Konuralp Journal of Mathematics
Volume 5 No. 2 pp. 181–191 (2017) ©KJM

HERMITE-HADAMARD AND HERMITE-HADAMARD-FEJÉR TYPE INEQUALITIES FOR (k,h)-CONVEX FUNCTION VIA KATUGAMPOLA FRACTIONAL INTEGRALS

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Abstract. In this paper, we obtain some new Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for (k,h)-convex functions via Katugampola fractionals which are a generalization of Riemann-Liouville and the Hadamard fractional integrals in to a single form.

1. Introduction

Definition 1.1. *The function $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y) \tag{1.1}$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Let $f:I\to\mathbb{R}$ be a convex function defined on a real interval I and fix $a,b\in I$ with $a<b$. The following double inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \tag{1.2}$$

is known in the literature as the Hermite-Hadamard inequality for convex functions (see [18] for the historical background). Note that some of the classical inequalities for means can be derived from (1.2) for appropriate particular selections of the function f. Both inequalities hold in the reversed direction if f is concave.

In [8] Fejér gave the important generalization of the inequality (1.2) as follows. If $f:[a,b]\to\mathbb{R}$ is a convex function and $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric with respect to the point $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\le\int_a^b f(x)g(x)dx\le\frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{1.3}$$

Date: June 6, 2017 and, in revised form, October 12, 2017.

2010 *Mathematics Subject Classification.* 26A33, 26A51, 26D10.

Key words and phrases. Hermite-Hadamard-Fejér inequality, Riemann-Liouville fractional integrals, Hadamard fractional integrals, Katugampola fractional integrals, (k,h)-convex function.

For various modifications of (1.2) and (1.3), we refer the reader to the recent papers (see [5, 6, 7, 11, 20, 21, 25, 27],).

We recall some previously known definitions of different type of convexity.

Definition 1.2. *The function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, is said to be Jensen-convex or J-convex if the following inequality holds*

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{1.4}$$

for all $x, y \in I$.

Definition 1.3. (see [4],[19]) *Let $0 < s \leq 1$. A function $f : [0,\infty) \to \mathbb{R}$, is said to be s-Orlicz convex or s-convex in the first sense, if for every $x, y \in [0,\infty)$ and $\alpha, \beta \geq 0$ with $\alpha^s + \beta^s = 1$, we have:*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y). \tag{1.5}$$

We denote the set of all s-convex functions in the first sense by K_s^1.

Definition 1.4. (see [1],[10]) *Let $0 < s \leq 1$. A function $f : [0,\infty) \to \mathbb{R}$, is said to be s-Breckner convex or s-convex in the second sense, if for every $x, y \in [0,\infty)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, we have inequality* (1.5)*.The set of all s-convex functions in the second sense is denoted by K_s^2.*

Definition 1.5. ([9]) *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class of $Q(I)$, if it is nonnegative and, for all $x, y \in I$ and $\lambda \in [0,1]$, satisfies the following inequality;*

$$f\big(\lambda x + (1-\lambda)y\big) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}. \tag{1.6}$$

Definition 1.6. ([16]) *A function $f : (0,1) \to \mathbb{R}$ is said to be subadditive if the following inequalities holds*

$$f(s+t) \leq f(s) + f(t) \tag{1.7}$$

$s, t \geq 0$.

Definition 1.7. ([7]) *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and, for all $x, y \in I$ and $\lambda \in [0,1]$, satisfies the following inequality;*

$$f\big(\lambda x + (1-\lambda)y\big) \leq f(x) + f(y). \tag{1.8}$$

Definition 1.8. ([30])*Let I be a real interval and let $h : (0,1) \to \mathbb{R}$ be a nonnegative function, $h \neq 0$. A nonnegative function $f : I \to \mathbb{R}$ is then called h-convex if, for all $x, y \in I$ and $t \in (0,1)$. We have*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{1.9}$$

We now give the more general concepts of some different type of convexity.

Definition 1.9. ([17]) *Let $k : (0,1) \to \mathbb{R}$ be a given function. Then a subset D of a real linear space X will be called k-convex if $k(t)x + k(1-t)y \in D$ for all $x, y \in D$ and $t \in (0,1)$.*

This definition agrees with the one of classical convexity for $k(t) = t$.

Definition 1.10. ([17]) *Let $k, h : (0,1) \to \mathbb{R}$ be two given functions and suppose that $D \subset X$ is a k-convex set.Then a function $f : D \to \mathbb{R}$ is (k,h)-convex, if for all $x, y \in D$ and $t \in (0,1)$,*

$$f(k(t)x + k(1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{1.10}$$

If (1.10) can be replaced with the corresponding equality, f will be called (k,h)-affine (more genarel functions of this type are subject of the paper [15]).

For suitable functions h and k, the condition (1.10) produces the families of convex, Jensen-convex, h-convex, s-Orlicz convex, s-Breckner convex, P-function, Godunova-Levin, Starshaped functions and subadditive mapping. In the following theorem a new inequality of Hermite-Hadamard-Fejér types for (k,h)-convex functions is proved.

Theorem 1.1. ([17])*(The first Fejér inequality for (k,h)-convex functions)*
Let $f : D \to \mathbb{R}$ be a (k,h)-convex function with $h(1/2) > 0$, fix $a < b$ such that $[a,b] \subset D$ and let $g : [a,b] \to \mathbb{R}$ be a nonnegative function which is symmetric with respect to $(a+b)/2$. Then

$$\frac{f\Big(k(1/2)(a+b)\Big)}{2h(1/2)} \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx. \tag{1.11}$$

Now, we give information about Riemann-Liouville fractional integrals.

Definition 1.11. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}$ and $J_{b^-}^{\alpha}$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ and $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

Because of the wide applications of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; (see [3, 23, 24, 25, 28, 29]).

Some important results that is related Riemann-Liouville fractional integrals are as follow;

Theorem 1.2. ([23])*Let f $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.12}$$

with $\alpha > 0$.

Theorem 1.3. ([11]) *Let $f : [a, b] \to \mathbb{R}$ be convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a + b)/2$, then the following inequalities for fractional integrals hold*

$$\begin{aligned} (1.13)\quad f\left(\frac{a+b}{2}\right)\left[\mathrm{J}^{\alpha}_{a+}g(b) + \mathrm{J}^{\alpha}_{b-}g(a)\right] &\leq \left[\mathrm{J}^{\alpha}_{a+}fg(b) + \mathrm{J}^{\alpha}_{b-}fg(a)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[\mathrm{J}^{\alpha}_{a+}g(b) + \mathrm{J}^{\alpha}_{b-}g(a)\right] \end{aligned}$$

with $\alpha > 0$.

Theorem 1.4. ([26])*Let $f : D \to \mathbb{R}$ be a (k, h)-convex function with $h(1/2) > 0$, fix $a < b$ such that $[a, b] \subset D$ and let $g : [a, b] \to \mathbb{R}$ be a nonnegative function which is symmetric with respect to $(a + b)/2$. Then the following inequality holds*

$$(1.14)\qquad \frac{f\left(k(1/2)(a+b)\right)}{2h(1/2)}\left[J^{\alpha}_{a+}g(b) + J^{\alpha}_{b-}g(a)\right] \leq \left[J^{\alpha}_{a+}fg(b) + J^{\alpha}_{b-}fg(a)\right]$$

with $\alpha > 0$

Now, we give information about Hadamard fractional integrals.

Definition 1.12. ([22])*Let $\alpha > 0$ with $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, and $a < x < b$. The left- and right-side Hadamard fractional integrals of order α of a function f are given bye*

$$\begin{aligned} (1.15)\qquad H^{\alpha}_{a+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1}\frac{f(t)}{t}dt \quad \text{and} \\ H^{\alpha}_{b-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1}\frac{f(t)}{t}dt. \end{aligned}$$

Recently, Katugampola introduced a new fractional integral that generalizes the Riemann-Liouville and the Hadamard fractional integrals in to a single form(see [12, 14]).

Definition 1.13. ([13]) *Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then, the left-and right-side Katugampola fractional integrals of order $\alpha(> 0)$ of $f \in X^p_c(a, b)$ are defined by,*

$${}^{\rho}\mathrm{I}^{\alpha}_{a+}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{\rho-1}}{\left[x^{\rho}-t^{\rho}\right]^{1-\alpha}}\, f(t)dt \ \text{ and } \ {}^{\rho}\mathrm{I}^{\alpha}_{b-}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{\rho-1}}{\left[t^{\rho}-x^{\rho}\right]^{1-\alpha}}\, f(t)dt$$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Theorem 1.5. ([13])*Let $\alpha > 0$ and $\rho > 0$. Then for $x > a$,*

1. $\lim_{\rho\to 1} {}^{\rho}I^{\alpha}_{a+}f(x) = J^{\alpha}_{a+}f(x)$,
2. $\lim_{\rho\to 0^+} {}^{\rho}I^{\alpha}_{a+}f(x) = H^{\alpha}_{a+}f(x)$.

In [2], Chen and Katugampola established the Hermite-Hadamard inequalities for Katugampola fractional integrals as follows.

Theorem 1.6. ([2]) *Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^{\rho}, b^{\rho}] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in X^p_c(a^{\rho}, b^{\rho})$. If f is also a convex function on $[a, b]$,* *then the following inequalities hold:*

$$f\left(\frac{a^\rho+b^\rho}{2}\right) \le \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho \mathrm{I}^\alpha_{a+} f(b^\rho) + {}^\rho \mathrm{I}^\alpha_{b-} f(a^\rho)\right] \le \frac{f(a^\rho)+f(b^\rho)}{2} \tag{1.16}$$

where the fractional integrals are considered for the function $f(x^\rho)$ and evaluated at a and b, respectively.

Theorem 1.7. ([2])*If f is convex function on $[a,b]$ and $f \in L[a,b]$. Then $F(x)$ is also integrable and the following inequalities hold*

$$F\left(\frac{a+b}{2}\right) \le \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)^\alpha}\left[{}^\rho \mathrm{I}^\alpha_{a+} F(b) + {}^\rho \mathrm{I}^\alpha_{b-} F(a)\right] \le \frac{F(a)+F(b)}{2} \tag{1.17}$$

with $\alpha > 0$ and $\rho > 0$, where $F(x) := f(x) + f(a+b-x)$.

Chen and Katugampola gave a generalization of the inequalities (1.16)-(1.17) as follows.

Theorem 1.8. ([2])*Let $f : [a,b] \to \mathbb{R}$ be convex function with $a < b$ and $f \in L[a,b]$. Then $F(x)$ is also convex and $F \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative and integrable,then the following inequalities hold:*

$$\begin{aligned} F\left(\frac{a+b}{2}\right)\left[{}^\rho \mathrm{I}^\alpha_{a+} g(b) + {}^\rho \mathrm{I}^\alpha_{b-} g(a)\right] &\le \left[{}^\rho \mathrm{I}^\alpha_{a+} (gF)(b) + {}^\rho \mathrm{I}^\alpha_{b-} (gF)(a)\right] \\ &\le \frac{F(a)+F(b)}{2}\left[{}^\rho \mathrm{I}^\alpha_{a+} g(b) + {}^\rho \mathrm{I}^\alpha_{b-} g(a)\right] \end{aligned} \tag{1.18}$$

with $\alpha > 0$ and $\rho > 0$, where $F(x) := f(x) + f(a+b-x)$.

The aim of this paper is to establish Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for (k,h)-convex functions via Katugampola fractional integrals.

2. Main Results

Theorem 2.1. *Let $\alpha > 0$ and $\rho > 0$. Let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in X^p_c(a^\rho, b^\rho)$. Let $f : D \to \mathbb{R}$ is a (k,h)-convex function with $h(1/2) > 0$, fix $a < b$ such that $[a^\rho, b^\rho] \subset D$ then the following inequality hold;*

$$f\left(k(1/2)(a^\rho+b^\rho)\right) \le \frac{h(1/2)\,\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho \mathrm{I}^\alpha_{a+} (f\circ g)(b) + {}^\rho \mathrm{I}^\alpha_{b-} (f\circ g)(a)\right] \tag{2.1}$$

where $g(x) = x^\rho$.

Proof. Let $t \in [0,1]$. Consider $x, y \in [a,b], a \ge 0$, defined by $x^\rho = w^\rho a^\rho + (1-w^\rho)b^\rho$, $y^\rho = (1-w^\rho)a^\rho + w^\rho b^\rho$ for $w \in [0,1]$. Since $f : D \to \mathbb{R}$ is (k,h)-convex function on $[a^\rho, b^\rho]$, we have

$$f(k(t)x^\rho + k(1-t)y^\rho) \le h(t)f(x^\rho) + h(1-t)f(y^\rho). \tag{2.2}$$

By writing $t = 1/2$ in (2.2), we get;

$$\begin{aligned} f\left(k(1/2)(a^\rho+b^\rho)\right) &= f\left(k(1/2)x^\rho + k(1/2)y^\rho\right) \\ &\le h(1/2)\Big[f\big(w^\rho a^\rho + (1-w^\rho)b^\rho\big) \\ &\quad + f\big((1-w^\rho)a^\rho + w^\rho b^\rho\big)\Big]. \end{aligned} \tag{2.3}$$

We may now multiply both sides of (2.3) by $w^{\alpha\rho-1}$, $\alpha > 0$ and then integrate it over $[0,1]$ with respect to w, getting;

$$\begin{aligned} & f\left(k(1/2)(a^\rho+b^\rho)\right)\int_0^1 w^{\alpha\rho-1}dw \\ \leq \quad & h(1/2)\left[\int_0^1 w^{\alpha\rho-1}f\big(w^\rho a^\rho+(1-w^\rho)b^\rho\big)dw \right. \\ & \left. +\int_0^1 w^{\alpha\rho-1}f\big((1-w^\rho)a^\rho+w^\rho b^\rho)\big)dw\right]. \end{aligned}$$

Then we have

$$\begin{aligned} & \frac{1}{\alpha\rho}f\left(k(1/2)(a^\rho+b^\rho)\right) \\ \leq \quad & h(1/2)\left[\int_b^a \left(\frac{x^\rho-b^\rho}{a^\rho-b^\rho}\right)^\alpha\left(\frac{a^\rho-b^\rho}{x^\rho-b^\rho}\right)f(x^\rho)\frac{x^{\rho-1}}{a^\rho-b^\rho}dx \right. \\ & \left. +\int_a^b \left(\frac{y^\rho-a^\rho}{b^\rho-a^\rho}\right)^\alpha\left(\frac{b^\rho-a^\rho}{y^\rho-a^\rho}\right)f(y^\rho)\frac{y^{\rho-1}}{b^\rho-a^\rho}dy\right]. \end{aligned}$$

Then

$$\begin{aligned} & \frac{1}{\alpha\rho}f\left(k(1/2)(a^\rho+b^\rho)\right) \\ \leq \quad & h(1/2)\left[\int_a^b \frac{(b^\rho-a^\rho)^{1-\alpha}}{(b^\rho-x^\rho)^{1-\alpha}}\;\frac{x^{\rho-1}}{(b^\rho-a^\rho)}\;f(x^\rho)dx \right. \\ & \left. +\int_a^b \frac{(b^\rho-a^\rho)^{1-\alpha}}{(y^\rho-a^\rho)^{1-\alpha}}\;\frac{y^{\rho-1}}{(b^\rho-a^\rho)}\;f(y^\rho)dy\right] \end{aligned}$$

i.e

$$\begin{aligned} & \frac{1}{\alpha\rho}f\left(k(1/2)(a^\rho+b^\rho)\right) \\ \leq \quad & \frac{h\left(1/2\right)\Gamma(\alpha)}{\left(b^\rho-a^\rho\right)^\alpha\rho^{1-\alpha}}\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b \frac{x^{\rho-1}}{(b^\rho-x^\rho)^{1-\alpha}}f(x^\rho)dx \right. \\ & \left. +\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b \frac{y^{\rho-1}}{(y^\rho-a^\rho)^{1-\alpha}}f(y^\rho)dy\right]. \end{aligned}$$

Using the definition of Katugampola fractional integrals, we can write;

$$\frac{1}{\alpha\rho}f\left(k(1/2)(a^\rho+b^\rho)\right)\leq\frac{h\left(1/2\right)\Gamma\left(\alpha\right)}{\left(b^\rho-a^\rho\right)^\alpha\rho^{1-\alpha}}\left[{}^\rho I^\alpha_{a+}(f\circ g)(b)+{}^\rho\, I^\alpha_{b-}(f\circ g)(a)\right].$$

This implies;

$$f\left(k(1/2)(a^\rho+b^\rho)\right)\leq\frac{h\left(1/2\right)\rho^\alpha\Gamma\left(\alpha+1\right)}{\left(b^\rho-a^\rho\right)^\alpha}\left[{}^\rho I^\alpha_{a+}(f\circ g)(b)+{}^\rho\, I^\alpha_{b-}(f\circ g)(a)\right]$$

the proof is complete. □

Corollary 2.1. *If we write $\rho = 1$ in inequality* (2.1)*, we obtain;*

$$f\left(k(1/2)(a+b)\right) \leq \frac{h\left(1/2\right)\Gamma\left(\alpha+1\right)}{\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]$$

with $\alpha > 0$.

Corollary 2.2. *If we write $\alpha = 1$ in inequality* (2.1)*, we obtain;*

$$\frac{1}{\rho}f\left(k(1/2)(a^{\rho}+b^{\rho})\right) \leq \frac{2\,h\left(1/2\right)}{\left(b^{\rho}-a^{\rho}\right)}\int_{a}^{b}x^{\rho-1}(f\circ g)(x)dx$$

with $\rho > 0$, where $g(x) = x^{\rho}$.

Remark 2.1. *If a function $f : D \to \mathbb{R}$ is convex i.e for $k(t) = t$ and $h(t) = t$ the inequality* (2.1) *becomes the left-hand side of the inequality* (1.16)*.*

Remark 2.2. *If we write $\rho = 1$, $k(t) = t$ and $h(t) = t$ in inequality* (2.1) *becomes the left-hand side of the inequality* (1.12)*.*

Theorem 2.2. *Let $f : D \to \mathbb{R}$ be a (k,h)-convex function with $h(1/2) > 0$, fix $0 < a < b < \infty$ such that $[a,b] \subset D$ and $f \in L[a,b]$. If f is nonnegative function which is symmetric with respect to $(a+b)/2$ then the following inequality holds*

$$f\left(k(1/2)(a+b)\right) \leq \frac{h(1/2)\rho^{\alpha}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha}}\left[{}^{\rho}\mathrm{I}_{a+}^{\alpha}f(b)+{}^{\rho}\,\mathrm{I}_{b-}^{\alpha}f(a)\right] \tag{2.4}$$

with $\alpha > 0$ and $\rho > 0$.

Proof. If a function $f : D \to \mathbb{R}$ is (k,h)-convex for all $x, y \in D$, $t \in (0,1)$ then;

$$f(k(t)x + k(1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{2.5}$$

If f is nonnegative function which is symmetric with respect to (a+b)/2 and writing (2.5) with $t = 1/2$, $x = wa + (1-w)b$ and $y = (1-w)a + wb$ for $w \in [0,1]$, we get;

$$\begin{aligned} f\left(k(1/2)(a+b)\right) &= f\left(k(1/2)x+k(1/2)y\right) \\ &\leq h(1/2)\left[f\big(wa+(1-w)b\big)+f\big((1-w)a+wb\big)\right] \\ &= 2\,h(1/2)f\big((1-w)a+wb\big). \end{aligned} \tag{2.6}$$

We may now multiply both sides of (2.6) by

$$\frac{\left((1-w)a+wb\right)^{\rho-1}}{\left[b^{\rho}-(1-w)a+wb)^{\rho}\right]^{1-\alpha}} \tag{2.7}$$

and integrating the resulting inequality over $[0,1]$, we get;

$$\begin{aligned} & f\left(k(1/2)(a+b)\right)\int_{0}^{1}\frac{\left((1-w)a+wb\right)^{\rho-1}}{\left[b^{\rho}-(1-w)a+wb)^{\rho}\right]^{1-\alpha}}dw \\ \leq\ & 2\,h(1/2)\int_{0}^{1}\frac{\left((1-w)a+wb\right)^{\rho-1}}{\left[b^{\rho}-(1-w)a+wb)^{\rho}\right]^{1-\alpha}}\ f\left((1-w)a)+wb\right)dw. \end{aligned}$$

Then we have

$$f\left(k(1/2)\ (a+b)\right)\int_{a}^{b}\frac{x^{\rho-1}}{\left[b^{\rho}-x^{\rho}\right]^{1-\alpha}}\ \frac{dx}{b-a} \leq 2\,h(1/2)\int_{a}^{b}\frac{x^{\rho-1}}{\left[b^{\rho}-x^{\rho}\right]^{1-\alpha}}\ \ f(x)\ \ \frac{dx}{b-a}$$

i.e

$$f\left(k(1/2)\ (a+b)\right)\frac{(b^\rho-a^\rho)^\alpha}{\alpha\rho(b-a)}\leq\frac{2\ h(1/2)}{(b-a)}\ \frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\frac{x^{\rho-1}}{\left[b^\rho-x^\rho\right]^{1-\alpha}}\ f(x)dx.$$

Using the definition of Katugampola fractional integrals, we can write

$$f\left(k(1/2)\ (a+b)\right)\leq\frac{2\ h(1/2)\ \rho^\alpha\ \Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\ {}^\rho I_{a+}^\alpha f(b). \tag{2.8}$$

Similarly multiplying both sides of (2.6) by

$$\frac{((1-w)a+wb)^{\rho-1}}{\left[(1-w)a+wb)^\rho-a^\rho\right]^{1-\alpha}} \tag{2.9}$$

and integrating the resulting inequality over $[0,1]$, we get;

$$f\left(k(1/2)\ (a+b)\right)\leq\frac{2\ h(1/2)\ \rho^\alpha\ \Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\ {}^\rho I_{b-}^\alpha f(a). \tag{2.10}$$

By adding inequalities (2.8) and (2.10), we obtain;

$$f\left(k(1/2)\ (a+b)\right)\leq\frac{h(1/2)\ \rho^\alpha\ \Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\ \left[{}^\rho I_{a+}^\alpha f(b)+{}^\rho\ I_{b-}^\alpha f(a)\right]$$

and the proof is complete. □

Corollary 2.3. *If a function $f:D\to\mathbb{R}$ is convex i.e for $k(t)=t$ and $h(t)=t$ inequality* (2.4) *becomes the following inequality*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\rho^\alpha\ \Gamma(\alpha+1)}{2\ (b^\rho-a^\rho)^\alpha}\ \left[{}^\rho\mathrm{I}_{a+}^\alpha f(b)+{}^\rho\ \mathrm{I}_{b-}^\alpha f(a)\right]$$

Corollary 2.4. *If we write $\rho=1$ in inequality* (2.4)*, we obtain;*

$$f\left(k(1/2)\ (a+b)\right)\leq\frac{h(1/2)\ \Gamma(\alpha+1)}{(b-a)^\alpha}\ \left[\mathrm{J}_{a+}^\alpha f(b)+\mathrm{J}_{b-}^\alpha f(a)\right]$$

with $\alpha>0$.

Corollary 2.5. *If we write $\alpha=1$ in inequality* (2.4)*, we obtain;*

$$\frac{1}{\rho}f\left(k(1/2)(a+b)\right)\leq\frac{2\ h\left(1/2\right)}{(b^\rho-a^\rho)}\int_a^b x^{\rho-1}f(x)dx$$

with $\rho>0$.

Remark 2.3. *If we write $\rho=1$, $k(t)=t$ and $h(t)=t$ in inequality* (2.4) *becomes the left-hand side of the inequality* (1.12).

Theorem 2.3. *Let $f:D\to\mathbb{R}$ be a (k,h)-convex function with $h(1/2)>0$, fix $0<a<b<\infty$ such that $[a,b]\subset D$ and $f\in L[a,b]$. If f is nonnegative function which is symmetric with respect to $(a+b)/2$ and $g:[a,b]\to\mathbb{R}$ is nonnegative and integrable, then the following inequality holds*

$$\frac{f\left(k(1/2)(a+b)\right)}{h(1/2)}\left[{}^\rho\mathrm{I}_{a+}^\alpha g(b)+{}^\rho\ \mathrm{I}_{b-}^\alpha g(a)\right]\leq\left[{}^\rho\mathrm{I}_{a+}^\alpha fg(b)+{}^\rho\ \mathrm{I}_{b-}^\alpha fg(a)\right] \tag{2.11}$$

with $\alpha>0$ and $\rho>0$.

Proof. If a function $f : D \to \mathbb{R}$ is (k,h)-convex for all $x, y \in D$, $t \in (0,1)$ then;

$$f(k(t)x + k(1-t)y) \leq h(t)f(x) + h(1-t)f(y). \tag{2.12}$$

If f is nonnegative function which is symmetric with respect to $(a+b)/2$ and writing (2.12) with $t = 1/2$, $x = wa + (1-w)b$ and $y = (1-w)a + wb$ for $w \in [0,1]$, we get;

$$\begin{aligned} f\left(k(1/2)(a+b)\right) &= f\left(k(1/2)x + k(1/2)y\right) \\ &\leq h(1/2)\big[f\big(wa + (1-w)b\big) + f\big((1-w)a + wb\big)\big] \\ &= 2\,h(1/2)f\big((1-w)a + wb\big). \end{aligned} \tag{2.13}$$

We may now multiply both sides of (2.13) by

$$\frac{((1-w)a + wb)^{\rho-1}}{\left[b^\rho - (1-w)a + wb)^\rho\right]^{1-\alpha}}\, g\big((1-w)a + wb\big) \tag{2.14}$$

and integrating the resulting inequality over $[0,1]$, we get;

$$\begin{aligned} & f\left(k(1/2)(a+b)\right)\int_0^1 \frac{((1-w)a + wb)^{\rho-1}}{\left[b^\rho - (1-w)a + wb)^\rho\right]^{1-\alpha}}\, g\big((1-w)a + wb\big)dw \\ \leq\; & 2\,h(1/2)\int_0^1 \frac{((1-w)a + wb)^{\rho-1}}{\left[b^\rho - (1-w)a + wb)^\rho\right]^{1-\alpha}}\, f\left((1-w)a) + wb\right)\; g\big((1-w)a + wb\big)dw. \end{aligned}$$

Then we have

$$f\left(k(1/2)\;(a+b)\right)\int_a^b \frac{x^{\rho-1}}{\left[b^\rho - x^\rho\right]^{1-\alpha}}\; g(x)\frac{dx}{b-a} \leq 2\;h(1/2)\int_a^b \frac{x^{\rho-1}}{\left[b^\rho - x^\rho\right]^{1-\alpha}}\;\; f(x)g(x)\;\;\frac{dx}{b-a}$$

i.e

$$\begin{aligned} & f\left(k(1/2)\;(a+b)\right)\frac{1}{(b-a)}\;\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\;\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b \frac{x^{\rho-1}}{\left[b^\rho - x^\rho\right]^{1-\alpha}}\;\; g(x)dx \\ \leq\; & \frac{2\;h(1/2)}{(b-a)}\;\frac{\Gamma(\alpha)}{\rho^{1-\alpha}}\;\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^b \frac{x^{\rho-1}}{\left[b^\rho - x^\rho\right]^{1-\alpha}}\;\; f(x)g(x)dx. \end{aligned}$$

Using the definition of Katugampola fractional integral, we can write

$$f\left(k(1/2)\;(a+b)\right)^\rho\, I^\alpha_{a+}g(b) \leq 2\;h(1/2)^\rho I^\alpha_{a+}fg(b). \tag{2.15}$$

Similarly multiplying both sides of (2.13) by

$$\frac{((1-w)a + wb)^{\rho-1}}{\left[(1-w)a + wb)^\rho - a^\rho\right]^{1-\alpha}}\, g\big((1-w)a + wb\big) \tag{2.16}$$

and integrating the resulting inequality over $[0,1]$, we get;

$$f\left(k(1/2)\;(a+b)\right)^\rho\, I^\alpha_{b-}g(a) \leq 2\;h(1/2)^\rho I^\alpha_{b-}fg(a). \tag{2.17}$$

By adding inequalities (2.15) and (2.17), we obtain;

$$\frac{f\left(k(1/2)\;(a+b)\right)}{2\;h(1/2)}\left[{}^\rho I^\alpha_{a+}g(b) +^\rho\, I^\alpha_{b-}g(a)\right] \leq \left[{}^\rho I^\alpha_{a+}fg(b) +^\rho\, I^\alpha_{b-}fg(a)\right]$$

and the proof is complete. □

Corollary 2.6. *If a function $f : D \to \mathbb{R}$ is convex i.e for $k(t) = t$ and $h(t) = t$ inequality* (2.11) *becomes the following inequality*

$$f\left(\frac{a+b}{2}\right)\left[{}^\rho \mathrm{I}^\alpha_{a+}g(b) +^\rho \mathrm{I}^\alpha_{b-}g(a)\right] \leq \left[{}^\rho \mathrm{I}^\alpha_{a+}fg(b) +^\rho \mathrm{I}^\alpha_{b-}fg(a)\right]$$

Corollary 2.7. *If we write $\alpha = 1$ in inequality* (2.11)*, we obtain;*

$$f\left(k(1/2)(a+b)\right)\int_a^b x^{\rho-1}g(x)dx \leq 2\ h\left(1/2\right)\int_a^b x^{\rho-1}f(x)g(x)dx$$

with $\rho > 0$.

Remark 2.4. *If we write $\rho = 1$ in inequality* (2.11) *we obtain inequality* (1.14)*.*

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