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Modified Adomian Decomposition Method for Thirteenth Order Boundary Value Problems

Abiodun A. OPANUGA*, Hilary I. OKAGBUE, Enahoro A. OWOLOKO, Olasunmbo O. AGBOOLA

Department of Mathematics, College of Science & Technology, Covenant University, Ota, Nigeria.

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Abstract

This work considers the numerical solution of thirteenth order boundary value problems using the modified Adomian decomposition method (MADM). Some examples are considered to illustrate the efficiency of the method. It was demonstrated that MADM converges more rapidly to the exact solution than the existing methods in literature and it reduces the computational involvement..

1. INTRODUCTION

Research has shown that boundary value problems arise in various fields of studies ranging from physical, biological and chemical processes. Proposing a numerical solution to various boundary value problems has posed a great challenge in the past years because most of the numerical methods are computationally intensive. Some proposed numerical methods which have been applied to solve some of these boundary value problems include: Spline method; Akram et al. [1], Lamnii et al. [2], Exp-Function method; Mohyud-Din et al. [3], Generalized Differential Quadrature rule (GDQR); Liua et al. [4], Homotopy Perturbation Method; Mohyud-Din et al. [5], Opanuga et al. [6], Ravi Kanth et al. [7], Variational Iteration Technique; Mohyud-Din et al., [8], Agboola et al. [9], Siddiqi et al. [10], Adeosun et al. [11], New Implicit Block Method; Omar and Kuboye [12], Finite-difference method; Khalsaraei and Jahandizi [13], Boutayeb et al. [14], Opanuga et al. [15], Differential Transform method; Iftikhar et al. [16], Opanuga et al. [17], Rung-kutta method; Mohamad-Jawad [18]. Others are Lucas polynomials-based method; Çetin et al. [19], kernel-based method; Uddin et al. [20], Taylor matrix method; Keşan [21] and Hybrid collocation method; Gürbüz and Sezer [22].

In 1980s, George Adomian (1923-1996) introduced a powerful method for solving linear and nonlinear differential equations. Since then, this method is known as the Adomian Decomposition Method (ADM) Adomian [23-24]. Later, Wazwaz [25] developed the modified form of the Adomian decomposition method. The modified technique provides a qualitative improvement over standard Adomian method, although it introduces a slight change in the formulation of Adomian recursive relation. The reason for this improvement rests on the fact that the technique accelerates the convergence of the solution and facilitates the formulation of Adomian polynomials.

In this work, the objective is the application of MADM for the solution of thirteenth order boundary value problems. Some examples are presented to illustrate the efficiency of the method and its rapid convergence to the exact solution.

2. ANALYSIS OF ADOMIAN DECOMPOSITION METHOD

Consider the generalized differential equation of the form

$$Ly + Ry + Ny = g \tag{1}$$

*Corresponding author, e-mail: abiodun.opanuga@covenantuniversity.edu.ng

L is the highest order derivative (L is invertible), R is a linear differential operator, Ny is the nonlinear term and g is the source term.

Applying L^{-1} on both sides of equation (1), we have

$$y = L^{-1}(Ry) - L^{-1}(Ny) + L^{-1}(g) \tag{2}$$

Equation (2) can then be written as

$$y = h - L^{-1}(Ry) - L^{-1}(Ny) \tag{3}$$

Note that h represents integral of the source term, ($L^{-1}(g)$) and boundary conditions.

Using the standard Adomian decomposition method, we identify the zeroth component as

$$y_0 = h \tag{4}$$

and the recurrent relation is

$$y_{n+1} = -L^{-1}(Ry_n) - L^{-1}(Ny_n), n \geq 0 \tag{5}$$

$$y_1 = -L^{-1}(Ry_0) - L^{-1}(Ny_0) \tag{6}$$

$$y_2 = -L^{-1}(Ry_1) - L^{-1}(Ny_1) \tag{7}$$

$$y_3 = -L^{-1}(Ry_2) - L^{-1}(Ny_2) \tag{8}$$

……………………………

Then the solution will be of the form

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{9}$$

The modification by Wazwaz [25] splits the function h into two parts say h_0 and h_1 ,

$$h = h_0 + h_1 \tag{10}$$

We will then have the zeroth component as

$$y_0 = h_0 \tag{11}$$

and the rest terms written as

$$y_1 = h_1 - L^{-1}(Ry_0) - L^{-1}(Ny_0) \tag{12}$$

$$y_2 = -L^{-1}(Ry_1) - L^{-1}(Ny_1) \tag{13}$$

The above modification reduces the size of computations involved in the method and thereby enhances the rapidity of its convergence. The nonlinear term Ny can be determined by an infinite series of Adomian polynomials.

$$Ny = \sum_{n=0}^{\infty} A_n \tag{14}$$

Where $An's$ are calculated by the relation

3. NUMERICAL EXAMPLES

3.1 Example 1: Consider the following non-linear thirteenth order two-point boundary value problems

$$u^{13}(t) = e^{-t}u^2(t), \qquad 0 \leq t \leq 1 \tag{16}$$

With the following boundary conditions

$$\begin{aligned} &u(0)=1, && u(1)=e \\ &u'(0)=1, && u'(1)=e \\ &u''(0)=1, && u''(1)=e \\ &u'''(0)=1, && u'''(1)=e \\ &u^{iv}(0)=1, && u^{iv}(1)=e \\ &u^{v}(0)=1, && u^{v}(1)=e \\ &u^{vi}(0)=1 \end{aligned} \tag{17}$$

The exact solution of the boundary value problems is

$$u(t)=e^{t} \tag{18}$$

To solve the boundary value problems by Adomian decomposition method, equation (16) is expressed in operator form as

$$Lu=e^{-t}u^{2} \tag{19}$$

Applying L^{-1} to both sides of equation (19) and imposing the boundary conditions (L^{-1} is a thirteenth-fold integral operator) to obtain

$$u(t)=1+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}+\frac{t^6}{6!}+\frac{At^7}{7!}+\frac{Bt^8}{8!}+\frac{Ct^9}{9!}+\frac{Dt^{10}}{10!}+ \frac{Et^{11}}{11!}+\frac{Ft^{12}}{12!}+L^{-1}(e^{-t}\cdot A_n) \tag{20}$$

and the constants

$$A=u^{vii}(0),\quad B=u^{viii}(0),\quad C=u^{ix}(0),\quad D=u^{x}(0),\quad E=u^{xi}(0),\quad F=u^{xii}(0) \tag{21}$$

which will be determined later.

By modified Adomian decomposition method, the zeroth component is

$$u_0(t)=1 \tag{22}$$

$$u_1(t)=\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}+\frac{t^5}{5!}-\frac{t^6}{6!}+\frac{At^7}{7!}+\frac{Bt^8}{8!}+\frac{Ct^9}{9!}+\frac{Dt^{10}}{10!}+ \frac{Et^{11}}{11!}+\frac{Ft^{12}}{12!}+L^{-1}(e^{-t}\cdot y^2{}_0) \tag{23}$$

Other components are determined by the recursive relations below

$$u_{n+1}(t)=L^{-1}(e^{-t}\cdot A_n) \tag{24}$$

$$u_2=L^{-1}(e^{-t}\cdot 2\cdot y_0y_1) \tag{25}$$

The series solution of equation (16) is given as

$$u(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+\frac{At^7}{5040}+\frac{Bt^8}{40320}+\frac{Ct^9}{362880}+\frac{Dt^{10}}{3628800}+ \frac{Et^{11}}{39916800}+\frac{Ft^{12}}{479001600}+1.605904383\times10^{-10}t^{13}+\cdots \tag{26}$$

To determine the constants, the boundary conditions (17) at $t=1$ is applied to obtain the following set of equations

$$
\begin{aligned}
&2.718055556+\frac{A}{5040}+\frac{B}{40320}+\frac{C}{362880}+\frac{D}{3628800}+\frac{E}{39916800}+\frac{F}{479001600}\\
&2.716666669+\frac{A}{720}+\frac{B}{5040}+\frac{C}{40320}+\frac{D}{362880}+\frac{E}{3628800}+\frac{F}{39916800}\\
&2.708333356+\frac{A}{120}+\frac{B}{720}+\frac{C}{5040}+\frac{D}{40320}+\frac{E}{362880}+\frac{F}{3628800}\\
&2.666666920+\frac{A}{24}+\frac{B}{120}+\frac{C}{720}+\frac{D}{5040}+\frac{E}{40320}+\frac{F}{362880}\\
&2.50000503+\frac{A}{6}+\frac{B}{24}+\frac{C}{120}+\frac{D}{720}+\frac{E}{5040}+\frac{F}{40320}\\
&2.000022299+\frac{A}{2}+\frac{B}{6}+\frac{C}{24}+\frac{D}{120}+\frac{E}{720}+\frac{F}{5040}
\end{aligned} \tag{27}
$$

Solving the system of equations, yields

$$
\begin{aligned}
&A=0.9996458699, \quad B=1.011394205, \quad C=0.8300462688,\\
&D=2.441964672, \quad E=-5.862746736, \quad F=15.51629883
\end{aligned}
$$

(28)

Substituting (28) in (26), we have the series solution in the form

$$
\begin{aligned}
u(t)=1+t+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+\frac{t^6}{720}+0.0001983424345t^7+0.00002508418167t^8+\\
2.287385000\times10^{-6}t^9+\cdots
\end{aligned} \tag{29}
$$

Table 1: Numerical Results for Example 1

t	u_{EXACT}	u_{ADM}	ABS ERROR
0	1.000000000	1.000000000	0
0.1	1.105170918	1.105170918	5.10703E-15
0.2	1.221402758	1.221402758	3.78586E-13
0.3	1.349858808	1.349858808	3.98614E-12
0.4	1.491824698	1.491824698	1.77294E-11
0.5	1.648721271	1.648721271	4.82907E-11
0.6	1.8221188	1.8221188	9.5365E-11
0.7	2.013752707	2.013752707	1.50626E-10
0.8	2.225540928	2.225540928	2.04268E-10
0.9	2.459603111	2.459603111	2.51792E-10
1	2.718281828	2.718281828	2.93647E-10

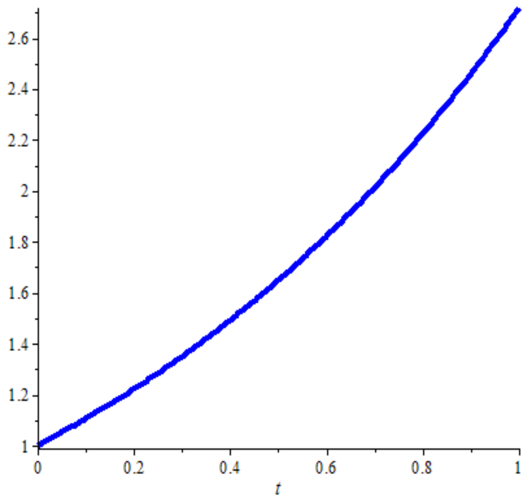

FIG1: PLOT OF NUMERICAL SOLUTION(ADM)

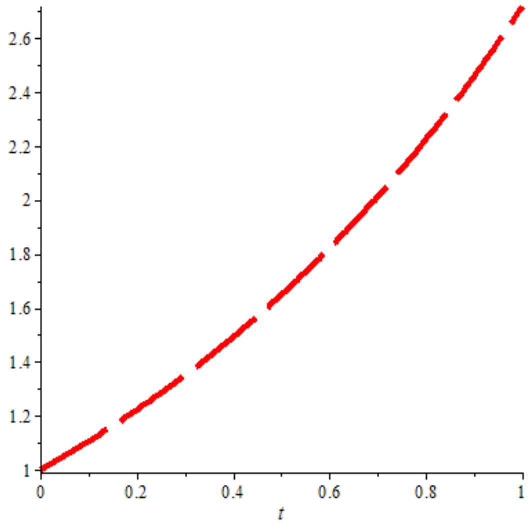

FIG 2: PLOT OF EXACT SOLUTION

3.2 Example 2: Also consider the linear thirteenth order two-point boundary value problems

$$u^{13}(t) = -9e^{t} + u(t) \tag{30}$$

subject to the boundary conditions

$$\begin{aligned}
&u(0) = 1, \\
&u'(0) = 0, && u(1) = 0, \\
&u''(0) = -1, && u'(1) = -e, \\
&u'''(0) = -2, && u''(1) = -2e, \\
&u^{iv}(0) = -3, && u'''(1) = -3e, \\
&u^{v}(0) = -4, && u^{iv}(1) = -4e, \\
&u^{vi}(0) = -5 && u^{v}(1) = -5e
\end{aligned} \tag{31}$$

The exact solution of the boundary value problem is

$$u(t) = (1-t)e^{t} \tag{32}$$

In operator form, equation (30) becomes

$$Lu = -9e^{t} + u \tag{33}$$

Multiplying both sides of equation (33) by L^{-1} and imposing the boundary conditions (L^{-1} is a thirteenth-fold integral operator), gives

$$u(t) = 1 - \frac{t^2}{2!} - \frac{2t^3}{3!} - \frac{3t^4}{4!} - \frac{4t^5}{5!} - \frac{5t^6}{6!} + \frac{At^7}{7!} + \frac{Bt^8}{8!} + \frac{Ct^9}{9!} + \frac{Dt^{10}}{10!} + \frac{Et^{11}}{11!} + \frac{Ft^{12}}{12!} - 9L^{-1}(e^{t}) + L^{-1}(y_n) \tag{34}$$

and the constants

$$A = u^{vii}(0), \quad B = u^{viii}(0), \quad C = u^{ix}(0), \quad D = u^{x}(0), \quad E = u^{xi}(0), \quad F = u^{xii}(0) \tag{35}$$

will be evaluated later.

By modified Adomian decomposition method, the zeroth component is

$$u_0(t) = 1 \tag{36}$$

$$u_1(t)=-\frac{t^2}{2!}-\frac{2t^3}{3!}-\frac{3t^4}{4!}-\frac{4t^5}{5!}-\frac{5t^6}{6!}+\frac{At^7}{7!}+\frac{Bt^8}{8!}+\frac{Ct^9}{9!}+\frac{Dt^{10}}{10!}+ \frac{Et^{11}}{11!}+\frac{Ft^{12}}{12!}-9L^{-1}(e^t)+L^{-1}(y_0) \quad (37)$$

The remaining components are determined recursively by

$$u_{n+1}(t)=-9L^{-1}(e^t)+L^{-1}(y_n) \quad (38)$$

$$u_2(t)=-9L^{-1}(e^t)+L^{-1}(y_1) \quad (39)$$

Collecting the terms together, the series solution is written as

$$u(t)=1-\frac{t^2}{2}-\frac{t^3}{3}-\frac{t^4}{8}-\frac{t^5}{30}-\frac{t^6}{144}+\frac{At^7}{5040}+\frac{Bt^8}{40320}+\frac{Ct^9}{362880}+\frac{Dt^{10}}{3628800}+ \frac{Et^{11}}{39916800}+\frac{Ft^{12}}{479001600}-\frac{t^{13}}{778377600}-\frac{t^{14}}{96864768800}+\cdots \quad (40)$$

The constants are evaluated by imposing the boundary conditions at $t=1$ to obtain the following system of equations

$$\begin{aligned}
&0.001388887561+\frac{A}{5040}+\frac{B}{40320}+\frac{C}{362880}+\frac{D}{3628800}+\frac{E}{39916800}+\frac{F}{479001600}\\
&-\frac{2470051600651}{912019046400}+\frac{A}{720}+\frac{B}{5040}+\frac{C}{40320}+\frac{D}{362880}+\frac{E}{3628800}+\frac{F}{39916800}\\
&-\frac{12495555585217}{2324754432000}+\frac{A}{120}+\frac{B}{720}+\frac{C}{5040}+\frac{E}{362880}+\frac{F}{3628800}\\
&-\frac{284540345009}{36324288000}+\frac{A}{24}+\frac{B}{120}+\frac{C}{720}+\frac{D}{5040}+\frac{E}{40320}+\frac{F}{362880}\\
&-\frac{55213061731}{5811886080}+\frac{A}{6}+\frac{B}{24}+\frac{C}{120}+\frac{D}{720}+\frac{E}{5040}+\frac{F}{40320}\\
&-\frac{3113588563}{345945600}+\frac{A}{2}+\frac{B}{6}+\frac{C}{24}+\frac{D}{120}+\frac{E}{720}+\frac{F}{5040}
\end{aligned} \quad (41)$$

Solving the system of equations above yields the following

$$A=-5.999900284,\quad B=-7.004010833,\quad C=-7.928474770,\quad D=-9.708044645, \quad E=-6.080274338,\quad F=-21.16601529 \quad (42)$$

Using equation (42) in equation (40) gives the solution as

$$u(t)=1-\frac{t^2}{2}-\frac{t^3}{3}-\frac{t^4}{8}-\frac{t^5}{30}-\frac{t^6}{144}-0.001190456406t^7-0.000173710586t^8- 0.00002184875102t^9-20675276853\times10^{-6}t^{10}+\cdots \quad (43)$$

Table 2: Numerical Results for Example 2

t	u_{EXACT}	u_{ADM}	ABS ERROR
0	1.000000000	1.000000000	0
0.1	0.994653826	0.994653826	1.11022E-15
0.2	0.977122207	0.977122207	8.11573E-14
0.3	0.944901165	0.944901165	6.90559E-13
0.4	0.895094819	0.895094819	2.20013E-12
0.5	0.824360635	0.824360635	3.26417E-12

0.6	0.72884752	0.72884752	5.29576E-13
0.7	0.604125812	0.604125812	8.45168E-12
0.8	0.445108186	0.445108186	2.16682E-11
0.9	0.245960311	0.245960311	3.31795E-11
1	0	-3.78511E-11	3.78511E-11

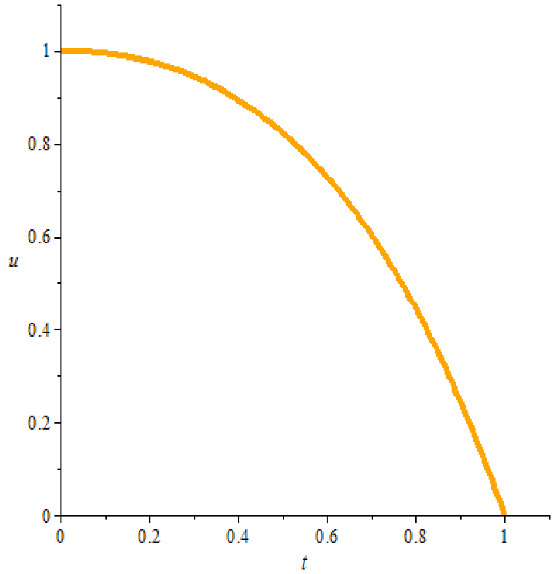

FIG 3:PLOT OF NUMERICAL SOLUTION(ADM)

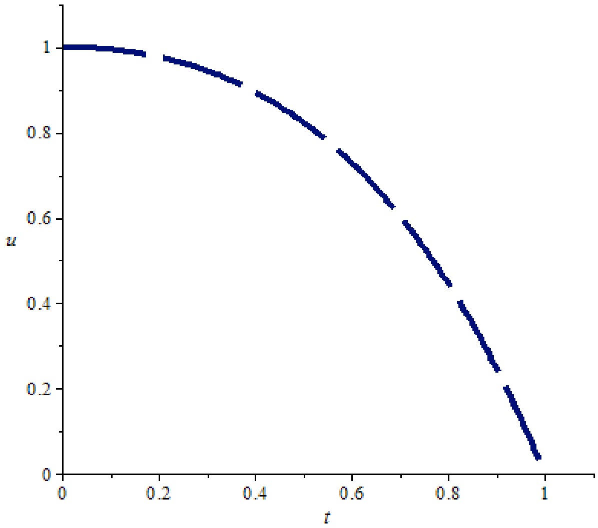

FIG 4: PLOT OF EXACT SOLUTION

4. CONCLUSION

In the present work, some examples of thirteenth order boundary value problems via modified Adomian decomposition method been solved. MADM was applied without any form of transformation, linearization, perturbation or discretization. The approximate solutions obtained are compared with those obtained using the variational iteration and differential transform techniques, it was shown that MADM reduces the computational involment and converges rapidly to the exact even with few terms.

CONFLICTS OF INTEREST: The authors declare no conflict of interest

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