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# THE MULTIPLE VALUES OF ALGEBROID FUNCTIONS AND UNIQUENESS ON ANNULI

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Abstract. In this paper, we present a unified approach of investigating the influence of multiple values and deficiencies on the uniqueness problem of algebroid functions on annuli and deduced several results on uniqueness of algebroid functions on annuli.

## 1. Introduction

The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. The uniqueness problem of algebroid functions was firstly considered by Valiron, afterwards some scholars have got several uniqueness theorems of algebroid functions in the complex plane $\mathbb{C}$ (see [2, 3, 5, 9, 10, 11, 18, 19]). In 2005, A. Ya. Khrystiyanyn and A. A. Kondratyuk have proposed on the Nevanlinna Theory for meromorphic functions annuli (see [7,8]). In 2009, Cao and Yi [1] investigated the uniqueness of meromorphic functions sharing some values on annuli. In 2015, Yang Tan [12], Yang Tan and Yue Wang [13] proved some interesting results on the multiple values and uniqueness of algebroid functions on annuli. In thsi paper, we mainly study doubly connected domain. By Doubly connected mapping theorem [17] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0$, $R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

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## 2. Basic Notations and Definitions

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [4] and [15]).

Let $A_v(z), A_{v-1}(z), ..., A_0(z)$ be analytic functions which have no common zeros and define on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$,

$$\psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + ... + A_1(z)W + A_0(z) = 0. \tag{2.1}$$

Then irreducible equation (2.1) defines a $v$-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$.

Let $W(z)$ be a $v$-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, we use the following notations

$$m(r, W) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r, w_j) = \frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

$$N_1(r, W) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{n_1(t, W)}{t}dt, \quad N_2(r, W) = \frac{1}{\nu}\int_1^r\frac{n_2(t, W)}{t}dt,$$

$$\overline{N}_1\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1\left(t, \frac{1}{W-a}\right)}{t}dt, \quad \overline{N}_2\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_1^r\frac{\overline{n}_2\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$\overline{N}_1^{k)}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1^{k)}\left(t, \frac{1}{W-a}\right)}{t}dt, \quad \overline{N}_2^{k)}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_1^r\frac{\overline{n}_2^{k)}\left(t, \frac{1}{w-a}\right)}{t}dt,$$

$$\overline{N}_1^{(k}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1^{(k}\left(t, \frac{1}{W-a}\right)}{t}dt, \quad \overline{N}_2^{(k}\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu}\int_1^r\frac{\overline{n}_2^{(k}\left(t, \frac{1}{W-a}\right)}{t}dt,$$

$$m_0(r, W) = m(r, W) + m\left(\frac{1}{r}, W\right) - 2m(1, W), \quad N_0(r, W) = N_1(r, W) + N_2(r, W),$$

$$\overline{N}_0\left(r, \frac{1}{W-a}\right) = \overline{N}_1\left(r, \frac{1}{W-a}\right) + \overline{N}_2\left(r, \frac{1}{W-a}\right),$$

$$\overline{N}_0^{k)}\left(r, \frac{1}{W-a}\right) = \overline{N}_1^{k)}\left(r, \frac{1}{W-a}\right) + \overline{N}_2^{k)}\left(r, \frac{1}{W-a}\right),$$

$$\overline{N}_0^{(k}\left(r, \frac{1}{W-a}\right) = \overline{N}_1^{(k}\left(r, \frac{1}{W-a}\right) + \overline{N}_2^{(k}\left(r, \frac{1}{W-a}\right),$$

where $w_j(z)(j = 1, 2, ..., \nu)$ is one valued branch of $W(z)$, $n_1(t, W)$ is the counting function of poles of the function $W(z)$ in $\{z : t < |z| \leq 1\}$ and $n_2(t, W)$ is the counting function of poles of the function $W(z)$ in $\{z : 1 < |z| \leq t\}$ (both counting multiplicity).$\overline{n}_1\left(t, \frac{1}{W-a}\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ in $\{z : t < |z| \leq 1\}$ and $\overline{n}_2\left(t, \frac{1}{W-a}\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ in $\{z : 1 < |z| \leq t\}$ (both ignoring multiplicity). $\overline{n}_1^{k)}\left(t, \frac{1}{W-a}\right)$ $\left(\overline{n}_1^{(k}\left(t, \frac{1}{W-a}\right)\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ with multiplicity $\leq k$ $(or \quad > k)$ in $\{z : t < |z| \leq 1\}$, each point count only once; $\overline{n}_2^{k)}\left(t, \frac{1}{W-a}\right)$ $\left(\overline{n}_2^{(k}\left(t, \frac{1}{W-a}\right)\right)$ is the counting function of poles of the function $\frac{1}{W-a}$

with multiplicity $\leq k$ (*or* $> k$) in $\{z : 1 < |z| \leq t\}$, each point count only once, respectively.

Let $W(z)$ be a $\nu$-valued algebroid function which determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, when $a \in \mathbb{C}$, $n_0\left(r, \frac{1}{W-a}\right) = n_0\left(r, \frac{1}{\psi(z,a)}\right)$, $N_0\left(r, \frac{1}{W-a}\right) = \frac{1}{\nu} N_0\left(r, \frac{1}{\psi(z,a)}\right)$. In particular, when $a = 0$, $N_0\left(r, \frac{1}{W}\right) = \frac{1}{\nu} N_0\left(r, \frac{1}{A_0}\right)$. When $a = \infty$, $N_0(r, W) = \frac{1}{\nu} N_0\left(r, \frac{1}{A_v}\right)$; where $n_0\left(r, \frac{1}{W-a}\right)$ and $n_0\left(r, \frac{1}{\psi(z,a)}\right)$ are the counting function of zeros of $W(z) - a$ and $\psi(z, a)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$.

**Definition 2.1.** [**12**] Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, the function

$$T_0(r, W) = m_0(r, W) + N_0(r, W), \qquad 1 \leq r < R_0$$

is called Nevanlinna characteristic of $W(z)$.

**Definition 2.2.** [**12**] Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, we denote the deficiency of $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by

$$\delta_0(a, W) = \delta_0(0, W - a) = \liminf_{r \to R_0} \frac{m_0\left(r, \frac{1}{W-a}\right)}{T_0(r, W)} = 1 - \limsup_{r \to R_0} \frac{N_0\left(r, \frac{1}{W-a}\right)}{T_0(R, W)},$$

and denote the reduced deficiency by

$$\Theta_0(a, W) = \Theta_0(0, W - a) = 1 - \limsup_{r \to R_0} \frac{\overline{N_0}\left(r, \frac{1}{W-a}\right)}{T_0(r, W)}.$$

**Definition 2.3.** Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, let $'a'$ be any arbitrary complex number. The Valiron deficiency of $W(z)$ on the annulus $\mathbb{A}$ with respect to the value $'a'$ will be defined by

$$\Delta_0(a, W) = \limsup_{r \to \infty} \frac{m_0(R, \frac{1}{W-a})}{T_0(R, W)} = 1 - \liminf_{r \to \infty} \frac{N_0(R, \frac{1}{W-a})}{T_0(R, W)}.$$

## Some Lemmas

**Lemma 2.1.** [**7**] *(Jensen theorem for meromorphic function on annuli) Let $f(z)$ be a meromorphic function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, then*

$$\begin{aligned} N_0\left(r, \frac{1}{f}\right) - N_0(r, f) &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta, \end{aligned}$$

*where $1 \leq r < R_0$.*

**Lemma 2.2.** [**12**] *(The first fundamental theorem on annuli) Let $W(z)$ be $\nu$-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ ($1 < R_0 \le +\infty$), $a \in \mathbb{C}$*

$$m_0(r,a) + N_0(r,a) = T_0\,(r,W) + O(1).$$

**Lemma 2.3.** [**13**] *(The second fundamental theorem on annuli). Let $W(z)$ be $\nu$-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ ($1 < R_0 \le +\infty$), $a_k$ ($k = 1, 2, .., p$) are $p$ distinct complex numbers (finite or infinite), then we have*

$$(p - 2v)T_0\,(r,W) \le \sum_{k=1}^{p} N_0\left(r, \frac{1}{W - a_k}\right) - N_1(r,W) + S_0(r,W) \tag{2.2}$$

*$N_1(r,W)$ is the density index of all multiple values including finite or infinite, every $\tau$ multiple value counts $\tau - 1$, and*

$$S_0(r,W) = m_0\left(r, \frac{W}{W'}\right) + \sum_{j=1}^{p} m_0\left(r, \frac{W}{W - a_k}\right) + O(1).$$

*The remainder of the second fundamental theorem is the following formula*

$$S_0(r,W) = O\left(\log T_0(r,W)\right) + O(\log r),$$

*outside a set of finite linear measure, if $r \to R_0 = +\infty$, while*

$$S_0(r,W) = O\left(\log T_0(r,W)\right) + O\left(log\frac{1}{R_0 - r}\right),$$

*outside a set $E$ of $r$ such that $\int_E \frac{dr}{R_0 - r} < +\infty$, when $r \to R_0 < +\infty$.*

*Remark* 2.1. [**13**] The second fundamental theorem on annuli has other forms, as the following:

$$\begin{aligned}(p-1)T_0\,(r,W) \quad &\le \quad N_0\,(r,W) + \sum_{k=1}^{p} N_0\left(r, \frac{1}{W - a_k}\right) \\ &\qquad -N_1(r,W) + Q_1(r,W),\end{aligned} \tag{2.3}$$

$$N_1(r,W) = 2N_0(r,W) - N_0(r,W') + N_0\left(r, \frac{1}{W'}\right),$$

$$Q_1(r,W) = \sum_{k=0}^{p} m_0\left(r, \frac{W}{W - a_k}\right) + O(1), a_0 = 0.$$

We notice that the following formula is true,

$$\sum_{k=1}^{p} N_0\left(r, \frac{1}{W - a_k}\right) - N_1(r,W) \le \sum_{k=1}^{p} \overline{N}_0\left(r, \frac{1}{W - a_k}\right). \tag{2.4}$$

$\overline{N}_0\left(r, \frac{1}{W - a_k}\right)$ is the reduced counting function of zeros(ignoring multiplicity). Then the second fundamental theorem can be rewritten as the following

$$(p - 2v)T_0\,(r,W) \le \sum_{k=1}^{p} N_0\left(r, \frac{1}{W - a_k}\right) + S_0(r,W). \tag{2.5}$$

**Lemma 2.4.** [**12**] *Let $W(z)$ be $\nu$-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, if the following conditions are satisfied*

$$\liminf_{r\to\infty} \frac{T_0\left(r, W\right)}{\log r} < \infty, \quad R_0 = +\infty,$$

$$\liminf_{r\to R_0^-} \frac{T_0\left(r, W\right)}{\log \frac{1}{(R_0 - r)}} < \infty, \quad R_0 < +\infty,$$

*then $W(z)$ is an algebraic function.*

*Remark* 2.2. [**12**] Let $W(z)$ be a $\nu$-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ and $\widehat{W}(z)$ be a $\mu$ valued algebroid function which is determined by the following equation on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$,

$$\varphi(z, \widehat{W}) = B_\mu(z)\widehat{W}^\mu + B_{\mu-1}(z)\widehat{W}^{\mu-1} + ... + B_1(z)\widehat{W} + B_0(z) = 0.$$

Without loss of generality, let $\mu \leq \nu, \overline{n}^{k)}_{\Delta}(r, a)$ denotes the counting function of the common values of $W(z) = a$ and $\widehat{W}(z) = a$ with multiplicity $\leq k$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, each point counts only once. And let

$$\begin{aligned} \overline{N}^{k_j)}_{\Delta}(r, a) &= \frac{\mu + \nu}{2\mu\nu} \int_{\frac{1}{r}}^{1} \frac{\overline{n}^{k)}_{\Delta_1}(t, a)}{t} dt + \frac{\mu + \nu}{2\mu\nu} \int_{1}^{r} \frac{\overline{n}^{k)}_{\Delta_2}(t, a)}{t} dt \\ \overline{N}^{k_j)}_{12}(r, a) &= \overline{N}^{k)}_{0}\left(r, \frac{1}{W - a}\right) + \overline{N}^{k)}_{0}\left(r, \frac{1}{\widehat{W} - a}\right) - 2\overline{N}^{k_j)}_{\Delta}(r, a). \end{aligned}$$

## 3. Main Results

Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ and $a$ be a complex number in the extended complex plane. Write $E(a, W) = \{z \in \mathbb{A} : W(z) - a = 0\}$, where each zero with multiplicity $m$ is counted $m$ times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a, W)$. We use $\overline{E}_{k)}(a, W)$ to denote the set of zeros of $W - a$ with multiplicities not greater than $k$, in which each zero is counted only once.

We now show our main results below which is an analog of a result on the plane $\mathbb{C}$ obtained by H. X. Yi [16](see Theorem 3.19 and 3.20 in [15]).

**Theorem 3.1.** *Let $W(z)$ and $\widehat{W}(z)$ be two $\nu$-valued and $\mu$ valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., q)$ be $q$ distinct complex numbers in $\overline{\mathbb{C}}$ and $k_j\,(j = 1, 2, ..., q)$ be $q$ positive integers or $\infty$ such that*

$$k_1 \geq k_2 \geq ... \geq k_q \tag{3.1}$$

*and*

$$\overline{E}_{k_j)}(a_j, W(z)) = \overline{E}_{k_j)}(a_j, \widehat{W}(z)), \qquad (j = 1, 2, .., q). \tag{3.2}$$

*Set*

$$A_1 = \frac{\delta_0(a_1, W(z)) + \delta_0(a_2, W(z)) + ... + \delta_0(a_{2\nu}, W(z))}{k_3 + 1} + \sum_{j=2\nu+1}^{q} \frac{k_j + \delta_0(a_j, W(z))}{k_j + 1} - 2\nu$$

*and*

$$A_2 = \frac{\delta_0(a_1, \widehat{W}(z)) + \delta_0(a_2, W(z)) + ... + \delta_0(a_{2\nu}, \widehat{W}(z))}{k_3 + 1} + \sum_{j=2\nu+1}^{q} \frac{k_j + \delta_0(a_j, \widehat{W}(z))}{k_j + 1} - 2\mu$$

*If*

$$min\{A_1, A_2\} \geq 0, \tag{3.3}$$

*and*

$$max\{A_1, A_2\} > 0, \tag{3.4}$$

*then* $W(z) \equiv \widehat{W}(z)$.

*Proof.* We may assume, without loss of generality, that all $a_j$ $(j = 1, 2, ..., q)$ are finite, otherwise, a suitable Mobius transformation will be done. From Remark 2.1, we have

$$(q - 2\nu)\, T_0(r, W) < \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \overline{N_0}^{k_j)} \left(r, \frac{1}{W - a_j}\right) + \sum_{j=1}^{q} \frac{1}{k_j + 1} N_0 \left(r, \frac{1}{W - a_j}\right) + S_0(R, W). \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{aligned}
&(q - 2\nu)\, T_0(r, W) \\
&< \frac{k_{2\nu+1}}{k_{2\nu+1} + 1} \sum_{j=1}^{q} \overline{N_0}^{k_j)} \left(r, \frac{1}{W - a_j}\right) + \sum_{j=1}^{2\nu} \left(\frac{k_j}{k_j + 1} - \frac{k_{2\nu+1}}{k_{2\nu+1} + 1}\right) N_0 \left(r, \frac{1}{W - a_j}\right) \\
&+ \sum_{j=1}^{q} \frac{1}{k_j + 1} N_0 \left(r, \frac{1}{W - a_j}\right) + S_0(r, W)
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
&= \frac{k_{2\nu+1}}{k_{2\nu+1} + 1} \sum_{j=1}^{q} \overline{N_0}^{k_j)} \left(r, \frac{1}{W - a_j}\right) + \sum_{j=1}^{2\nu} \left(\frac{k_j}{k_j + 1} - \frac{k_{2\nu+1}}{k_{2\nu+1} + 1}\right) (1 - \delta_0(a_j, W)) T_0(r, W) \\
&+ \sum_{j=1}^{q} \frac{1}{k_j + 1} (1 - \delta_0(a_j, W)) T_0(r, W) + S_0(r, W) \\
&= \frac{k_{2\nu+1}}{k_{2\nu+1} + 1} \sum_{j=1}^{q} \overline{N_0}^{k_j)} \left(r, \frac{1}{W - a_j}\right) + \sum_{j=1}^{2\nu} \left(\frac{k_j}{k_j + 1} + \frac{1}{k_j + 1}\right) (1 - \delta_0(a_j, W)) T_0(r, W) \\
&- \sum_{j=1}^{2\nu} \frac{k_{2\nu+1}}{k_{2\nu+1} + 1} (1 - \delta_0(a_j, W)) T_0(r, W) + \sum_{j=2\nu+1}^{q} \frac{1}{k_j + 1} (1 - \delta_0(a_j, W)) T_0(r, W) + S_0(r, W) \\
&= \frac{k_3}{k_{2\nu+1} + 1} \sum_{j=1}^{q} \overline{N_0}^{k_j)} \left(r, \frac{1}{W - a_j}\right) + 2\nu T_0(r, W) - \sum_{j=1}^{2\nu} \delta_0(a_j, W) T_0(R, W) - \frac{2\nu\, k_{2\nu+1}}{k_{2\nu+1} + 1} T_0(R, W) \\
&+ \sum_{j=1}^{2\nu} \frac{k_{2\nu+1}}{k_{2\nu+1} + 1} \delta_0(a_j, W) T_0(r, W) + \sum_{j=2\nu+1}^{q} \frac{1}{k_j + 1} (1 - \delta_0(a_j, W)) T_0(r, W) + S_0(r, W)
\end{aligned} \tag{3.7}$$

$$= \frac{k_{2\nu+1}}{k_{2\nu+1}+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(r,\frac{1}{W-a_j}\right)+2\nu T_0(r,W)-\sum_{j=1}^{2\nu}\frac{\delta_0(a_j,W)}{k_{2\nu+1}+1}T_0(r,W)$$

$$-\frac{2\nu k_{2\nu+1}}{k_{2\nu+1}+1}T_0(r,W)+\sum_{j=2\nu+1}^{q}\frac{1}{k_j+1}T_0(r,W)-\sum_{j=2\nu+1}^{q}\frac{1}{k_j+1}(\delta_0(a_j,W))T_0(r,W)+S_0(r,W)$$

$$\left(\frac{\delta_0(a_1,W)+\delta_0(a_2,W)+...+\delta_0(a_{2\nu},W)}{k_{2\nu+1}+1}+\sum_{j=2\nu+1}^{q}\frac{\delta_0(a_j,W)}{k_j+1}+\frac{2\nu k_{2\nu+1}}{k_{2\nu+1}+1}+\sum_{j=2\nu+1}^{q}\frac{k_j}{k_j+1}-2\nu\right)T_0(r,W)$$

$$< \frac{k_{2\nu+1}}{k_{2\nu+1}+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(r,\frac{1}{W-a_j}\right)+S_0(r,W).$$

Therefore,

(3.8)
$$\left(C_1+\frac{2\nu k_{2\nu+1}}{k_{2\nu+1}+1}+\sum_{j=2\nu+1}^{q}\frac{k_j}{k_j+1}-2\nu\right)T_0(r,W)<\frac{k_{2\nu+1}}{k_{2\nu+1}+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(r,\frac{1}{W-a_j}\right)+S_0(r,W),$$

where

$$C_1=\frac{\delta_0(a_1,W(z))+\delta_0(a_2,W(z))+...+\delta_0(a_{2\nu},W(z))}{k_{2\nu+1}+1}+\sum_{j=2\nu+1}^{q}\frac{\delta_0(a_j,W(z))}{k_j+1}.$$

Similarly

(3.9)
$$\left(C_2+\frac{2\nu k_{2\nu+1}}{k_{2\nu+1}+1}+\sum_{j=\nu+1}^{q}\frac{k_j}{k_j+1}-2\mu\right)T_0(r,\widehat{W})<\frac{k_{2\nu+1}}{k_{2\nu+1}+1}\sum_{j=1}^{q}\overline{N_0}^{k_j)}\left(r,\frac{1}{\widehat{W}-a_j}\right)+S_0(r,\widehat{W}),$$

where

$$C_2=\frac{\delta_0(a_1,\widehat{W}(z))+\delta_0(a_2,\widehat{W}(z))+...+\delta_0(a_{2\nu},\widehat{W}(z))}{k_{2\nu+1}+1}+\sum_{j=2\nu+1}^{q}\frac{\delta_0(a_j,\widehat{W}(z))}{k_j+1}.$$

By (3.8), (3.9) and Remark 3.2, we have

$$\begin{aligned}
&\left(C_1+\frac{2\nu k_{2\nu+1}}{k_{2\nu+1}+1}+\sum_{j=2\nu+1}^{q}\frac{k_j}{k_j+1}-2\nu\right)T_0(r,W)\\
&+\left(C_2+\frac{2\nu k_{2\nu+1}}{k_{2\nu+1}+1}+\sum_{j=2\nu+1}^{q}\frac{k_j}{k_j+1}-2\mu\right)T_0(r,\widehat{W})\\
&\le \frac{k_{2\nu+1}}{k_{2\nu+1}+1}\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{W-a_j}\right)+\frac{k_{2\nu+1}}{k_{2\nu+1}+1}\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{\widehat{W}-a_j}\right)\\
&+S_0(r,W)+S_0(r,\widehat{W}),\\
&\le \frac{k_{2\nu+1}}{k_{2\nu+1}+1}\left(\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{W-a_j}\right)+\sum_{j=1}^{q}\overline{N}_0^{k_j)}\left(r,\frac{1}{\widehat{W}-a_j}\right)\right)\\
&+S_0(r,W)+S_0(r,\widehat{W}),
\end{aligned}$$

$$\le \frac{k_{2\nu+1}}{k_{2\nu+1}+1}\left(\sum_{j=1}^{q}\overline{N}_{12}(r,a_j)+2\sum_{j=1}^{q}\overline{N}_{\Delta}(r,a_j)\right)+S_0(r,W)+S_0(r,\widehat{W}). \tag{3.10}$$

If $W(z) \not\equiv \widehat{W}(z)$, then we have

$$\sum \overline{n}_{\Delta}(r,a) \leq n_0\left(r, \frac{1}{R(\varphi,\psi)}\right),$$

$R(\varphi,\psi)$ denotes the resultant of $\varphi(z,W)$ and $\psi(z,W)$, it can be written as the following

$$R(\varphi,\psi) = [A_\nu(z)]^\mu [B_\mu(z)]^\nu \prod_{\substack{1\leq j\leq \nu\\ 1\leq k\leq \mu}} [w_j(z) - \widehat{w}_j(z)].$$

It can be written in the another form

$$R(\varphi,\psi) = \begin{vmatrix} A_\nu(z) & A_{\nu-1}(z) & \dots & \dots & A_0(z) & 0 & \dots & 0 \\ 0 & A_\nu(z) & A_{\nu-1}(z) & \dots & A_1(z) & A_0(z) & 0 & \\ \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & A_\nu(z) & A_{\nu-1}(z) & \dots & \dots & A_0(z) \\ B_\mu(z) & B_{\mu-1}(z) & \dots & \dots & B_0(z) & 0 & \dots & 0 \\ 0 & B_\mu(z) & B_{\mu-1}(z) & \dots & B_1(z) & B_0(z) & 0 & \\ \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & B_\mu(z) & B_{\mu-1}(z) & \dots & \dots & B_0(z) \end{vmatrix}$$

So we know that $R(\varphi,\psi)$ is a holomorphic function and using Jensen Theorem for meromorphic function on annuli, we have

$$\begin{aligned}
& N_0\left(r, \frac{1}{R(\varphi,\psi)}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(re^{i\theta},W), \varphi(re^{i\theta},\widehat{W})]|d\theta \\
+ \quad & \frac{1}{2\pi}\int_0^{2\pi} \log\left|R\left[\psi\left(re^{i\theta},W\right), \varphi\left(re^{i\theta},\widehat{W}\right)\right]\right| d\theta \\
+ \quad & 2.\frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(e^{i\theta},W), \varphi(e^{i\theta},\widehat{W})]|d\theta \\
= \quad & \frac{\mu}{2\pi}\int_0^{2\pi} \log|A_\nu(re^{i\theta})|d\theta + \frac{\nu}{2\pi}\int_0^{2\pi} \log|B_\mu(re^{i\theta})|d\theta \\
+ \quad & \frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\leq j\leq \nu\\ 1\leq k\leq \mu}} [w_j(re^{i\theta}) - \widehat{w}_j(re^{i\theta})]\right| d\theta \\
+ \quad & \frac{\mu}{2\pi}\int_0^{2\pi} \log\left|A_\nu\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta + \frac{\nu}{2\pi}\int_0^{2\pi} \log\left|B_\mu\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta \\
+ \quad & \frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\leq j\leq \nu\\ 1\leq k\leq \mu}} \left[w_j\left(\frac{1}{r}e^{i\theta}\right) - \widehat{w}_j\left(\frac{1}{r}e^{i\theta}\right)\right]\right| d\theta - 2.\frac{\mu}{2\pi}\int_0^{2\pi} \log|A_\nu(e^{i\theta})|d\theta \\
- \quad & 2.\frac{\nu}{2\pi}\int_0^{2\pi} \log|B_\mu(e^{i\theta})|d\theta - 2.\frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\leq j\leq \nu\\ 1\leq k\leq \mu}} [w_j(e^{i\theta}) - \widehat{w}_j(e^{i\theta})]\right| d\theta \\
= \quad & \frac{\mu}{2\pi}\int_0^{2\pi} \log|A_\nu(re^{i\theta})|d\theta + \frac{\mu}{2\pi}\int_0^{2\pi} \log\left|A_\nu\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta - 2.\frac{\mu}{2\pi}\int_0^{2\pi} \log|A_\nu(e^{i\theta})|d\theta \\
+ \quad & \frac{\nu}{2\pi}\int_0^{2\pi} \log|B_\mu(re^{i\theta})|d\theta + \frac{\nu}{2\pi}\int_0^{2\pi} \log\left|B_\mu\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta - 2.\frac{\nu}{2\pi}\int_0^{2\pi} \log|B_\mu(e^{i\theta})|d\theta
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\le j\le \nu\\ 1\le k\le \mu}} [w_j(re^{i\theta}) - \widehat{w}_j(re^{i\theta})]\right| d\theta\\
&+ \quad \frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\le j\le \nu\\ 1\le k\le \mu}} \left[w_j\left(\frac{1}{r}e^{i\theta}\right) - \widehat{w}_j\left(\frac{1}{r}e^{i\theta}\right)\right]\right| d\theta\\
&- \quad 2.\frac{1}{2\pi}\int_0^{2\pi} \log\left|\prod_{\substack{1\le j\le \nu\\ 1\le k\le \mu}} [w_j(e^{i\theta}) - \widehat{w}_j(e^{i\theta})]\right| d\theta\\
&\le \quad \mu\left[m_0(r, A_\nu) - m_0\left(r, \frac{1}{A_\nu}\right)\right] + \nu\left[m_0(r, B_\mu) - m_0\left(r, \frac{1}{B_\mu}\right)\right]\\
&\qquad +\mu\nu[m_0(r, W) + m_0(r, \widehat{W})] + O(1)\\
&= \quad \mu\nu[T_0(r, W) + T_0(r, \widehat{W})] + O(1).
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\sum \overline{N}_{\Delta}^{k_j)}(r, a_j) &\le \frac{2\mu\nu}{\mu+\nu}[T_0(r, W) + T_0(r, \widehat{W})] + O(1)\\
&\le \nu[T_0(r, W) + T_0(r, \widehat{W})] + O(1). \qquad (3.11)
\end{aligned}
$$

By the condition of Theorem 3.1, we know that $W(z)$ and $\widehat{W}(z)$ take the same values with multiplicity $< k_j$ about $q$ distinct $a_j$, each point counts only once , at the same time we get $\overline{N}_{12}^{k_j)}(r, a_j) = 0$. From (3.10) and (3.11) and Remark 2.2

$$
\begin{aligned}
&\left(C_1 + \frac{2\nu k_{2\nu+1}}{k_{2\nu+1}+1} + \sum_{j=2\nu+1}^{q} \frac{k_j}{k_j+1} - 2\nu\right) T_0(r, W)\\
&+ \left(C_2 + \frac{2\nu k_{2\nu+1}}{k_{2\nu+1}+1} + \sum_{j=2\nu+1}^{q} \frac{k_j}{k_j+1} - 2\mu\right) T_0(r, \widehat{W})\\
&\le \frac{2\nu\, k_{2\nu+1}}{k_{2\nu+1}+1}[T_0(r, W) + T_0(r, \widehat{W})] + S_0(r, W) + S_0(r, \widehat{W})
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left(C_1 + \sum_{j=2\nu+1}^{q} \frac{k_j}{k_j+1} - 2\nu\right) T_0(r, W) + \left(C_2 + \sum_{j=2\nu+1}^{q} \frac{k_j}{k_j+1} - 2\mu\right) T_0(r, \widehat{W})\\
&\le S_0(r, W) + S_0(r, \widehat{W}).
\end{aligned}
$$

Therefore

$$A_1 T_0(r, W) + A_2 T_0(r, \widehat{W}) = S_0(r, W) + S_0(r, \widehat{W}).$$

From Lemma 2.4 we know that this is not true, so it must be $W(z) \equiv \widehat{W}(z)$. Therefore we complete the proof of theorem.

$\square$

From Theorem 3.1, we get the following corollaries

**Corollary 3.1.** *Let $W(z)$ and $\widehat{W}(z)$ be two $\nu$-valued and $\mu$ valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., q)$ be $q$ distinct complex numbers in $\overline{\mathbb{C}}$ and $k_j\,(j = 1, 2, ..., q)$ be $q$ positive integers or $\infty$ such that*

$$k_1 \geq k_2 \geq ... \geq k_q$$

*and*

$$\overline{E}_{k_j)}(a_j, W(z)) = \overline{E}_{k_j)}(a_j, \widehat{W}(z)), \qquad (j = 1, 2, .., q).$$

*Set*

$$A_1 = \sum_{j=2\nu+1}^{q} \frac{k_j}{k_j + 1} - 2\nu$$

*and*

$$A_2 = \sum_{j=2\nu+1}^{q} \frac{k_j}{k_j + 1} - 2\mu$$

*If*

$$min\{A_1, A_2\} \geq 0,$$

*and*

$$max\{A_1, A_2\} > 0,$$

*then $W(z) \equiv \widehat{W}(z)$.*

From Corollary 3.1, we obtained Corollary 3.2.

**Corollary 3.2.** *Let $W(z)$ and $\widehat{W}(z)$ be two $\nu$-valued and $\mu$ valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., q)$ be $q$ distinct complex numbers in $\overline{\mathbb{C}}$ and $k_j\,(j = 1, 2, ..., q)$ be $q$ positive integers or $\infty$ such that*

$$k_1 \geq k_2 \geq ... \geq k_q$$

*and*

$$\overline{E}_{k_j)}(a_j, W(z)) = \overline{E}_{k_j)}(a_j, \widehat{W}(z)), \qquad (j = 1, 2, .., q).$$

*If*

$$\sum_{j=2\nu+1}^{q} \frac{k_j}{k_j + 1} > -2\nu,$$

*then $W(z) \equiv \widehat{W}(z)$.*

As a consequence of Corollary 3.2, we get the following corollary

**Corollary 3.3.** *Let $W(z)$ and $\widehat{W}(z)$ be two $\nu$-valued and $\mu$ valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., q)$ be $q$ distinct complex numbers in $\overline{\mathbb{C}}$ and $k_j\,(j = 1, 2, ..., q)$ be $q$ positive integers or $\infty$ such that*

$$k_1 \geq k_2 \geq ... \geq k_{6\nu+1}$$

*and*

$$\overline{E}_{k_j)}(a_j, W(z)) = \overline{E}_{k_j)}(a_j, \widehat{W}(z)), \qquad (j = 1, 2, .., q),$$

*(i) if* $q = 6\nu + 1$, *then* $W(z) \equiv \widehat{W}(z))$;
*(ii) if* $q = 6\nu$ *and* $k_{2\nu+1} \geq 2\nu$, *then* $W(z) \equiv \widehat{W}(z))$;
*(iii) if* $q = 4\nu + 1$, $k_{2\nu+1} \geq 2\nu + 1$ *and* $k_5 \geq 2\nu$, *then* $W(z) \equiv \widehat{W}(z))$;
*(iv) if* $q = 4\nu + 1$ *and* $k_{4\nu} \geq 4\nu$, *then* $W(z) \equiv \widehat{W}(z))$.

From Corollary 3.3, we obtain the following result

**Theorem 3.2.** *Let* $W(z)$ *and* $\widehat{W}(z)$ *be two* $\nu$*-valued and* $\mu$ *valued algebroid functions on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ *respectively and* $\mu \leq \nu$, *let* $a_j\,(j = 1, 2, ..., 6\nu + 1)$ *be seven distinct complex numbers in* $\overline{\mathbb{C}}$.*If* $\overline{E}_{k_j)}(a_j, W(z)) = \overline{E}_{k_j)}(a_j, \widehat{W}(z)), \qquad (j = 1, 2, .., 6\nu + 1)$, *then* $W(z) \equiv \widehat{W}(z))$.

Also from Corollary 3.3, we obtain an analogue of Nevanlinna's five value theorem for algebroid functions on annuli as follows

**Theorem 3.3.** *Let* $W(z)$ *and* $\widehat{W}(z)$ *be two* $\nu$*-valued and* $\mu$ *valued algebroid functions on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ *respectively and* $\mu \leq \nu$, *let* $a_j\,(j = 1, 2, ..., 4\nu + 1)$ *be seven distinct complex numbers in* $\overline{\mathbb{C}}$.*If* $\overline{E}_{k_j)}(a_j, W(z)) = \overline{E}_{k_j)}(a_j, \widehat{W}(z)), \qquad (j = 1, 2, .., 4\nu + 1)$, *then* $W(z) \equiv \widehat{W}(z))$.

**Theorem 3.4.** *Let* $W(z)$ *and* $\widehat{W}(z)$ *be two* $\nu$*-valued and* $\mu$ *valued algebroid functions on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ *respectively and* $\mu \leq \nu$, *let* $a_j\,(j = 1, 2, ..., q)$ *be* $q$ *distinct complex numbers in* $\overline{\mathbb{C}}$ *and* $k_j\,(j = 1, 2, ..., q)$ *be* $q$ *positive integers or* $\infty$ *such that*

$$k_1 \geq k_2 \geq ... \geq k_q$$

*and*

$$\overline{E}_{k_j)}(a_j, W(z)) = \overline{E}_{k_j)}(a_j, \widehat{W}(z)), \qquad (j = 1, 2, .., q).$$

*Set*

$$B_1 = \frac{\Delta_0(a_1, W(z)) + \Delta_0(a_2, W(z)) + ... + \Delta_0(a_{2\nu}, W(z))}{k_{2\nu+1} + 1} + \sum_{j=2\nu+1}^{q} \frac{\Delta_0(a_j, W(z))}{k_j + 1}$$

*and*

$$B_2 = \frac{\Delta_0(a_1, \widehat{W}(z)) + \Delta_0(a_2, \widehat{W}(z)) + ... + \Delta_0(a_{2\nu}, \widehat{W}(z))}{k_{2\nu+1} + 1} + \sum_{j=2\nu+1}^{q} \frac{\Delta_0(a_j, \widehat{W}(z))}{k_j + 1}.$$

*If*

$$\sum_{j=2\nu+1}^{q} \frac{k_j}{k_j + 1} = 2\nu. \tag{3.12}$$

*and*

$$max\{B_1, B_2\} > 0. \tag{3.13}$$

*Then* $W(z) \equiv \widehat{W}(z)$.

*Proof.* Using the similar argument as in Theorem 3.1, we can prove Theorem 3.2 □

As a consequence of Theorem 3.4, we get the following corollary

**Corollary 3.4.** *Let $W(z)$ and $\widehat{W}(z)$ be two $\nu$-valued and $\mu$ valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ respectively and $\mu \leq \nu$, let $a_j\,(j = 1, 2, ..., q)$ be $q$ distinct complex numbers in $\overline{\mathbb{C}}$. Then*
*(i) If $q = 6\nu$ and $\overline{E}_{1)}(a_j, W) = \overline{E}_{1)}(a_j, \widehat{W})$, $(j = 1, 2, .., 6\nu)$,*

$$\sum_{j=1}^{6\nu} max\{\Delta_0(a_j, W), \Delta_0(a_j, \widehat{W})\} > 0,$$

*then $W(z) \equiv \widehat{W}(z)$;*
*(ii) If $q = 4\nu + 1$ and $\overline{E}_{2)}(a_j, W) = \overline{E}_{2)}(a_j, \widehat{W})$, $(j = 1, 2, .., 4\nu + 1)$,*

$$\sum_{j=1}^{4\nu+1} max\{\Delta_0(a_j, W), \Delta_0(a_j, \widehat{W})\} > 0,$$

*then $W(z) \equiv \widehat{W}(z)$.*

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