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Primes of The Form $4m + 1$

Oznur Oztunc Kaymak

Balıkesir University, 10145, Balıkesir, Turkey.

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Abstract. In this paper, a new equation related to the sums of the squares of the first n k-Fibonacci numbers has been found. From this equation, the problem of existing infinitely many primes exist p such that $p \equiv 1$ $(mod\ 4)$ of elementary number theory is obtained.

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1. Introduction

Prime numbers and their applications were studied by the ancient Greek mathematicians. Since then, these numbers have been of great importance in mathematics. Let $k \geq 1$ be any integer number. k-Fibonacci numbers are defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1 \tag{1.1}$$

with the initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$. As particular cases, if $k = 1$, we obtain the classical Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, ...\}$ and if $k = 2$, the Pell sequence appears $\{0, 1, 2, 5, 12, 29, ...\}$ (see fore more details [3, 4, 6, 7] and the references therein). Using matrix methods, some sum formulas for these numbers are obtained in [5].

In number theory, Dirichlet's theorem states that there are infinitely many primes of the form $am + b$ when m is a natural number. ($(a, b) = 1$ where a, b are natural numbers) At its proof, a difficult technical was used. In 1994, this problem was studied with terrific logic using the properties fundamental of Fibonacci numbers and Fermat numbers by Robbins [8] in the special case $a = 4$ and $b = 1$. In this study, after finding a sum formula of k- Fibonacci numbers, we use this equation to derive existing infinitely many primes exist p such that $p \equiv 1$ $(mod\ 4)$ of elemantary number theory.

Email address: oztunc@balikesir.edu.tr

2. Sums of The Squares of The First n k-Fibonacci Numbers

In this section, we consider the sums of the squares of the first n k-Fibonacci numbers.

Theorem 2.1. *For any integer $n \geq 1$, we obtain*

$$\sum_{i=1}^{n} F_{k,i}^2 = \frac{1}{k} F_{k,n} F_{k,n+1}. \tag{2.1}$$

Proof. We apply the principle of mathematical induction. For $n = 1$, we find

$$\sum_{i=1}^{n} F_{k,i}^2 = F_{k,1}^2 = \frac{1}{k} F_{k,1} F_{k,2} = \frac{1}{k} 1k = 1$$

since we have $F_{k,1} = 1$ and $F_{k,2} = k$. Now suppose that the equation (2.1) is true for n. Then by (1.1) we get

$$\begin{aligned} \sum_{i=1}^{n+1} F_{k,i}^2 &= \sum_{i=1}^{n} F_{k,i}^2 + F_{k,n+1}^2 = \frac{1}{k} F_{k,n} F_{k,n+1} + F_{k,n+1}^2 \\ &= F_{k,n+1}\left(\frac{1}{k} F_{k,n} + F_{k,n+1}\right) \\ &= \frac{1}{k} F_{k,n+1}(F_{k,n} + kF_{k,n+1}) \\ &= \frac{1}{k} F_{k,n+1} F_{k,n+2}. \end{aligned}$$

□

If $k = 1$, we have the classical Fibonacci sequence and the equation (2.1) becomes

$$\sum_{i=1}^{n} F_i^2 = F_n \, F_{n+1}.$$

If $k = 2$, we get the Pell sequence defined by

$$P_0 = 0, P_1 = 1 \text{ and } P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geq 1$$

and the equation (2.1) becomes

$$\sum_{i=1}^{n} P_i^2 = \frac{1}{2} P_n P_{n+1}.$$

Now, we reconfirm Theorem 2.1 using the Euclidean algorithm and the following fact (see [6]): Let a and b be any two positive integers for $a \geq b$ with the equations

$$\begin{aligned} a &= q_0 r_0 + r_1, \\ r_0 &= q_1 r_1 + r_2, \\ &\cdots \\ r_i &= q_i r_i + r_{i+1}, \\ &\cdots \\ r_{n-2} &= q_{n-1} r_{n-1} + r_n, \\ r_{n-1} &= q_n r_n + 0. \end{aligned}$$

The above equations imply that

$$ab = \sum_{i=0}^{n} q_i r_i^2. \tag{2.2}$$

This last equation is true for any positive integer n. In our case, let $a = F_{k,n}$ and $b = F_{k,n+1}$. By the Euclidean algorithm, we have

$$\begin{aligned} F_{k,n+1} &= k\,F_{k,n} + F_{k,n-1}, \\ F_{k,n} &= k\,F_{k,n-1} + F_{k,n-2}, \\ &\dots \\ F_{k,3} &= k\,F_{k,2} + F_{k,1}, \\ F_{k,2} &= k\,F_{k,1} + 0. \end{aligned}$$

For $0 \le i < n$, we obtain $q_i = q_n = k$. Using (2.2), we get

$$ab = F_{k,n}F_{k,n+1} = \sum_{i=1}^{n} kF_{k,i}^2 = k\sum_{i=1}^{n} F_{k,i}^2. \tag{2.3}$$

So, if we rearrange the equation (2.3) we obtain

$$\sum_{i=1}^{n} F_{k,i}^2 = \frac{1}{k} F_{k,n}\,F_{k,n+1}.$$

which is identity (2.1).

3. Primes of The Form $4k + 1$

In order to show that there exist infinitely many primes p such that $p \equiv 1\ (mod\ 4)$, $\{U_n\}$ which is a sequence of natural numbers, is constructed as follows

(i) $U_n > 1$ for all $n \ge 1$,

(ii) If q is prime and $q \mid U_n$, then $q \equiv 1\ (mod\ 4)$,

(iii) $(U_m, U_n) = 1$ for all $m \ne n$.

If P_n be least prime divisor of U_n for all $n \ge 1$, then an infinite sequence $\{P_n\}$ consisting of distinct primes such that $P_n \equiv 1 (mod\ 4)$ for all $n \ge 1$ exists. Let $U_n = a_n^2 + b_n^2$ where a_n and b_n are natural numbers such that $(a_n, b_n) = 1$ and $a_n \not\equiv b_n (mod\ 2)$. Then the sequence $\{U_n\}$ satisfies (i) and (ii). If (iii) also holds, then $\{U_n\}$ fulfills all requirements in [8].

In order to see that infinitely many primes exist p such that $p \equiv 1\ (mod\ 4)$, firstly we shall prove the following lemma.

Lemma 3.1. *For any integer number $n \ge 1$, we find*

$$F_{k,2n+1} = F_{k,n+1}^2 + F_{k,n}^2.$$

Proof. Let us consider the equation (2.1) for $n \to 2n, 2n+1$, then we obtain

$$\sum_{i=1}^{2n} F_{k,i}^2 = \frac{1}{k}\, F_{k,2n}F_{k,2n+1} \tag{3.1}$$

and

$$\sum_{i=1}^{2n+1} F_{k,i}^2 = \frac{1}{k}\, F_{k,2n+1}F_{k,2n+2}. \tag{3.2}$$

By multiplying the equation (3.2) with (-1) and by adding the equation (3.1) to new equation, we obtain

$$F_{k,2n+1} = -\frac{1}{k}\left(F_{k,2n} - F_{k,2n+2}\right).$$

After some algebra, the desired result is obtained. □

Lemma 3.2 ([1]). *For any integer number $m, n > 0$, we have*

$$(F_{k,m}, F_{k,n}) = F_{k(m,n)}. \tag{3.3}$$

Lemma 3.3. *If $n \geq 3$, then we have $F_{k,n} > k$.*

Proof. From the definition (1.1), we can easily see that $F_{k,n} > k$. □

Now, the result finding for Generalized Fibonacci Polynomials in [2] will be adapted to k-Fibonacci Numbers with the following theorem.

Lemma 3.4. *For any positive real number k,*

$$F_{k,3} = (k^2 + 1) \mid F_{k,n} \iff 3 \mid n\,.$$

Proof. For the first part of theorem, clearly we have

$$(k^2 + 1) \mid (k^2 + 1). \tag{3.4}$$

For $t \geq 1$, let

$$F_{k,3} \mid F_{k,3t}. \tag{3.5}$$

It is known that

$$\begin{aligned} F_{k,3(t+1)} &= F_{k,3t+3} \\ &= F_{k,3t}\, F_{k,4} + F_{k,3t-1}\, F_{k,3}. \end{aligned}$$

As seen in the equations (3.4) and (3.5), we find that

$$F_{k,3} \mid F_{k,3(t+1)}.$$

Thus if $3 \mid n$, we find that

$$F_{k,3} = (k^2 + 1) \mid F_{k,n}.$$

As for another part of the theorem, let

$$F_{k,3} \mid F_{k,n}.$$

Conversely, $3 \nmid n$. Then there exist integers q and r with $0 < r < 3$, such that

$$n = 3q + r.$$

We get

$$\begin{aligned} F_{k,n} &= F_{k,3q+r} \\ &= F_{k,3q+1} F_{k,r} + F_{k,3q} F_{k,r-1}. \end{aligned}$$

From the fact that $F_{k,3} \mid F_{k,3q}$ where $q \geq 1$ is fixed, this shows that $F_{k,3} \mid F_{k,3q+1} F_{k,r}$. We know that $\left(F_{k,3q}, F_{k,3q+1}\right) = 1$ is true by the Lemma 3.2. This case shows that $F_{k,3} \mid F_{k,r}$. But, this situation is impossible. Consequently, we can find that $r = 0$ and $3 \mid n$. □

Let $U_n = F_{k,n}$ and $n \geq 5$ be is a prime. By Lemma 3.1, we have

$$F_{k,n} = F^2_{k,1/2(n-1)} + F^2_{k,1/2(n+1)} \text{ for all } n \geq 1.$$

Since $(1/2(n-1), 1/2(n+1)) = 1$ is true, Lemma 3.2 implies

$$(F_{k,1/2(n-1)}, F_{k,1/2(n+1)}) = F_{k,1} = 1\,.$$

Since $n > 3$ and n is a prime, Lemma 3.4 implies $F_{k,3} = (k^2 + 1) \nmid F_{k,n}$ and so

$$F_{k,1/2(n-1)} \not\equiv F_{k,1/2(n+1)}\ (mod\ k^2 + 1).$$

Consequently, we find that $(m, n) = 1$ for all $m \neq n$. Thus, Lemma 3.2 implies $(F_{k,m}, F_{k,n}) = 1$.

To sum up, an infinitude of primes p such that $p \equiv 1(mod\ 4)$ can be obtained by taking into account the least prime divisor of the k-Fibonacci numbers $F_{k,n}$, where n is prime and $n \geq 5$.

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