Euler-Riesz Difference Sequence Spaces

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ABSTRACT. Başar and Braha [9], introduced the sequence spaces $\ell_\infty$, $\ell_c$ and $\ell_0$ of Euler- Cesáro bounded, convergent and null difference sequences and studied their some properties. The main purpose of this study is to introduce the sequence spaces $[\ell_\infty]_{e.r}$, $[\ell_c]_{e.r}$ and $[\ell_0]_{e.r}$ of Euler- Riesz bounded, convergent and null difference sequences by using the composition of the Euler mean $E_1$ and Riesz mean $R_q$ with backward difference operator $\Delta$. Furthermore, the inclusions $\ell_\infty \subset [\ell_\infty]_{e.r}$, $\ell_c \subset [\ell_c]_{e.r}$ and $\ell_0 \subset [\ell_0]_{e.r}$ strictly hold, the basis of the sequence spaces $[\ell_0]_{e.r}$ and $[\ell_c]_{e.r}$ is constructed and alpha-, beta- and gamma-duals of these spaces are determined. Finally, the classes of matrix transformations from the Euler- Riesz difference sequence spaces to the spaces $\ell_\infty$, $\ell_c$ and $\ell_0$ are characterized.

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1. Preliminaries, Background and Notation

In this section, we give some basic definitions and notations for which we refer to [7,12,17,23].

By a sequence space, we understand a linear subspace of the space $w = \mathbb{C}^\infty$ of all complex sequences which contains $\phi$, the set of all finitely non-zero sequences, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We shall write $\ell_\infty$, $\ell_c$ and $\ell_0$ for the spaces of all bounded, convergent and null sequences, respectively. Also by $bs, cs, \ell_1$ and $\ell_p$, we denote the spaces of all bounded, convergent, absolutely and $p$–absolutely convergent series, respectively, where $1 < p < \infty$.

We shall assume throughout unless stated otherwise that $p, q > 1$ with $p^{-1} + q^{-1} = 1$ and $0 < r < 1$, and use the convention that any term with negative subscript is equal to naught.

Let $\lambda$, $\mu$ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then, we say that $A$ defines a matrix mapping from $\lambda$ into $\mu$, and we denote it by writing $A : \lambda \rightarrow \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the $A$–transform of $x$, is in $\mu$; where

$$
(Ax)_n = \sum_k a_{nk}x_k \quad (n \in \mathbb{N}).
$$

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By \((\lambda, \mu)\), we denote the class of all matrices \(A\) such that \(A : \lambda \to \mu\). Thus, \(A \in (\lambda, \mu)\) if and only if the series on the right hand side of (1.1) converges for each \(n \in \mathbb{N}\) and every \(x \in \lambda\), and we have \(Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu\) for all \(x \in \lambda\). A sequence \(x\) is said to be \(A\)–summable to \(\alpha\) if \(Ax\) converges to \(\alpha\) which is called the \(A\)–limit of \(x\).

Let \(X\) be a sequence space and \(A\) be an infinite matrix. The sequence space
\[
X_A = \{x = (x_k) \in w : Ax \in X\}
\]
is called the domain of \(A\) in \(X\) which is a sequence space.

A sequence space \(\lambda\) with a linear topology is called a \(K\)–space provided each of the maps \(p_i : \lambda \to \mathbb{C}\) defined by \(p_i(x) = x_i\) is continuous for all \(i \in \mathbb{N}\). A \(K\)–space is called an \(FK\)–space provided \(\lambda\) is a complete linear metric space. An \(FK\)–space whose topology is normable is called a \(BK\)–space. If a normed sequence space \(\lambda\) contains a sequence \((b_n)\) with the property that for every \(x \in \lambda\) there is a unique sequence of scalars \((\alpha_n)\) such that
\[
\lim_{n \to \infty} \|x - (\alpha_0b_0 + \alpha_1b_1 + \cdots + \alpha_nb_n)\| = 0
\]
then \((b_n)\) is called a Schauder basis (or briefly basis) for \(\lambda\). The series \(\sum \alpha_kb_k\) which has the sum \(x\) is then called the expansion of \(x\) with respect to \((b_n)\), and is written as \(x = \sum \alpha_kb_k\).

Given a \(BK\)–space \(\lambda \supset \phi\), we denote the \(n\)th section of a sequence \(x = (x_k) \in \lambda\) by \(x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}\), and we say that \(x\) has the property
\[
AK \text{ if } \lim_{n \to \infty} \|x - x^{[n]}\| = 0 \text{ (abschnittskonvergenz)},
\]
\[
AB \text{ if } \sup_{n \in \mathbb{N}} \|x^{[n]}\| < \infty \text{ (abschnittsbeschränktheit)},
\]
\[
AD \text{ if } x \in \phi \text{ (closure of } \phi \subset \lambda) \text{ (abschnittsdichte)},
\]
\[
KB \text{ if the set } \{x_k e^{(k)}\} \text{ is bounded in } \lambda \text{ (koordinatenweise beschränkt)},
\]
where \(e^{(k)}\) is a sequence whose only non-zero term is a 1 in \(k\)th place for each \(k \in \mathbb{N}\). If one of these properties holds for every \(x \in \lambda\) then we say that the space \(\lambda\) has that property \([16, 23]\). It is trivial that \(AK\) implies \(AD\) and \(AK\) iff \(AB + AD\). For example, \(c_0\) and \(\ell_p\) are \(AK\)–spaces and, \(c\) and \(\ell_\infty\) are not \(AD\)–spaces.

A matrix \(A = (a_{nk})\) is called a triangle if \(a_{nk} = 0\) for \(k > n\) and \(a_{nn} \neq 0\) for all \(n \in \mathbb{N}\). It is trivial that \(A(Bx) = (AB)x\) holds for the triangle matrices \(A, B\) and a sequence \(x\). Further, a triangle matrix \(U\) uniquely has an inverse \(U^{-1} = V\) which is also a triangle matrix. Then, \(x = U(Vx) = V(Ux)\) holds for all \(x \in w\).

Let us give the definition of some triangle limitation matrices which are needed in the text. \(\Delta\) denotes the backward difference matrix \(\Delta = (\Delta_{nk})\) and \(\Delta' = (\Delta'_{nk})\) denotes the transpose of the matrix \(\Delta\), the forward difference matrix, which are defined by
\[
\Delta_{nk} = \begin{cases} (-1)^{n-k}, & n - 1 \leq k \leq n, \\ 0, & 0 \leq k < n - 1 \text{ or } k > n, \end{cases}
\]
\[
\Delta'_{nk} = \begin{cases} (-1)^{n-k}, & n \leq k \leq n + 1, \\ 0, & 0 \leq k < n \text{ or } k > n + 1, \end{cases}
\]
for all \(k, n \in \mathbb{N}\); respectively.

Then, let us define the Euler mean \(E_1 = (e_{nk})\) of order one and Riesz mean \(R_q = (r_{nk})\) with respect to the sequence \(q = (q_k)\)
\[
e_{nk} = \begin{cases} \binom{n}{k}, & 0 \leq k \leq n, \\ \frac{0}{k}, & k > n, \end{cases}
\]
\[
r_{nk} = \begin{cases} \frac{q_k}{q_n}, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}
\]
for all \(k, n \in \mathbb{N}\) and where \((q_k)\) is a sequence of positive numbers and \(Q_n = \sum_{k=0}^{n} q_k\) for all \(n \in \mathbb{N}\). Their inverses \(E_1^{-1} = (g_{nk})\) and \(R_q^{-1} = (h_{nk})\) are given by
\[
g_{nk} = \begin{cases} \binom{n}{k} (-1)^{n-k} q_k, & 0 \leq k \leq n, \\ 0, & k > n, \end{cases}
\]
\[
h_{nk} = \begin{cases} (-1)^{n-k} \frac{q_k}{q_n}, & n - 1 \leq k \leq n, \\ 0, & \text{otherwise,} \end{cases}
\]
for all \(k, n \in \mathbb{N}\).
We define the matrix $B = (\hat{b}_{nk})$ by the composition of the matrices $E_1, R_q$ and $\Delta$ as

$$\hat{b}_{nk} = \begin{cases} \binom{(n_q)k}{q} & 0 \leq k \leq n, \\ 0 & k > n, \end{cases} \quad (1.2)$$

for all $k, n \in \mathbb{N}$.

In the literature, the notion of difference sequence spaces was introduced by Kızmaz [18], who defined the sequence spaces

$$X(\Delta) = \{x = (x_k) \in w : \Delta^j x = (x_k - x_{k+1}) \in X\}$$

for $X \in \{\ell_\infty, c, c_0\}$. The difference space $bv_p$, consisting of all sequences $x = (x_k)$ such that $\Delta x = (x_k - x_{k-1})$ is in the sequence space $\ell_p$, was studied in the case $0 < p < 1$ by Altay and Başar [5] and in the case $1 \leq p \leq \infty$ by Başar and Altay [6], and Çolak et al. [13]. Kirişçi and Başar [19] have introduced and studied the generalized difference sequence space

$$\hat{X} = \{x = (x_k) \in w : B(r, s)x \in X\},$$

where $X$ denotes any of the spaces $\ell_\infty, c, c_0$ and $\ell_p$ with $1 \leq p < \infty$, and $B(r, s)x = (sx_{k-1} + rx_k)$ with $r, s \in \mathbb{R} \setminus \{0\}$. Following Kirişçi and Başar [19], Sönmez [21] has examined the sequence space $X(B)$ as the set of all sequences whose $B(r, s, t)$-trasforms are in the space $X \in \{\ell_\infty, c, c_0, \ell_p\}$, where $B(r, s, t)$ denotes the triple band matrix $B(r, s, t) = \{b_{nk}\{r, s, t\}^{\infty}\}$ defined by

$$b_{nk}\{r, s, t\} = \begin{cases} r & n = k \\ s & n = k + 1 \\ t & n = k + 2 \\ 0 & \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s, t \in \mathbb{R} \setminus \{0\}$. Quite recently, Başar has studied the spaces $\ell_p$ of $p$-absolutely $\hat{B}$-summable sequences, in [8]. In [11], Choudhary and Mishra have defined the sequence space $\ell(\hat{p})$ which consists of all sequences whose $S$-transforms are in the space $\ell(\hat{p})$. Also, many authors have constructed new sequence spaces by using matrix domain of infinite matrices. For instance, $e_0^r$ and $e^s_\infty$ in [1], $e_0^r$ and $e_\infty^r$ in [3], $e_0^r(u, p)$, $e^r_\infty(u, p)$ in [14], $e_0^r(\Delta^{(m)})$, $e^r_\infty(\Delta^{(m)})$ and $e_0^r(\Delta^{(m)})$ in [13], $c_0(\Delta^{(m)})$, $c_r(\Delta^{(m)})$ and $\ell_\infty(\Delta^{(m)})$ in [15], $r_0^r(p)$, $r^s_\infty(r)$ and $r_\infty^s(p)$ in [2], $r^q(p, \Delta)$ in [10]. Finally, the new technique for deducing certain topological properties, for example $AB-$, $KB-$, $AD-$properties, solidity and monotonicity etc., and determining the $\beta-$ and $\alpha-$duals of the domain of a triangle matrix in a sequence space is given by Altay and Başar [4].

Then, as a natural continuation of Başar [8], Başar and Braha [9] introduce the spaces $\hat{\ell}_\infty, \hat{c}$ and $\hat{c}_0$ of Euler-Cesáro bounded, convergent and null difference sequences by using the composition of the Euler mean $E_1$ and Cesáro mean $C_1$ of order one with backward difference operator $\Delta$.

In the present paper, we introduce the $[\ell_\infty]_{e, r}, [c]_{e, r}$ and $[c_0]_{e, r}$ of Euler-Riesz bounded, convergent and null difference sequences by using the composition of the Euler mean $E_1$ and Riesz mean $R_q$ with respect to the sequence $q = (q_k)$ with backward difference operator $\Delta$ and prove that the inclusions $\ell_\infty \subset [\ell_\infty]_{e, r}, c \subset [c]_{e, r}$ and $c_0 \subset [c_0]_{e, r}$ strictly hold. We show that the spaces $[c_0]_{e, r}$ and $[c]_{e, r}$ turn out to be the separable BK spaces such that $[c]_{e, r}$ does not possess any of the following: AK property and monotonicity. Furthermore, we investigate some properties and compute alpha-, beta- and gamma-duals of these spaces. Afterwards, we characterize some matrix classes related to Euler-Riesz sequence spaces.
2. The Euler-Riesz Sequence Spaces

In this section, we give some new sequence spaces and investigate their certain properties.

\[ [c_0]_{e.r} = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(n)q_k}{2^nQ_n} x_k = 0 \right\} \]
\[ [c]_{e.r} = \left\{ x = (x_k) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(n)q_k}{2^nQ_n} x_k \right\} \]
\[ [\ell_\infty]_{e.r} = \left\{ x = (x_k) \in w : \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} \frac{(n)q_k}{2^nQ_n} x_k \right| < \infty \right\} \]

With the notation (1.2), we may redefine the spaces \([c_0]_{e.r}, [c]_{e.r}\) and \([\ell_\infty]_{e.r}\) as follows:

\[ [c_0]_{e.r} = (c_0)_B, \quad [c]_{e.r} = c_B \quad \text{and} \quad [\ell_\infty]_{e.r} = (\ell_\infty)_B. \]

In the case \((q_k) = e = (1, 1, \ldots)\); the sequence spaces \([c_0]_{e.r}, [c]_{e.r}\) and \([\ell_\infty]_{e.r}\) are, respectively, reduced to the sequence spaces \(\tilde{c}_0, \tilde{c}\) and \(\tilde{\ell}_\infty\) which are introduced by Başar and Braha [9]. Define the sequence \(y = (y_k)\), which will be frequently used, as the \(\tilde{B}\)–transform of a sequence \(x = (x_k)\), i.e.,

\[ y_k = \sum_{j=0}^{k} \frac{(k)q_j}{2^kQ_k} x_j, \quad k \in \mathbb{N}. \quad (2.1) \]

Throughout the text, we suppose that the sequences \(x = (x_k)\) and \(y = (y_k)\) are connected with the relation (2.1). One can obtain by a straightforward calculation from (2.1) that

\[ x_k = \frac{1}{q_k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j Q_j y_j, \quad k \in \mathbb{N}. \quad (2.2) \]

**Theorem 2.1.** The sets \([c_0]_{e.r}, [c]_{e.r}\) and \([\ell_\infty]_{e.r}\) are linear spaces with coordinatewise addition and scalar multiplication that are BK–spaces with norm \(\|x\|_{[c_0]_{e.r}} = \|x\|_{[c]_{e.r}} = \|x\|_{[\ell_\infty]_{e.r}} = \|\tilde{B}x\|_\infty\)

**Proof.** The proof of the first part of the theorem is a routine verification, and so we omit it. Furthermore, since (2.1) holds, \(c_0, c\) and \(\ell_\infty\) are BK–spaces with respect to their natural norm, and the matrix \(\tilde{B}\) is a triangle, Theorem 4.3.2 of Wilansky [23] implies that the spaces \([c_0]_{e.r}, [c]_{e.r}\) and \([\ell_\infty]_{e.r}\) are BK–spaces. \(\Box\)

Therefore, one can easily check that the absolute property does not hold on the spaces \([c_0]_{e.r}, [c]_{e.r}\) and \([\ell_\infty]_{e.r}\), because \(\|x\|_{[c_0]_{e.r}} \neq \|x\|_{[c]_{e.r}}\) and \(\|x\|_{[\ell_\infty]_{e.r}} \neq \|x\|_{[\ell_\infty]_{e.r}}\) for at least one sequence in the spaces \([c_0]_{e.r}, [c]_{e.r}\) and \([\ell_\infty]_{e.r}\), where \(|x| = (|x_k|)\). This says that \([c_0]_{e.r}, [c]_{e.r}\) and \([\ell_\infty]_{e.r}\) are the sequence spaces of nonabsolute type.

**Theorem 2.2.** \([c_0]_{e.r}, [c]_{e.r}\) and \([\ell_\infty]_{e.r}\) are linearly isomorphic to the spaces \(c_0, c\) and \(\ell_\infty\), respectively, i.e., \([c_0]_{e.r} \cong c_0, [c]_{e.r} \cong c\) and \([\ell_\infty]_{e.r} \cong \ell_\infty\).

**Proof.** To prove this theorem, we should show the existence of a linear bijection between the spaces \([c_0]_{e.r}\) and \(c_0\). Consider the transformation \(S\) defined, with the notation of (2.1), from \([c_0]_{e.r}\) to \(c_0\) by \(y = Sx = \tilde{B}x\). The linearity of \(S\) is clear. Further, it is obvious that \(x = \theta\) whenever \(Sx = \theta\) and hence \(S\) is injective, where \(\theta = (0, 0, 0, \ldots)\).

Let \(y \in c_0\) and define the sequence \(x = \{x_n\}\) by

\[ x_n = \frac{1}{q_n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k Q_k y_k, \quad \text{for all} \ n \in \mathbb{N}. \]
Then, we have
\[
\lim_{n \to \infty} (\tilde{B} x)_n = \lim_{n \to \infty} \left[ \sum_{k=0}^{n} \left( \frac{n}{k} \right) Q_n x_k \right] = \lim_{n \to \infty} \left[ \sum_{k=0}^{n} \frac{n}{k} q_k \frac{1}{k} \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) (-1)^{k-j} 2^j y_j \right] = \lim_{n \to \infty} y_n = 0
\]
which says us that \( x \in [c_0]_{e.r} \). Additionally, we observe that
\[
\|x\|_{[c_0]_{e.r}} = \sup_{n \in \mathbb{N}} \left\| \sum_{k=0}^{n} \frac{n}{k} q_k \frac{1}{k} \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) (-1)^{k-j} 2^j y_j \right\| = \sup_{n \in \mathbb{N}} |y_n| = \|y\|_\infty < \infty.
\]
Consequently, \( S \) is surjective and is norm preserving. Hence, \( S \) is a linear bijection which therefore says us that the spaces \([c_0]_{e,r} \) and \( c_0 \) are linearly isomorphic, as desired.

It is clear that if the spaces \([c_0]_{e.r} \) and \( c_0 \) are replaced by the spaces \([c]_{e,r} \) and \( c \) or \([\ell_\infty]_{e.r} \) and \( \ell_\infty \) respectively, then we obtain the fact that \([c]_{e,r} \cong c \) and \([\ell_\infty]_{e.r} \cong \ell_\infty \). This completes the proof. \( \square \)

We wish to exhibit some inclusion relations concerning with the spaces \([c_0]_{e,r} \), \([c]_{e,r} \) and \([\ell_\infty]_{e,r} \), in the present section. Here and after, by \( \lambda \) we denote any of the sets \([c_0]_{e,r} \), \([c]_{e,r} \) and \([\ell_\infty]_{e,r} \) and \( \mu \) denotes any of the spaces \( c_0, c \) or \( \ell_\infty \).

**Theorem 2.3.** The inclusions \( \mu \subset \lambda \) hold.

**Proof.** Let \( x = (x_k) \in \mu \). Then, since it is immediate that
\[
\|x\|_\lambda = \|\tilde{B} x\|_\infty = \sup_{n \in \mathbb{N}} \left\| \sum_{k=0}^{n} \frac{n}{k} q_k \frac{1}{k} \sum_{j=0}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) (-1)^{k-j} 2^j y_j \right\| \leq \|x\|_\infty \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \frac{n}{k} = \|x\|_\infty.
\]
The inclusion \( \mu \subset \lambda \) holds. \( \square \)

**Theorem 2.4.** The space \([c_0]_{e,r} \) has \( AK \)-property.

**Proof.** Let \( x = (x_k) \in [c_0]_{e.r} \) and \( x^{[n]} = \{x_1, x_2, \ldots, x_n, 0, 0, \ldots\} \). Hence,
\[
x - x^{[n]} = \{0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots\} \Rightarrow \|x - x^{[n]}\|_{[c_0]_{e.r}} = \|(0, 0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots)\|
\]
and since \( x \in [c_0]_{e,r} \),
\[
\|x - x^{[n]}\|_{[c_0]_{e.r}} = \sup_{k \geq n+1} \left\| \sum_{j=0}^{k} \frac{k}{j} q_j x_j \right\| = \sup_{k \geq n+1} \left\{ \sum_{j=0}^{k} \frac{k}{j} q_j x_j \right\}
\]
Then the space \([c_0]_{e,r} \) has \( AK \)-property. \( \square \)

Since the isomorphism \( S \), defined in Theorem 2.1, is surjective, the inverse image of the basis of the spaces \( c_0 \) and \( c \) are the basis of the new spaces \([c]_{e,r} \) and \([c_0]_{e.r} \), respectively. Since the space \( \ell_\infty \) has no Schauder basis, \([\ell_\infty]_{e,r} \) has no Schauder basis. Therefore, we have the following theorem without proof.

**Theorem 2.5.** Define the sequence \( b^{(k)} = \{b_n^{(k)}\}_{n \in \mathbb{N}} \) of elements of the space \([c_0]_{e,r} \) for every fixed \( k \in \mathbb{N} \) by
\[
b_n^{(k)} = \begin{cases} \frac{(-1)^{k-n} 2^n Q_k}{q_n}, & 0 \leq k < n, \\ 0, & k \geq n. \end{cases}
\]
Let \( \lambda_k = (\tilde{B}x)_k \) for all \( k \in \mathbb{N} \). Then the following assertions are true:

(i): The sequence \( \{b^{(k)}\}_{k \in \mathbb{N}} \) is a basis for the space \([c_{0}]_{e,r}\) and any \( x \in [c_{0}]_{e,r} \) has a unique representation of the form

\[
x = \sum_{k} \lambda_k b^{(k)}.
\]

(ii): The set \( \{e, b^{(k)}\}_{k \in \mathbb{N}} \) is a basis for the space \([c]_{e,r}\) and any \( x \in [c]_{e,r} \) has a unique representation of the form

\[
x = le + \sum_{k} [\lambda_k - l] b^{(k)},
\]

where \( l = \lim_{k \to \infty} (\tilde{B}x)_k \).

**Remark 2.6.** It is well known that every Banach space \( X \) with a Schauder basis is separable.

From Theorem 2.5 and Remark 2.6, we can give the following corollary:

**Corollary 2.7.** The spaces \([c_{0}]_{e,r}\) and \([c]_{e,r}\) are separable.

### 3. Duals of The New Sequence Spaces

In this section, we state and prove the theorems determining the \( \alpha-, \beta- \) and \( \gamma- \) duals of the sequence spaces \([c_{0}]_{e,r}, [c]_{e,r}\) and \([\ell_{\infty}]_{e,r}\) of non-absolute type.

The set \( S(\lambda, \mu) \) defined by

\[
S(\lambda, \mu) = \{ z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda \}
\]

is called the multiplier space of the sequence spaces \( \lambda \) and \( \mu \). One can easily observe for a sequence space \( \nu \) with \( \lambda \supset \nu \supset \mu \) that the inclusions

\[
S(\lambda, \mu) \subset S(\nu, \mu) \text{ and } S(\lambda, \mu) \subset S(\lambda, \nu)
\]

hold. With the notation of (3.1), the alpha-, beta- and gamma-duals of a sequence space \( \lambda \), which are respectively denoted by \( \lambda^{\alpha}, \lambda^{\beta} \) and \( \lambda^{\gamma} \), are defined by

\[
\lambda^{\alpha} = S(\lambda, \ell_1), \lambda^{\beta} = S(\lambda, cs) \text{ and } \lambda^{\gamma} = S(\lambda, bs).
\]

For giving the alpha-, beta- and gamma-duals of the spaces \([c_{0}]_{e,r}, [c]_{e,r}\) and \([\ell_{\infty}]_{e,r}\) of non-absolute type, we need the following Lemma:

**Lemma 3.1.** [22]

(i): \( A \in (c_0 : \ell_1) = (c : \ell_1) = (\ell_{\infty} : \ell_1) \) if and only if

\[
\sup_{K \in \mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{k \in K} a_{nk} \right| < \infty.
\]

(ii): \( A \in (c_0 : \ell_{\infty}) = (c : \ell_{\infty}) = (\ell_{\infty} : \ell_{\infty}) \) if and only if

\[
\sup_{n \in \mathbb{N}} \sum_{k=0}^{\infty} |a_{nk}| < \infty.
\]

(iii): \( A \in (c : c) \) if and only if (3.2) holds, and

\[
\exists (\alpha_k) \in w \text{ such that } \lim_{n \to \infty} a_{nk} = \alpha_k \text{ for all } k \in \mathbb{N},
\]

\[
\exists \alpha \in \mathbb{C} \text{ such that } \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = \alpha.
\]

Now, we may give the theorems determining the \( \alpha-, \beta- \) and \( \gamma- \) duals of the Euler-Riesz sequence spaces \([c_{0}]_{e,r}, [c]_{e,r}\) and \([\ell_{\infty}]_{e,r}\).
Theorem 3.2. Define the set $a_q$ as follows:

$$a_q = \left\{ a = (a_k) \in w : \sup_{K \subset F} \sum_{n=0}^{\infty} \left( \sum_{k \in K} \binom{n}{k} (-1)^{n-k} 2^k a_n \frac{Q_k}{q_n} \right) < \infty \right\}.$$ 

Then, $\{[c_0]_{e,r} \}^\alpha = \{[c]_{e,r} \}^\alpha = \{[\ell_\infty]_{e,r} \}^\alpha = a_q$.

Proof. We give the proof for the space $[c_0]_{e,r}$. We chose the sequence $a = (a_k) \in w$. We can easily derive with (2.2) that

$$a_n x_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k a_n \frac{Q_k}{q_n} k y_k = (By)_n, \quad (n \in \mathbb{N});$$

where $B = (b_{nk})$ is defined by the formula

$$b_{nk} = \left\{ \binom{n}{k} (-1)^{n-k} 2^k a_n \frac{Q_k}{q_n} , \quad (0 \leq k \leq n), \quad (0 \leq n, k \in \mathbb{N}). \right\}$$

It follows from (3.4) that $ax = (a_n x_n) \in \ell_1$ whenever $x \in [c_0]_{e,r}$ if and only if $By \in \ell_1$ whenever $y \in c_0$. This gives the result that $\{[c_0]_{e,r} \}^\alpha = a_q$. □

Theorem 3.3. The matrix $D(r) = (d_{nk})$ is defined by

$$d_{nk} = \left\{ \sum_{j=k}^{n} \binom{n}{j} (-1)^{j-k} 2^j a_j \frac{Q_j}{q_j} , \quad (0 \leq k \leq n), \quad (k > n). \right\}$$

for all $k, n \in \mathbb{N}$. Then, $\{[c_0]_{e,r} \}^\beta = b_1 \cap b_2$ and $\{[c_{e,r}] \}^\beta = b_1 \cap b_2 \cap b_3$ where

$$b_1 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |d_{nk}| < \infty \right\},$$

$$b_2 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} d_{nk} = \alpha_k \right\},$$

$$b_3 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} d_{nk} \text{ exists} \right\}.$$

Proof. We give the proof for the space $[c_0]_{e,r}$. Consider the equation

$$\sum_{k=0}^{n} a_{k} x_k = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j a_j \frac{Q_j}{q_j} y_k = (Dy)_n,$$

where $D = (d_{nk})$ defined by (3.5).

Thus, we decude by (3.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in [c_0]_{e,r}$ if and only if $Dy \in c$ whenever $y = (y_k) \in c_0$. Therefore, we derive from (3.2) and (3.3) that

$$\lim_{n \to \infty} d_{nk} \text{ exists for each } k \in \mathbb{N},$$

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |d_{nk}| < \infty$$

which shows that $\{[c_0]_{e,r} \}^\beta = b_1 \cap b_2$. □

Theorem 3.4. $\{[c_0]_{e,r} \}^\gamma = \{[c_{e,r}] \}^\gamma = b_1$.

Proof. This is obtained in the similar way used in the proof of Theorem 3.3. □
4. Matrix Transformations Related to the New Sequence Spaces

In this section, we characterize the matrix transformations from the spaces \([c_0]_{e,r}, [c]_{e,r}\) and \(\ell_\infty\) into any given sequence space \(\mu\) and from the sequence space \(\mu\) into the spaces \([c_0]_{e,r}, [c]_{e,r}\) and \(\ell_\infty\).

Since \([c_0]_{e,r} \cong c_0\) (or \([c]_{e,r} \cong c\) and \(\ell_\infty \cong \ell_\infty\)), we can say: The equivalence “\(x \in [c_0]_{e,r}\) (or \(x \in [c]_{e,r}\) and \(x \in \ell_\infty\))”, if and only if \(y \in c_0\) (or \(y \in c\) and \(y \in \ell_\infty\)” holds.

In what follows, for brevity, we write,

\[
\tilde{a}_{nk} := \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} 2^k Q_k a_{nk}
\]

for all \(k, n \in \mathbb{N}\).

Theorem 4.1. Suppose that the entries of the infinite matrices \(A = (a_{nk})\) and \(E = (e_{nk})\) are connected with the relation

\[
e_{nk} := \tilde{a}_{nk}
\]

for all \(k, n \in \mathbb{N}\) and \(\mu\) be any given sequence space. Then,

(i): \(A \in ([c_0]_{e,r} : \mu)\) if and only if \(\{a_{nk}\}_{k \in \mathbb{N}} \in [c_0]_{e,r}^{\beta}\) for all \(n \in \mathbb{N}\) and \(E \in (c_0 : \mu)\).

(ii): \(A \in ([c]_{e,r} : \mu)\) if and only if \(\{a_{nk}\}_{k \in \mathbb{N}} \in \{(c)_{e,r}\}^\beta\) for all \(n \in \mathbb{N}\) and \(E \in (c : \mu)\).

(iii): \(A \in (\ell_\infty : \mu)\) if and only if \(\{a_{nk}\}_{k \in \mathbb{N}} \in \{(\ell_\infty)_{e,r}\}^\beta\) for all \(n \in \mathbb{N}\) and \(E \in (\ell_\infty : \mu)\).

Proof. We prove only Part (i). Let \(\mu\) be any given sequence space. Suppose that (4.1) holds between \(A = (a_{nk})\) and \(E = (e_{nk})\), and take into account that the spaces \([c_0]_{e,r}\) and \(c_0\) are linearly isomorphic.

Let \(A \in ([c_0]_{e,r} : \mu)\) and take any \(y = (y_k) \in c_0\). Then \(EB\) exists and \(\{a_{nk}\}_{k \in \mathbb{N}} \in b_1 \cap b_2\) which yields that \(\{e_{nk}\}_{k \in \mathbb{N}} \in c_0\) for each \(n \in \mathbb{N}\). Hence, \(Ey\) exists and thus

\[
\sum_k e_{nk} y_k = \sum_k a_{nk} x_k
\]

for all \(n \in \mathbb{N}\).

We have that \(Ey = Ax\) which leads us to the consequence \(E \in (c_0 : \mu)\).

Conversely, let \(\{a_{nk}\}_{k \in \mathbb{N}} \in \{(c)_{e,r}\}^\beta\) for each \(n \in \mathbb{N}\) and \(E \in (c_0 : \mu)\), and take any \(x = (x_k) \in [c_0]_{e,r}\). Then, \(Ax\) exists. Therefore, we obtain from the equality

\[
\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^j Q_j a_{kj} y_k
\]

for all \(n \in \mathbb{N}\), that \(Ey = Ax\) and this shows that \(A \in ([c_0]_{e,r} : \mu)\). This completes the proof of Part (i). \(\square\)

Theorem 4.2. Suppose that the elements of the infinite matrices \(A = (a_{nk})\) and \(B = (b_{nk})\) are connected with the relation

\[
b_{nk} := \sum_{j=0}^{k} \binom{k}{j} Q_j a_{jk} 2^{k} Q_k a_{nk} \quad \text{for all } k, n \in \mathbb{N}.
\]

Let \(\mu\) be any given sequence space. Then,

(i): \(A \in (\mu : [c_0]_{e,r})\) if and only if \(B \in (\mu : c_0)\).

(ii): \(A \in (\mu : [c]_{e,r})\) if and only if \(B \in (\mu : c)\).

(iii): \(A \in (\mu : [\ell_\infty]_{e,r})\) if and only if \(B \in (\mu : \ell_\infty)\).

Proof. We prove only Part (iii). Let \(z = (z_k) \in \mu\) and consider the following equality.

\[
\sum_{k=0}^{m} b_{nk} z_k = \sum_{j=0}^{k} \binom{k}{j} Q_j a_{jk} z_k \quad \text{for all } m, n \in \mathbb{N}
\]

which yields as \(m \to \infty\) that \((Bz)_n = \{B(Az)_n\}_n\) for all \(n \in \mathbb{N}\). Therefore, one can observe from here that \(Az \in [\ell_\infty]_{e,r}\) whenever \(z \in \mu\) if and only if \(Bz \in \ell_\infty\) whenever \(z \in \mu\). This completes the proof of Part (iii). \(\square\)
The following results were taken from Stieglitz and Tietz [22]:

$$\lim_{k} a_{nk} = 0 \text{ for all } n,$$  \hspace{1cm}  (4.2)

$$\lim_{n} \left| \sum_{k} a_{nk} \right| \text{ exist,}$$  \hspace{1cm}  (4.3)

$$\lim_{n \to \infty} \sum_{k} |a_{nk}| = \sum_{k} \left| \lim_{n \to \infty} a_{nk} \right|,$$  \hspace{1cm}  (4.4)

$$\lim_{n \to \infty} \sum_{k} |a_{nk}| = 0,$$  \hspace{1cm}  (4.5)

**Lemma 4.3.** Let $A = (a_{nk})$ be an infinite matrix. Then

(i): $A = (a_{nk}) \in (c_0 : \ell_\infty) = (c : \ell_\infty) = (\ell_\infty : \ell_\infty)$ if and only if (3.2) holds.

(ii): $A = (\hat{a}_{nk}) \in (c_0 : c_0)$ if and only if (3.2) and (4.2) hold.

(iii): $A = (a_{nk}) \in (c : c_0)$ if and only if (3.2), (4.2) and (4.5) hold.

(iv): $A = (a_{nk}) \in (\ell_\infty : c_0)$ if and only if (4.5) holds.

(v): $A = (\hat{a}_{nk}) \in (c_0 : c)$ if and only if (3.2) and (3.3) hold.

(vi): $A = (a_{nk}) \in (c : c)$ if and only if (3.2), (3.3) and (4.3) hold.

(vii): $A = (a_{nk}) \in (\ell_\infty : c)$ if and only if (3.3) and (4.4) hold.

Now, we can give the following results:

**Corollary 4.4.** Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:

(i): $A \in ([c_0]_{c,r} : c_0)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([c_0]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (3.2) and (4.2) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.

(ii): $A \in ([c_0]_{c,r} : c)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([c_0]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (3.2) and (3.3) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.

(iii): $A \in ([c_0]_{c,r} : \ell_\infty)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([c_0]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (3.2) holds with $\hat{a}_{nk}$ instead of $a_{nk}$.

**Corollary 4.5.** Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:

(i): $A \in ([c]_{c,r} : c_0)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([c]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (3.2), (4.2) and (4.5) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.

(ii): $A \in ([c]_{c,r} : c)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([c]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (3.2), (3.3) and (4.2) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.

(iii): $A \in ([c]_{c,r} : \ell_\infty)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([c]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (3.2) holds with $\hat{a}_{nk}$ instead of $a_{nk}$.

**Corollary 4.6.** Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:

(i): $A \in ([\ell_\infty]_{c,r} : c_0)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([\ell_\infty]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (4.5) holds with $\hat{a}_{nk}$ instead of $a_{nk}$.

(ii): $A \in ([\ell_\infty]_{c,r} : c)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([\ell_\infty]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (3.3) and (4.4) hold with $\hat{a}_{nk}$ instead of $a_{nk}$.

(iii): $A \in ([\ell_\infty]_{c,r} : \ell_\infty)$ if and only if $\{a_{nk}\}_{k \in \mathbb{N}} \in ([\ell_\infty]_{c,r})^\beta$ for all $n \in \mathbb{N}$ and (3.2) holds with $\hat{a}_{nk}$ instead of $a_{nk}$.

**Corollary 4.7.** Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:

(i): $A = (a_{nk}) \in (c_0 : [c_0]_{c,r})$ if and only if (3.2) and (4.2) hold with $b_{nk}$ instead of $a_{nk}$.

(ii): $A = (a_{nk}) \in (c : [c_0]_{c,r})$ if and only if (3.2), (4.2) and (4.5) hold with $b_{nk}$ instead of $a_{nk}$.

(iii): $A = (a_{nk}) \in (\ell_\infty : [c_0]_{c,r})$ if and only if (4.5) holds with $b_{nk}$ instead of $a_{nk}$.

(iv): $A = (a_{nk}) \in (c_0 : [c]_{c,r}) = (c : [c]_{c,r}) = (\ell_\infty : [c]_{c,r})$ if and only if (3.2) and (3.3) hold with $b_{nk}$ instead of $a_{nk}$.

(v): $A = (a_{nk}) \in (c : [c]_{c,r})$ if and only if (3.2), (3.3) and (4.3) hold with $b_{nk}$ instead of $a_{nk}$.

(vi): $A = (a_{nk}) \in (\ell_\infty : [c]_{c,r})$ if and only if (3.3) and (4.4) hold with $b_{nk}$ instead of $a_{nk}$.
(vii): $A = (a_{nk}) \in (c_0 : [\ell_\infty]_{c,r}) = (c : [\ell_\infty]_{c,r}) = (\ell_\infty : [\ell_\infty]_{c,r})$ if and only if (3.2) holds with $b_{nk}$ instead of $a_{nk}$.

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