and Lopez [13],[14], Matthews [31] and a recent result of Shukla [32]. Examples are given which illustrate the results and show that the generalizations are proper.

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2. PRELIMINARIES

First, we recall some definitions from b-metric, partial metric and partial b-metric spaces (see [8],[31],[32]).

Definition 1. Let X be a nonempty set and the mapping$d: X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) satisfies:

(bM1)$d(x, y) = 0$ if and only if $x = y$;

(bM2)$d(x, y) = d(y, x)$;

(bM3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$,

for all $x, y \in X$. Then d is called a b-metric on X and (X, d) is called a b-metric space with coefficient s.

Definition 2. A partial metric on a nonempty set X is a function $p: X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$:

(P1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;

(P2) $p(x, x) \leq p(x, y)$;

(P3) $p(x, y) = p(y, x)$;

(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Definition 3. A partial b-metric on a nonempty set X is a function $b: X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(Pb1) $x = y$ if and only if $b(x, x) = b(x, y) = b(y, y)$;

(Pb2) $b(x, x) \leq b(x, y)$;

(Pb3) $b(x, y) = b(y, x)$;

(Pb4) there exists a real number $s \geq 1$ such that $b(x, y) \leq s[b(x, z) + b(z, y)] - b(z, z)$.

A partial b-metric space is a pair (X, b) such that X is a nonempty set and b is a partial b-metric on X. The number s is called the coefficient of (X, b).

Remark 1 ([32]). In a partial b-metric space (X, b) if $x, y \in X$ and $b(x, y) = 0$ then $x = y$, but converse may not be true.

Remark 2 ([32]). It is clear that every partial metric space is a partial b-metric space with coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

Example 1 ([32]). Let $X = \mathbb{R}^+$, $p > 1$ a constant and $b: X \times X \to \mathbb{R}^+$ be defined by

$$b(x, y) = [\max\{x, y\}]^p + |x - y|^p \text{ for all } x, y \in X.$$

Then (X, b) is a partial b-metric space with coefficient $s = 2^p > 1$, but it is neither a b-metric nor a partial metric space. Indeed, for any $x > 0$ we have $b(x, x) = x^p \neq 0$, therefore, b is not a b-metric on X. Also, for $x = 5, y = 1, z = 4$ we have $b(x, y) = 5^p + 4^p$ and $b(x, z) + b(z, y) - b(z, z) = 5^p + 1 + 4^p + 3^p - 4^p = 5^p + 1 + 3^p$, so $b(x, y) > b(x, z) + b(z, y) - b(z, z)$ for all $p > 1$, therefore b is not a partial metric on X.

For some more examples of partial b-metric space we refer to [32].

For each $x \in X$ and $\varepsilon > 0$, put $B_p(x, \varepsilon) = \{y \in X: b(x, y) < \varepsilon + b(x, x)\}$ and $\mathcal{B} = \{B_b(x, \varepsilon): x \in X, \varepsilon > 0\}$. Although, $\mathcal{B}$ is not a base for any topology on X, hence is not a topology on X. However, $\mathcal{B}$ can be a sub-base for some topology τ on X which is T_0 but need not to be T_1 (see, [36]).

Now we define Cauchy sequence and convergent sequence in partial b-metric spaces.

Definition 4. ([32, 36]**).** Let (X, b) be a partial b-metric space with coefficient s. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ is said to be convergent and converges to x, if $\lim_{n\to\infty} b(x_n, x) = b(x, x)$.

(ii) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, b) if $\lim_{n,m\to\infty} b(x_n, x_m)$ exists and is finite.

(iii) (X, b) is said to be a complete partial b-metric space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x).$$

Note that in a partial b-metric space the limit of convergent sequence may not be unique (see Example 2 in [32]).

If a nonempty set X is equipped with a partial order "$\sqsubseteq$" such that (X, b) is a partial b-metric space with coefficient $s \geq 1$, then the triple $(X, b, \sqsubseteq)$ is called an ordered partial b-metric space. Elements $x, y \in X$ are called comparable, if $x \sqsubseteq y$ or $y \sqsubseteq x$. A subset A of X is called well ordered if all the elements of A are comparable. A sequence $\{x_n\}$ in X is called non-decreasing with respect to $\sqsubseteq$, if $x_n \sqsubseteq x_{n+1}$ for all $n \in \mathbb{N}$. A mapping $T: X \to X$ is called non-decreasing with respect to $\sqsubseteq$, if $x \sqsubseteq y$ implies $Tx \sqsubseteq Ty$. We denote the set of all fixed points of T by Fix(T), that is, Fix $(T) = \{x \in X: Tx = x\}$.

Definition 5. Let $(X, b, \sqsubseteq)$ be an ordered partial b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a mapping. Then T is called an ordered Banach contraction if the following condition holds: there exists $\lambda \in [0,1)$ such that

$$x \sqsubseteq y \text{ implies } b(Tx, Ty) \leq \lambda b(x, y) \text{ for all } x, y \in X. \tag{1}$$

The constant λ is called the contractive constant of T.

Definition 6. Let (X, b) be a partial b-metric space and $f: X \to X$ be a mapping. Then f is called:

(i) continuous, if for a sequence $\{x_n\}$ in X, $\lim_{n\to\infty} b(x_n, x) = b(x, x)$ for some $x \in X$ implies $\lim_{n\to\infty} b(fx_n, fx) = b(fx, fx)$;

(ii) sequentially convergent, if for a sequence $\{x_n\}$ in X, $\lim_{n\to\infty} b(x_n, x) = b(x, x)$ for some $x \in X$, whenever $\lim_{n\to\infty} b(fx_n, y) = b(y, y)$ for some $y \in X$.

Now we can state our main results.

3. FIXED POINT THEOREMS

The following lemma will be useful in the sequel.

Lemma 1. Let $(X, b, \sqsubseteq)$ be an ordered partial b-metric space and $T: X \to X$ be a mapping. If T is non-decreasing with respect to $\sqsubseteq$ and it is an ordered Banach contraction with contractive constant λ. Then for any $k \in \mathbb{N}$ the mapping $F: X \to X$ defined by $Fx = T^k x$ for all $x \in X$ is also non-decreasing with respect to $\sqsubseteq$ and it is an ordered Banach contraction with contractive constant λ^k.

Proof. Since T is nondecreasing with respect to $\sqsubseteq$, for $x, y \in X$ with $x \sqsubseteq y$ we have $Tx \sqsubseteq Ty$. Continuing in this manner, we obtain $T^k x \sqsubseteq T^k y$, that is, $Fx \sqsubseteq Fy$. Therefore F is non-decreasing with respect to $\sqsubseteq$.

If $x \sqsubseteq y$ then since T is non-decreasing with respect to $\sqsubseteq$ we have $T^n x \sqsubseteq T^n y$ for all $n \in \mathbb{N}$, so, using (1) we obtain

$$\begin{aligned} b(Fx, Fy) &= b(T^k x, T^k y) = b(TT^{k-1}x, TT^{k-1}y) \\ &\leq \lambda b(T^{k-1}x, T^{k-1}y) \\ &\vdots \\ &\leq \lambda^k b(x, y). \end{aligned}$$

Therefore, F is an ordered Banach contraction with contractive constant λ^k.

Now we state the ordered version of Banach contraction principle in partial b-metric spaces.

Theorem 2. Let $(X, b, \sqsubseteq)$ be an ordered and complete partial b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a mapping such that the following conditions hold:

(I) T is an ordered Banach contraction with contractive constant λ;

(II) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;

(III) T is non-decreasing with respect to $\sqsubseteq$;

(IV) if $\{x_n\}$ is a non-decreasing sequence in X and converging to some z, then $x_n \sqsubseteq z$ for all $n \in \mathbb{N}$.

Then T has a fixed point $u \in X$ and $b(u, u) = 0$. In addition, the set of fixed points of T, $\text{Fix}(T)$ is well ordered if and only if the fixed point of T is unique.

Proof. As, $\lambda \in [0,1)$ we can choose $n_0 \in \mathbb{N}$ such that, for given $0 < \varepsilon < 1$ we have $\lambda^{n_0} < \frac{\varepsilon}{4s^2}$. Let $T^{n_0} \equiv F$ and $F^k x_0 = x_k$ for all $k \in \mathbb{N}$. By Lemma 1, F is also non-decreasing with respect to $\sqsubseteq$ and it is an ordered Banach contraction with contractive constant λ^{n_0}. Since $x_0 \sqsubseteq Tx_0$ and T is non-decreasing with respect to $\sqsubseteq$, we have $Tx_0 \sqsubseteq TTx_0$ and so $x_0 \sqsubseteq Tx_0 \sqsubseteq T^2x_0$. Continuing in this manner, we obtain

$$x_0 \sqsubseteq Tx_0 \sqsubseteq T^2x_0 \sqsubseteq \cdots \sqsubseteq T^n x_0 \sqsubseteq T^{n+1}x_0 \sqsubseteq \cdots \quad \text{for all } n \in \mathbb{N}.$$

Therefore, $x_0 \sqsubseteq T^{n_0}x_0 \sqsubseteq T^{2n_0}x_0 \sqsubseteq \cdots \sqsubseteq T^{nn_0}x_0 \sqsubseteq \cdots$ for all $n \in \mathbb{N}$, that is,

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k \sqsubseteq x_{k+1} \sqsubseteq \cdots \quad \text{for all } k \in \mathbb{N}.$$

Therefore the sequence $\{x_n\}$ is non-decreasing with respect to "$\sqsubseteq$".

Now for any $k \in \mathbb{N}$, since $x_{k-1} \sqsubseteq x_k$ and F is an ordered Banach contraction with contractive constant λ^{n_0}, therefore we have

$$\begin{aligned} b(x_k, x_{k+1}) &= b(Fx_{k-1}, Fx_k) \leq \lambda^{n_0} b(x_{k-1}, x_k) \\ &= \lambda^{n_0} b(Fx_{k-2}, Fx_{k-1}) \leq \lambda^{2n_0} b(x_{k-2}, x_{k-1}) \\ &\vdots \\ &\leq \lambda^{kn_0} b(x_0, x_1) \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

So, we can choose $l \in \mathbb{N}$ such that

$$b(x_l, x_{l+1}) < \frac{\varepsilon}{8s^2}.$$

Now, let

$$B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right] := \{y \in X: x_l \sqsubseteq y, b(x_l, y) \leq \frac{\varepsilon}{4s} + b(x_l, x_l)\}.$$

We shall show that F maps $B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$ into itself. Now it is obvious that $x_l \in B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$ therefore $B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right] \neq \emptyset$. Let $z \in B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$ be arbitrary then $x_l \sqsubseteq z$ and since F is non-decreasing with respect to $\sqsubseteq$, we have $Fx_l \sqsubseteq Fz$, that is, $x_{l+1} \sqsubseteq Fz$ and so $x_l \sqsubseteq x_{l+1} \sqsubseteq Fz$ which implies $x_l \sqsubseteq Fz$. Now, since F is an ordered Banach contraction with contractive constant λ^{n_0}, we have

$$b(Fx_l, Fz) = \lambda^{n_0} b(x_l, z) \leq \frac{\varepsilon}{4s^2}\left[\frac{\varepsilon}{4s} + b(x_l, x_l)\right],$$

also

$$b(x_l, Fx_l) = b(x_l, x_{l+1}) < \frac{\varepsilon}{8s^2}.$$

Therefore,

$$
\begin{aligned}
b(x_l, Fz) &\le s[b(x_l, Fx_l) + b(Fx_l, Fz)] - b(Fx_l, Fx_l)\\
&< s\left[\frac{\varepsilon}{8s^2} + \frac{\varepsilon}{4s^2}\left\{\frac{\varepsilon}{4s} + b(x_l, x_l)\right\}\right]\\
&= \frac{\varepsilon}{8s} + \frac{\varepsilon^2}{16s^2} + \frac{\varepsilon}{4s} b(x_l, x_l)\\
&< \frac{\varepsilon}{4s} + b(x_l, x_l),
\end{aligned}
$$

so, $Fz \in B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$. Thus, F maps $B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$ into itself.

Note that, $x_l \in B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$ therefore $Fx_l \in B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$ and repetition of this process gives $F^n x_l \in B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$ for all $n \in \mathbb{N}$, that is, $x_m \in B_b^{\sqsubseteq}\left[x_l, \frac{\varepsilon}{4s}\right]$ for all $m \ge l$. Thus, for $n, m > l$ we have

$$
\begin{aligned}
b(x_n, x_m) &\le s[b(x_n, x_l) + b(x_l, x_m)] - b(x_l, x_l)\\
&< s\left[\frac{\varepsilon}{4s} + b(x_l, x_l) + \frac{\varepsilon}{4s} + b(x_l, x_l)\right]\\
&= \frac{\varepsilon}{2} + 2sb(x_l, x_l) \le \frac{\varepsilon}{2} + 2sb(x_l, x_{l+1})\\
&< \frac{\varepsilon}{2} + \frac{2s\varepsilon}{8s^2} < \varepsilon.
\end{aligned}
$$

Thus, the sequence $\{x_n\}$ is a Cauchy sequence and

$$b(x_n, x_m) < \varepsilon \ \text{ for all } \ n, m > l.$$

By completeness of X there exists $u \in X$ such that

$$\lim_{n\to\infty} b(x_n, u) = \lim_{n,m\to\infty} b(x_n, x_m) = b(u, u) = 0. \tag{2}$$

We shall show that u is a fixed point of T.

By assumption (IV) we have $x_n \sqsubseteq u$ for all $n \in \mathbb{N}$. Therefore, for any $n \in \mathbb{N}$ by (1) we have

$$
\begin{aligned}
b(u, Tu) &\le s[b(u, Tx_n) + b(Tx_n, Tu)] - b(Tx_n, Tx_n)\\
&\le s[s\{b(u, x_n) + b(x_n, Tx_n)\} - b(x_n, x_n)] + s\lambda b(x_n, u)\\
&\le (s^2 + \lambda s) b(x_n, u) + s^2 b(F^n x_0, F^n T x_0). \qquad (3)
\end{aligned}
$$

Again since $x_0 \sqsubseteq Tx_0$ therefore $F^n x_0 \sqsubseteq F^n T x_0$ for all $n \in \mathbb{N}$ (as F is non-decreasing with respect to $\sqsubseteq$) and F is an ordered Banach contraction with contractive constant λ^{n_0}, so we have

$$
\begin{aligned}
b(F^n x_0, F^n T x_0) &= b(FF^{n-1}x_0, FF^{n-1}Tx_0) \le \lambda^{n_0} b(F^{n-1}x_0, F^{n-1}Tx_0)\\
&\ \vdots\\
&\le \lambda^{nn_0} b(x_0, Tx_0).
\end{aligned}
$$

Therefore we obtain from (3) that

$$b(u, Tu) \le (s^2 + \lambda s) b(x_n, u) + s^2 \lambda^{nn_0} b(x_0, Tx_0),$$

which together with (2) yields $b(u, Tu) = 0$. Thus, u is a fixed point of T.

For uniqueness, suppose Fix(T) is well ordered and $u, v \in$ Fix(T), then $Tu = u, Tv = v$. Suppose $b(u, v) > 0$, then since Fix(T) is well ordered, assume that $u \sqsubseteq v$. Now it follows from (1) that

$$b(u, v) = b(Tu, Tu) \le \lambda b(u, v) < b(u, v).$$

This contradiction shows that $b(u, v) = 0$, therefore $u = v$. Similarly, if $v \sqsubseteq u$ we obtain $u = v$. Hence, fixed point of T is unique. Further, if fixed point of T is unique then Fix(T) is singleton, and so well ordered.

The following is a simple example which illustrates the above theorem and shows that the condition of well orderedness of $\mathrm{Fix}(T)$ for uniqueness of fixed point is not superfluous. Also it shows that the above theorem is a proper generalization of known results.

Example 2. Let $X = \{1,2,3,4\}$ and $b: X \times X \to \mathbb{R}$ be defined by

$$\mathrm{b(x,y)} = \begin{cases} |x-y|^2 + \max\{x,y\}, & \text{if } x \neq y; \\ x, & \text{if } x = y \in \{2,3\}; \\ 0, & \text{if } x = y \in \{1,4\}. \end{cases}$$

Then (X,b) is a complete partial b-metric space with coefficient $s = 4 > 1$. We note that, $b(2,2) = 2 \neq 0$ therefore b is not a b-metric. Also b is not a partial metric. Indeed, $b(4,1) = 13 > 9 = b(4,3) + b(3,1) - b(3,3)$.

Define a mapping $T: X \to X$ and a partial order $\sqsubseteq$ by

$$T1 = 1, T2 = 1, T3 = 2, T4 = 4$$

and $\sqsubseteq = \{(1,1),(2,2)(3,3),(4,4),(1,2),(2,3),(1,3)\}$.Now note that, T does not satisfy the contractive condition of Ran and Reurings [4] in usual metric space (X,d), for example, $d(T2,T3) = |T2 - T3| = 1 = |2-3| = d(2,3)$, therefore there exists no $\lambda \in [0,1)$ such that $d(T2,T3) \leq \lambda d(2,3)$. So, the result of [4] cannot be applied to T. Also, (X,b) is neither a b-metric nor a partial metric space therefore the results of [29], and [31] are not applicable. Again, since $b(T2,T4) = 13 > 8 = b(2,4)$, therefore T does not satisfy (1) for all $x,y \in X$, and so the result of [32] is not applicable. Note that, all the conditions of Theorem 2 (except that the set $\mathrm{Fix}(T)$ is well ordered) are satisfied with $\lambda \in \left[\frac{3}{4}, 1\right)$ and T has two fixed point. Indeed, $\mathrm{Fix}(T) = \{1,4\}$ and $(1,4),(4,1) \notin \sqsubseteq$, that is, $\mathrm{Fix}(T)$ is not well ordered. Therefore, when we consider the uniqueness of fixed point of an ordered Banach contraction in partial b-metric space then the well orderedness of $\mathrm{Fix}(T)$ cannot be omitted.

Definition 7 (see [22] and the references therein)**.** Let $(X,b,\sqsubseteq)$ be an ordered and complete partial b-metric space with coefficient $s \geq 1$ and $f: X \to X$ and $T: X \to X$ be two mappings. The mapping T is called an ordered f-contraction if there exists $\lambda \in [0,1)$ such that

$$x \sqsubseteq y \text{ implies } b(fTx, fTy) \leq \lambda b(fx, fy) \text{for all } x, y \in X. \tag{4}$$

The constant λ is called the contractive constant of T.

Next, we prove a common fixed point result for two mappings as a consequence of Theorem 2.

Corollary 3. Let $(X,b,\sqsubseteq)$ be an ordered and complete partial b-metric space with coefficient $s \geq 1$. Let $f: X \to X$ and $T: X \to X$ be two mappings such that the following conditions hold:

(I) T is an ordered f-contraction;

(II) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;

(III)T is non-decreasing with respect to $\sqsubseteq$;

(IV) if $\{fx_n\}$ is a nondecreasing sequence in X and converging to some fz, then $x_n \sqsubseteq z$ for all $n \in \mathbb{N}$.

If f is continuous, injective and sequentially convergent then T has a fixed point $u \in X$ and $b(u,u) = 0$. In addition, the set of fixed points of T, $\mathrm{Fix}(T)$ is well ordered if and only if the fixed point of T is unique.

Proof. Define $b_1: X \times X \to \mathbb{R}^+$ by

$$b_1(x,y) = b(fx,fy) \text{ for all } x,y \in X.$$

We shall show that (X,b_1) is a complete partial b-metric space with same coefficient $s \geq 1$. Then, for all $x,y,z \in X$, we have

(Pb1) $b_1(x,y) = b_1(x,x) = b_1(y,y)$ implies $b(fx,fy) = b(fx,fx) = b(fy,fy)$, i.e., $fx = fy$ and f is injective, so $x = y$;

(Pb2) $b_1(x,x) = b(fx,fx) \leq b(fx,fy) = b_1(x,y)$;

(Pb3) $b_1(x,y) = b(fx,fy) = b(fy,fx) = b_1(y,x)$;

(Pb4) $b_1(x,y) = b(fx,fy) \leq s[b(fx,fz) + b(fz,fy)] - b(fz,fz) = s[b_1(x,z) + b_1(z,y)] - b_1(z,z)$.

Thus (X,b_1) is a partial b-metric space with coefficient $s \geq 1$.

Let $\{x_n\}$ be a Cauchy sequence in (X,b_1), then $\lim_{n,m\to\infty} b_1(x_n,x_m) = \lim_{n,m\to\infty} b(fx_n,fx_m)$ exists. Therefore, $\{fx_n\}$ is Cauchy sequence in (X,b) and (X,b) is complete, so there exists $y \in X$ such that $\lim_{n\to\infty} b(fx_n,y) = b(y,y)$. Thus $\{fx_n\}$ is convergent in (X,b), therefore by choice of f, there exists $x \in X$ such that $\lim_{n\to\infty} b(x_n,x) = b(x,x)$. Again by choice of f, we have $\lim_{n\to\infty} b(fx_n,fx) = b(fx,fx)$, i.e., $\lim_{n\to\infty} b_1(x_n,x) = b_1(x,x)$. Thus $\{x_n\}$ converges in (X,b_1) and so it is complete.

Note that, the contractive conditions (a) is reduced into the following condition:

$$(a')\ \ x \sqsubseteq y \text{ implies } b_1(Tx,Ty) \leq \lambda b_1(x,y) \text{ for all } x,y \in X.$$

Thus, T is an ordered Banach contraction in (X,b_1). Therefore, by Theorem 2, T has a fixed point $u \in X$ and $b_1(u,u) = b(fu,fu) = 0$. Again, the condition for uniqueness follows from Theorem 2. □

Remark 3. A pair $(T,f)$of self-maps of a nonempty set X is called a Banach pair if $Tfx = fTx$ for all $x \in \mathrm{Fix}(T)$. If all the conditions of the above corollary are satisfied and in addition, (T,f) is a Banach pair then T and f have a unique common fixed point. Indeed, if (T,f) is a Banach pair and if T has a unique fixed point $u \in X$ (which is ensured by Corollary 3) then $Tfu = fTu = fu$, and by uniqueness of fixed point of T we have $fu = u$. Thus, the pair (T,f) has a unique common fixed point.

In the above Corollary, for $u \in \mathrm{Fix}(T)$ the self-distance $b_1(u,u) = 0$, but $b(u,u)$ need not be zero, also, when we consider the existence of common fixed point of the pair (T,f) then the condition that (T,f) is a Banach pair cannot be omitted, as shown in the following example.

Example 3. Let $X = \{1,2,3,4\}$ and $b: X \times X \to \mathbb{R}$ be defined by

$$b(x,y) = \begin{cases} 0, & if \quad x = y = 4; \\ |x-y|^2 + \max\{x,y\}, & \text{otherwise.} \end{cases}$$

Then (X,b) is a complete partial b-metric space with coefficient $s = 4 > 1$. Define the mappings $T,f: X \to X$ by

$$T1 = 1, T2 = 1, T3 = 2, T4 = 1 \ \text{ and } \ f1 = 4, f2 = 2, f3 = 3, f4 = 1,$$

and a partial order by $\sqsubseteq = \{(1,1),(2,2),(3,3),(4,4),(1,2),(4,2)\}$. Then f is continuous, injective and sequentially convergent mapping. It is easy to see that T is an ordered f-contraction with contractive constant $\lambda \in \left[\frac{2}{3}, 1\right)$. All the conditions of Corollary 3 are satisfied and T has a unique fixed point, namely $\mathrm{Fix}(T) = \{1\}$. Note that $b(f1,f1) = b(4,4) = 0$ but $b(1,1) = 1 \neq 0$. Also, since $Tf1 \neq fT1$ so (T,f) is not a Banach pair and T and f have no common fixed point. Finally, T is not an ordered Banach contraction since $b(T4,T4) = b(1,1) = 1 \nleq \lambda \cdot 0 = \lambda b(4,4)$ for any real λ.

In the next theorem the completeness of space and the monotonicity of T are replaced by another condition on T.

Theorem 4. Let $(X,b,\sqsubseteq)$ be an ordered partial b-metric space and $T: X \to X$ be a mapping satisfies the following condition:

$$x \sqsubseteq y \text{ implies } b(Tx,Ty) \leq \lambda b(x,y) \tag{5}$$

for all $x,y \in X$, where $\lambda \in [0,1)$. Suppose, there exists $u \in X$ such that $u \sqsubseteq Tu$ and $b(u,Tu) \leq b(x,Tx)$ for all $x \in X$. Then u becomes a fixed point of T and $b(u,u) = 0$. In addition, the set of fixed points of T, $\mathrm{Fix}(T)$ is well ordered if and only if the fixed point of T is unique.

Proof. Let $F(x) = b(x,Tx)$ for all $x \in X$. Then by the assumption we have

$$F(u) \leq F(x) \text{for all } x \in X. (6)$$

Suppose $F(u) > 0$, then since $u \sqsubseteq Tu$, it follows from (5) that

$$
\begin{aligned}
F(Tu) &= b(Tu, TTu) \\
&\leq \lambda b(u, Tu) = \lambda F(u) \\
&< F(u).
\end{aligned}
$$

So, we have $F(Tu) < F(u)$, which contradicts the inequality (6). Thus, we must have $F(u) = b(u, Tu) = 0$, that is, $Tu = u$. Therefore,u is a fixed point of T.

Now for any fixed point $z \in X$ of T, if $b(z, z) > 0$, then from (5) we have

$$b(z, z) = b(Tz, Tz) \leq \lambda b(z, z) < b(z, z).$$

This contradiction shows that $b(z, z) = 0$.

For uniqueness, suppose $\mathrm{Fix}(T)$ is well ordered and $u, v \in \mathrm{Fix}(T)$, then $Tu = u, Tv = v$ and $b(u, u) = b(v, v) = 0$. Suppose $b(u, v) > 0$, then since $\mathrm{Fix}(T)$ is well ordered, assume that $u \sqsubseteq v$. Now it follows from (5) that

$$
\begin{aligned}
b(u, v) &= b(Tu, Tv) \leq \lambda b(u, v) \\
&< b(u, v).
\end{aligned}
$$

Therefore we must have $b(u, v) = 0$, that is, $u = v$. Similarly if $v \sqsubseteq u$ we have $u = v$. Hence fixed point of T is unique. Further, if fixed point of T is unique then $\mathrm{Fix}(T)$ is singleton, and so well ordered. □

CONFLICT OF INTEREST

No conflict of interest was declared by the authors

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