

# The Concept of Entropy on $D$ -Posets

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**Özet.** Bu makalede  $D$ -posetlerde parçalanış ve parçalanış entropisi kavramları ortaya atılmış ve bunların özellikleri araştırılmıştır. Ayrıca, koşullu entropi kavramını ortaya attık ve bunların bazı sonuçlarını inceledik. Son olarak, bir dinamik sistemin entropisini tanımladık, ilgili bazı sonuçları gösterdik ve değişmezliğini kanıtladık.<sup>†</sup>

**Anahtar Kelimeler.**  $D$ -poset, etki cebiri, entropi, dinamik sistem, eşyapı dönüşümü.

**Abstract.** In this paper, partition and entropy of partitions in a  $D$ -poset are introduced and their properties are investigated. Also we introduce the conditional entropy and then we study some their results. At the end we define the entropy of a dynamical system, we prove some results on that, and we show its invariance.

**Keywords.**  $D$ -poset, effect algebra, entropy, dynamical system, isomorphism.

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## 1. Introduction and Preliminaries

With the development of the theory of quantum logics, new algebraic structures have been proposed as their models. As a quantum structure generalizing orthomodular lattices, orthomodular posets, and orthoalgebras, effect algebras in which the primary operation is partial sum, are regarded as a mathematical model of quantum logic [5, 9]. From a completely different starting point, Kopka and Chovanec [8] defined  $D$ -posets as an axiomatic model for quantum logics, where the primary operation is partial difference. This is important for modelling unsharp measurement in quantum mechanics [2]. Moreover, the two models are equivalent. By using the notion of a state on a  $D$ -poset one can introduce the entropy of partitions in  $D$ -poset, which is a useful tool in the study of the dynamical systems and their isomorphism. If two dynamical systems are isomorphic, they have the same entropy.

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Therefore, systems with different entropies cannot be isomorphic. The concept of entropy plays a major role in thermodynamics and statistical mechanics. It serves to describe the amount of uncertainty in physical systems. In 2000, Rybarik [18] introduced entropy of partitions on  $MV$ -algebras. And recently in 2003, Riečan [16] constructed the entropy of a dynamical system on an arbitrary  $MV$ -algebra, while Yuan [21] introduced the notion of entropy of partitions on quantum logic and Zhao and Ma [22] introduced conditional entropy of partitions on quantum logic. Then in 2008 Khare and Roy [6, 7] introduced entropy of a quantum dynamical system.

In the present paper, we introduce the corresponding definitions of the partition and the entropy of partitions on  $D$ -posets, where  $s$  is a Bayesian state on a  $D$ -poset. In Section 2, some basic facts about a  $D$ -poset  $D$  and some results are collected. In Section 3, notions of a partition  $A$  of a  $D$ -poset  $D$ , common refinement of partitions, entropy  $H(A)$  of  $A$ , conditional entropy  $H(A|B)$ , where  $A$  and  $B$  are partitions on  $D$ , are introduced and studied; some results are proved which are necessary for the study made in the subsequent section. In subsequent section we introduce and study the notion of entropy  $h(\phi)$  of a dynamical system  $(D, s, \phi)$ . With the help of a bijective mapping between two dynamical systems we give the concept of their isomorphism and then prove that the entropy of dynamical systems is isomorphism-invariant.

## 2. $D$ -Posets

The concept of an effect algebra was introduced by Foulis and Bennet [5]. We will work with an equivalent algebraic structure, called  $D$ -poset introduced by Kopka and Chovanec [8].

**Definition 2.1.** Effect algebra is a system  $(E, +, 0, 1)$ , where  $0, 1$  are distinguished elements of  $E$  and  $+$  is a partial binary operation on  $E$  such that

1.  $x + y = y + x$  if one side is defined;
2.  $(x + y) + z = x + (y + z)$  if one side is defined;
3. for every  $x \in E$  there exists a unique  $x'$  with  $x' + x = 1$ ;
4. if  $x + 1$  is defined then  $x = 0$ .

Every effect algebra bears a natural partial ordering given by  $x \leq y$  if and only if  $y = x + z$  for some  $z \in E$ . The poset  $(E, \leq)$  is bounded,  $0$  is the bottom element

and 1 is the top element. In every effect algebra, a partial subtraction  $-$  can be defined as follows:

$$x - y \text{ exists and is equal to } z \text{ if and only if } x = y + z.$$

The system  $(E, \leq, -, 0, 1)$  so obtained is a  $D$ -poset defined by Kopka and Chovanec [8].

**Definition 2.2.** The structure  $(D, \leq, -, 0, 1)$  is called  $D$ -poset if the relation  $\leq$  is a partial ordering on  $D$ , 0 is the smallest and 1 is the largest element on  $D$  and  $-$  is a partial binary operation satisfying the following conditions:

$$1. \ b - a \text{ is defined if and only if } a \leq b; \tag{D1}$$

$$2. \ \text{if } a \leq b \text{ then } b - a \leq b \text{ and } b - (b - a) = a; \tag{D2}$$

$$3. \ a \leq b \leq c \implies c - b \leq c - a, (c - a) - (c - b) = b - a. \tag{D3}$$

For any element  $a$  in a  $D$ -poset  $D$ , the element  $1 - a$  is called the orthosupplement of  $a$  and is denoted by  $a'$ .

**Example 2.3.** Let  $(L, \leq, ', 1, 0)$  be an orthomodular poset [15]. We put  $b - a = b \wedge a'$  for every  $a, b \in L$ ,  $a \leq b$ . Then  $L$  is a  $D$ -poset.

**Example 2.4.** Let  $H$  be a Hilbert space. A positive Hermitian operator  $A$  on  $H$  such that  $O \leq A \leq I$ , where  $O$  and  $I$  are operators on  $H$  defined by the formulas  $Ox = 0$ ,  $Ix = x$  for any  $x \in H$ , is said to be an effect. A system  $E(H)$  of effects closed with respect to the difference  $B - A$  of operators  $A, B \in E(H)$ ,  $A \leq B$ , is a  $D$ -poset.

**Lemma 2.5.** Let  $D$  be a  $D$ -poset and  $a, b, c \in D$ . The following assertions are true:

1. If  $a \leq b \leq c$ , then  $b - a \leq c - a$  and  $(c - a) - (b - a) = c - b$ ;
2. If  $b \leq c$  and  $a \leq c - b$ , then  $b \leq c - a$  and  $(c - b) - a = (c - a) - b$ ;
3. If  $a \leq b \leq c$ , then  $a \leq c - (b - a)$  and  $(c - (b - a)) - a = c - b$ ;
4.  $a - 0 = a$  for all  $a \in D$ ;
5.  $a - a = 0$  for all  $a \in D$ .

*Proof.* See [8]. □

**Definition 2.6.** Let  $(D, \leq, -, 0, 1)$  be a  $D$ -poset. Define a partial binary operation  $\oplus$  and a binary operation  $\odot$  as follows, for any  $a, b \in D$ .

$$a \oplus b = (a' - b)' \text{ if } a \leq b',$$

and

$$a \odot b = \begin{cases} a - b' & \text{if } a' \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that for each  $a, b, c \in D$ , the operations  $\oplus$  and  $\odot$  have the following properties:

1. If  $a \oplus b$  is defined, then  $b \oplus a$  is defined and  $a \oplus b = b \oplus a$ ;
2.  $a \odot b = b \odot a$ ;
3. If  $a \oplus b, (a \oplus b) \oplus c$  are defined, then  $b \oplus c, a \oplus (b \oplus c)$  are defined and  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ ;
4.  $(a \odot b) \odot c = a \odot (b \odot c)$ ;
5.  $a \oplus a' = 1$  and  $a \odot a' = 0$ ;
6.  $a \oplus 0 = a$  and  $a \odot 1 = a$ .

**Lemma 2.7.** *Let  $D$  be a  $D$ -poset then the following assertions are true:*

1. If  $a \leq b' \implies a \oplus b \geq a, a \oplus b \geq b$ ;
2.  $a \odot b \leq a, a \odot b \leq b$  for every  $a, b \in D$ ;
3. If  $a' \leq b \implies (a \odot b)' = a' \oplus b'$ ;
4. If  $a \leq b' \implies (a \oplus b)' = a' \odot b'$ ;
5.  $a \leq b'$  and  $a \leq c'$  imply  $a \oplus b \leq a \oplus c \iff b \leq c$ ;
6.  $a \leq c \implies b \odot a \leq b \odot c$ .

*Proof.* It is clear. □

**Definition 2.8.** A state on a  $D$ -poset  $D$  is a map  $s : D \longrightarrow [0, 1]$  such that :

1.  $s(1) = 1$ ;
2. if  $a \leq b$  then  $s(a) \leq s(b)$ ;
3. if  $a \leq b$  then  $s(b - a) = s(b) - s(a)$ ;
4. if  $(a_n)_{n=1}^{\infty} \subseteq D, a \in D, a_n \nearrow a$ , then  $s(a_n) \nearrow s(a)$ .

**Definition 2.9.** Two elements  $a, b \in D$  are orthogonal if  $a \leq b'$ , and we denote this by the symbol  $a \perp b$ .

A finite system  $A = \{a_1, a_2, \dots, a_n\}$  of elements of a  $D$ -poset  $D$  is said to be a  $\oplus$ -orthogonal system if

$$(\bigoplus_{i=1}^k a_i) \perp a_{k+1} \text{ for } k = 1, 2, \dots, n-1.$$

If  $a \leq b'$  then by (3) of Definition 2.8, we have  $s(a \oplus b) = s(a) + s(b)$ . Also for any  $\oplus$ -orthogonal system  $A = \{a_1, a_2, \dots, a_n\}$  of elements of a  $D$ -poset  $D$  and any state  $s$  on  $D$ ,

$$s(\oplus_{i=1}^n a_i) = \sum_{i=1}^n s(a_i).$$

### 3. The Entropy of a Partition

Now we can introduce a partition on a  $D$ -poset. let  $D$  be a  $D$ -poset and  $s$  be a state on  $D$ .

**Definition 3.1.** A system  $A = \{a_1, a_2, \dots, a_n\}$  in a  $D$ -poset  $D$  is said to be a finite partition of  $D$  corresponding to the state  $s$  if:

1.  $A$  is a  $\oplus$ -orthogonal system;
2.  $s(\oplus_{i=1}^n a_i) = 1$ .

**Definition 3.2.** [18, 21] Let  $A = \{b_1, b_2, \dots, b_n\}$  be any finite partition of  $D$  corresponding to a state  $s$  and  $a \in D$ . We say that the state  $s$  has Bayes'property if

$$s(\oplus_{i=1}^n (a \odot b_i)) = s(a).$$

**Lemma 3.3.** *Let  $A = \{b_1, b_2, \dots, b_n\}$  be a finite partition of  $D$ ,  $a \in D$ , and the state  $s$  has Bayes' property. Then*

$$\sum_{i=1}^n s(a \odot b_i) = s(a).$$

*Proof.* We first show that  $\{a \odot b_1, a \odot b_2, \dots, a \odot b_n\}$  is a  $\oplus$ -orthogonal system. From the  $\oplus$ -orthogonality of the system  $A$ , we have  $b_1 \leq b_2'$ . Using the monotonicity of operations  $\oplus$  and  $\odot$ ,

- 1) if  $a' \leq b_2$ , then  $a \odot b_1 \leq b_1 \leq b_2' \leq a' \oplus b_2' = (a \odot b_2)'$ ,
- 2) if  $a' \not\leq b_2$ , then  $(a \odot b_2)' = 0' = 1$ , therefore  $a \odot b_1 \leq (a \odot b_2)'$ .

Similarly

- 1) if  $a' \leq b_3$ , then  $(a \odot b_1) \oplus (a \odot b_2) \leq b_1 \oplus b_2 \leq b_3' \leq a' \oplus b_3' = (a \odot b_3)'$ ,
- 2) if  $a' \not\leq b_3$ , then  $(a \odot b_3)' = 0' = 1$ , therefore  $(a \odot b_1) \oplus (a \odot b_2) \leq (a \odot b_3)'$ .

Then from Bayes' property of a state  $s$ . we get that

$$\sum_{i=1}^n s(a \odot b_i) = s(\oplus_{i=1}^n (a \odot b_i)) = s(a).$$

□

**Lemma 3.4.** *Let  $B = \{b_1, b_2, \dots, b_n\}$  be a finite partition of a  $D$ -poset  $D$ . Then for every  $a \in D$ ,*

$$(a \odot b_1) \oplus (a \odot b_2) \oplus \dots \oplus (a \odot b_n) \leq a.$$

*Proof.* If for each  $i = 1, \dots, n$ ,  $a' \not\leq b_i$ , then  $a \odot b_i = 0$  for each  $i = 1, \dots, n$ , then the proof is finished, if not then there exists  $1 \leq j \leq n$  such that  $a' \leq b_j$ , then

$$\begin{aligned} (a \odot b_1) \oplus (a \odot b_2) \oplus \dots \oplus (a \odot b_n) &= \oplus_{i \neq j} (a \odot b_i) \oplus (a \odot b_j) \\ &\leq \oplus_{i \neq j} b_i \oplus (a \odot b_j) \leq b'_j \oplus (a \odot b_j) \\ &= b'_j \oplus (a - b'_j) = (b_j - (a - b'_j))' \\ &= (b_j - (b_j - a'))' = (a')' = a. \end{aligned}$$

□

**Definition 3.5.** Let  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  be two finite partitions of a  $D$ -poset  $D$  corresponding to a state  $s$ . Then the common refinement of these partitions is defined as the system

$$A \vee B = \{a_i \odot b_j : a_i \in A, b_j \in B, i = 1, 2, \dots, n, j = 1, 2, \dots, m\}.$$

**Definition 3.6.** If  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  be two finite partitions of a  $D$ -poset,  $D$ , corresponding to a state  $s$ . Then  $B$  is called a refinement of  $A$ , written as  $A \leq B$ , if there exists a partition  $I(1), \dots, I(n)$  of the set  $\{1, \dots, m\}$  such that

$$s(a_i) = \sum_{j \in I(i)} s(b_j),$$

for every  $i = 1, \dots, n$ .

**Lemma 3.7.** *If the state  $s$  has the Bayes' property, and  $A, B$  are two finite partitions of a  $D$ -poset  $D$ , then the system  $A \vee B$  is also a partition of  $D$ , further  $A \leq A \vee B$ .*

*Proof.* Let  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  be two finite partitions of  $D$  corresponding to a state  $s$ . By Lemma 3.4, we can prove that the system  $A \vee B = \{a_i \odot b_j : a_i \in A, b_j \in B, i = 1, 2, \dots, n, j = 1, 2, \dots, m\}$  is  $\oplus$ -orthogonal, and from the Bayes' property of the state  $s$  and the  $\oplus$ -orthogonal of the system  $A \vee B$ , we have

$$\begin{aligned} s(\oplus_{i,j=1}^{n,m} (a_i \odot b_j)) &= s(\oplus_{i=1}^n (\oplus_{j=1}^m (a_i \odot b_j))) \\ &= \sum_{i=1}^n s(\oplus_{j=1}^m (a_i \odot b_j)) = \sum_{i=1}^n s(a_i) = s(\oplus_{i=1}^n a_i) = 1. \end{aligned}$$

Finally, let us mention that  $A \vee B$  is indexed by  $\{(i, j) : i = 1, \dots, n; j = 1, \dots, m\}$ . Therefore, if we put  $I(i) = \{(i, 1), \dots, (i, m)\}$ , then

$$s(a_i) = s(\oplus_{j=1}^m (a_i \odot b_j)) = \sum_{j=1}^m s(a_i \odot b_j) = \sum_{(i,j) \in I(i)} s(a_i \odot b_j)$$

for every  $i = 1, \dots, n$ . It follows that  $A \leq A \vee B$ . □

**Definition 3.8.** If  $A = \{a_1, a_2, \dots, a_n\}$  and  $B = \{b_1, b_2, \dots, b_m\}$  are two finite partitions of a  $D$ -poset,  $D$ , corresponding to a state  $s$ , we define the entropy of  $A$  by

$$H(A) = \sum_{i=1}^n f(s(a_i)),$$

where  $f(x) = -x \log x$ , if  $x > 0$ ,  $f(0) = 0$  and conditional entropy by

$$H(A|B) = \sum_{i=1}^n \sum_{j=1}^m s(b_j) f\left(\frac{s(a_i \odot b_j)}{s(b_j)}\right).$$

Omitting the  $j$ -terms when  $s(b_j) = 0$ .

**Lemma 3.9.** Let  $A = \{a_1, a_2, \dots, a_n\}$  be a finite partition of a  $D$ -poset,  $D$ , corresponding to a state  $s$  and  $c = \oplus_{i \in I} a_i$ ,  $I \subseteq \{1, 2, \dots, n\}$ . Then for each  $b \in D$ ,

$$s(b \odot c) = s(\oplus_{i \in I} (b \odot a_i)).$$

*Proof.* Put  $d = \oplus_{i \in I'} a_i$ , where  $I' = \{1, \dots, n\} - I$ , of Bayes' property of the state  $s$ , we have

$$\begin{aligned} s(b) &= s(\oplus_{i=1}^n (b \odot a_i)) = \sum_{i=1}^n s(b \odot a_i) \\ &= \sum_{i \in I} s(b \odot a_i) + \sum_{i \in I'} s(b \odot a_i) \\ &= s(\oplus_{i \in I} (b \odot a_i)) + s(\oplus_{i \in I'} (b \odot a_i)). \end{aligned} \tag{1}$$

Since  $\{c, d\}$  is a partition of  $D$ ,

$$s(b) = s(b \odot c) + s(b \odot d) \tag{2}$$

Equations (1) and (2) imply that

$$s(b \odot c) + s(b \odot d) = s(\oplus_{i \in I} (b \odot a_i)) + s(\oplus_{i \in I'} (b \odot a_i)),$$

also  $(b \odot c \odot a_i) = 0$  for every  $i \in I'$  and  $(b \odot d \odot a_i) = 0$  for every  $i \in I$ , thus

$$\begin{aligned} s(b \odot c) &= s(\bigoplus_{i=1}^n (b \odot c \odot a_i)) \\ &= s(\bigoplus_{i \in I} (b \odot c \odot a_i)) \leq s(\bigoplus_{i \in I} (b \odot a_i)), \end{aligned}$$

and

$$\begin{aligned} s(b \odot d) &= s(\bigoplus_{i=1}^n (b \odot d \odot a_i)) \\ &= s(\bigoplus_{i \in I'} (b \odot d \odot a_i)) \leq s(\bigoplus_{i \in I'} (b \odot a_i)), \end{aligned}$$

therefore  $s(b \odot c) = s(\bigoplus_{i \in I} (b \odot a_i))$ .  $\square$

**Proposition 3.10.** *Let  $A$ ,  $B$  and  $C$  be finite partitions of a  $D$ -poset,  $D$ , corresponding to a state  $s$  and  $B \leq C$ , then  $H(A|C) \leq H(A|B)$ .*

*Proof.* Let  $b_j = \bigoplus_{t \in I(j)} c_t$ , where  $\{I(1), \dots, I(k)\}$  is the corresponding partition and put (for fixed  $j$ )  $\alpha_t = s(c_t)/s(b_j)$ . Then

$$\sum_{t \in I(j)} \alpha_t = \frac{1}{s(b_j)} s(\bigoplus_{t \in I(j)} c_t) = 1;$$

hence by the concaveness of  $f$ ,

$$\sum_t \alpha_t f(x_t) \leq f\left(\sum_t \alpha_t x_t\right).$$

Therefore,

$$\begin{aligned} H(A|C) &= \sum_i \sum_t s(c_t) f\left(\frac{s(a_i \odot c_t)}{s(c_t)}\right) \\ &= \sum_i \sum_j s(b_j) \sum_{t \in I(j)} \frac{s(c_t)}{s(b_j)} f\left(\frac{s(a_i \odot c_t)}{s(c_t)}\right) \\ &\leq \sum_i \sum_j s(b_j) f\left(\sum_{t \in I(j)} \frac{s(a_i \odot c_t)}{s(b_j)}\right) \\ &= \sum_i \sum_j s(b_j) f\left(\frac{s(\bigoplus_{t \in I(j)} (a_i \odot c_t))}{s(b_j)}\right), \end{aligned}$$

by Lemma 3.9, we have

$$= \sum_i \sum_j s(b_j) f\left(\frac{s(a_i \odot b_j)}{s(b_j)}\right) = H(A|B).$$

$\square$

**Proposition 3.11.** *Let  $A, B$  and  $C$  be finite partitions of a  $D$ -poset  $D$  corresponding to a state  $s$ . Then  $H(B \vee C|A) = H(B|A) + H(C|B \vee A)$ .*

*Proof.* By the definition

$$\begin{aligned} H(B \vee C|A) &= \sum_{i,j,k} s(a_i) f\left(\frac{s(b_j \odot c_k \odot a_i) s(b_j \odot a_i)}{s(a_i) s(b_j \odot a_i)}\right) \\ &= - \sum s(b_j \odot c_k \odot a_i) \log \frac{s(b_j \odot c_k \odot a_i)}{s(b_j \odot a_i)} \\ &\quad - \sum s(b_j \odot c_k \odot a_i) \log \frac{s(b_j \odot a_i)}{s(a_i)}, \end{aligned}$$

from Bayes' property of a state  $s$ , we have  $\sum_k s(b_j \odot c_k \odot a_i) = s(b_j \odot a_i)$ , therefore

$$H(B \vee C|A) = H(C|B \vee A) + H(B|A).$$

□

**Proposition 3.12.** *Let  $A, B, C$  be finite partitions of a  $D$ -poset  $D$  corresponding to a state  $s$ . Then*

1.  $H(B \vee C|A) \leq H(B|A) + H(C|A)$ ;
2.  $H(B \vee C) = H(B) + H(C|B)$ ;
3.  $H(B \vee C) \leq H(B) + H(C)$ .

*Proof.* 1. It is clear.

2. Let  $A = \{1\}$ ; then  $H(B|A) = H(B)$ ; and by Proposition 3.11

$$\begin{aligned} H(B \vee C) &= H(B \vee C|A) \\ &= H(B|A) + H(C|B \vee A) \\ &= H(B) + H(C|B). \end{aligned}$$

3. By the Part 2 and Proposition 3.10,

$$H(B \vee C) = H(B) + H(C|B) \leq H(B) + H(C).$$

□

**Proposition 3.13.** *Let  $A, B$  and  $C$  be finite partitions of a  $D$ -poset  $D$  corresponding to a state  $s$ . Then  $H(A|B) + H(B|C) \geq H(A|C)$ .*

*Proof.* By Propositions 3.10 and 3.12, we obtain

$$\begin{aligned}
H(A|B) + H(B|C) &= H(A \vee B) + H(B \vee C) - H(B) - H(C) \\
&= H(A \vee B) + H(C|B) - H(C) \\
&\geq H(A \vee B) + H(C|A \vee B) - H(C) \\
&= H(A \vee B \vee C) - H(C) \\
&\geq H(A \vee C) - H(C) = H(A|C).
\end{aligned}$$

□

## 4. Entropy of Dynamical Systems

**Definition 4.1.** If  $D$  is a  $D$ -poset then by a dynamical system on  $D$  we mean a triple  $(D, s, \phi)$ , where  $s : D \rightarrow [0, 1]$  is a state on  $D$  with the Bayes' property and  $\phi : D \rightarrow D$  is a mapping satisfying the following conditions:

1. If  $a \leq b'$  then  $\phi(a) \leq \phi(b)'$  and  $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$ ;
2.  $\phi(a \odot b) = \phi(a) \odot \phi(b)$ ;
3.  $\phi(1) = 1$ ;
4.  $s(\phi(a)) = s(a)$  for any  $a \in D$ .

**Proposition 4.2.** If  $(D, s, \phi)$  is a dynamical system on  $D$  and  $A = \{a_1, a_2, \dots, a_n\}$  is a finite partition, then  $\phi(A)$  is a partition, too. If  $B = \{b_1, b_2, \dots, b_m\}$  is another finite partition, then  $H(\phi(A)|\phi(B)) = H(A|B)$ .

*Proof.* It is clear. □

**Lemma 4.3.** Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of non-negative numbers such that  $a_{r+t} \leq a_r + a_t$ , for any  $r, t \in \mathbb{N}$ . Then  $\lim_{n \rightarrow \infty} (a_n/n)$  exists.

*Proof.* See [12, 19]. □

**Proposition 4.4.** For any finite partition  $A = \{a_1, a_2, \dots, a_m\}$  there exists

$$\lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right).$$

*Proof.* The proof is immediate from Part 3 of Proposition 3.12 and Lemma 4.3. □

**Definition 4.5.** For every finite partition  $A$  we define

$$h(\phi, A) = \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right).$$

Further we define the entropy of dynamical system by

$$h(\phi) = \sup\{h(\phi, A); A \text{ is a partition}\}.$$

**Proposition 4.6.** Let  $A$  be a finite partition of  $D$ . Then for every  $k \in \mathbb{N}$ ,

$$h(\phi, A) = h\left(\phi, \bigvee_{j=0}^k \phi^j(A)\right).$$

*Proof.* We obtain immediately

$$\begin{aligned} h\left(\phi, \bigvee_{j=0}^k \phi^j(A)\right) &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi^i\left(\bigvee_{j=0}^k \phi^j(A)\right)\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{t=0}^{n+k-1} \phi^t(A)\right) \\ &= \lim_{p \rightarrow \infty} \frac{p}{p-k} \frac{1}{p} H\left(\bigvee_{t=0}^{p-1} \phi^t(A)\right) \\ &= h(\phi, A). \end{aligned}$$

□

**Proposition 4.7.** Let  $A$  be a finite partition of  $D$ . Then

$$h(\phi, A) = \lim_{n \rightarrow \infty} H\left(A \mid \bigvee_{i=1}^n \phi^i(A)\right).$$

*Proof.* By Proposition 3.12

$$\begin{aligned} H\left(\bigvee_{i=0}^k \phi^i(A)\right) &= H\left(A \vee \phi\left(\bigvee_{i=0}^{k-1} \phi^i(A)\right)\right) \\ &= H\left(\phi\left(\bigvee_{i=0}^{k-1} \phi^i(A)\right)\right) + H\left(A \mid \phi\left(\bigvee_{i=0}^{k-1} \phi^i(A)\right)\right) \\ &= H\left(\bigvee_{i=0}^{k-1} \phi^i(A)\right) + H\left(A \mid \bigvee_{i=1}^k \phi^i(A)\right). \end{aligned}$$

Now, by induction we obtain

$$H\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right) = H(A) + \sum_{k=1}^{n-1} H\left(A \mid \bigvee_{i=1}^k \phi^i(A)\right). \quad (3)$$

By Proposition 3.10 we obtain that  $\{H(A \mid \bigvee_{i=1}^n \phi^i(A))\}$  is decreasing, so that

$$\lim_{n \rightarrow \infty} H\left(A \mid \bigvee_{i=1}^n \phi^i(A)\right)$$

exists. But then there exists also the limit of the Cesaro means

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n H\left(A \mid \bigvee_{i=1}^k \phi^i(A)\right).$$

By (3) we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} H\left(A \mid \bigvee_{i=1}^n \phi^i(A)\right) &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n H\left(A \mid \bigvee_{i=1}^k \phi^i(A)\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ H\left(\bigvee_{i=1}^n \phi^i(A)\right) - H(A) \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} \phi^j(A)\right) = h(\phi, A). \end{aligned}$$

□

**Proposition 4.8.** *Let  $A$  be a finite partition of  $D$ . Then for every  $k \in \mathbb{N}$ ,*

$$h(\phi^k) = kh(\phi).$$

*Proof.* By Proposition 4.6, we have

$$\begin{aligned} h(\phi^k, A) &= h\left(\phi^k, \bigvee_{i=0}^{k-1} \phi^i(A)\right) = \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} \phi^{jk} \left(\bigvee_{i=0}^{k-1} \phi^i(A)\right)\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{nk-1} \phi^i(A)\right) = k \lim_{n \rightarrow \infty} \frac{1}{nk} H\left(\bigvee_{i=0}^{nk-1} \phi^i(A)\right) = kh(\phi, A). \end{aligned}$$

□

**Proposition 4.9.** *Let  $A, C$  be finite partitions of  $D$ . Then*

$$h(\phi, A) \leq h(\phi, C) + H(A|C).$$

*Proof.* Since  $H(B \vee D) = H(B) + H(D|B)$ ,

$$\begin{aligned} H\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right) &\leq H\left[\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right) \vee \left(\bigvee_{j=0}^{n-1} \phi^j(C)\right)\right] \\ &= H\left(\bigvee_{j=0}^{n-1} \phi^j(C)\right) + H\left(\bigvee_{i=0}^{n-1} \phi^i(A) \middle| \bigvee_{j=0}^{n-1} \phi^j(C)\right). \end{aligned}$$

Further  $H(D \vee E|B) \leq H(D|B) + H(E|B)$ ,  $H(D|B \vee E) \leq H(D|B)$ , hence

$$\begin{aligned} H\left(\bigvee_{i=0}^{n-1} \phi^i(A) \middle| \bigvee_{j=0}^{n-1} \phi^j(C)\right) &\leq \sum_{i=0}^{n-1} H(\phi^i(A) \middle| \bigvee_{j=0}^{n-1} \phi^j(C)) \\ &\leq \sum_{i=0}^{n-1} H(\phi^i(A) \middle| \phi^i(C)) = nH(A|C). \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi^i(A)\right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} \phi^j(C)\right) + H(A|C)$$

and finally

$$h(\phi, A) \leq h(\phi, C) + H(A|C).$$

□

**Corollary 4.10.** *If  $C_n = \bigvee_{i=0}^n \phi^i(C)$ . Then for any finite partition  $A$ ,*

$$h(\phi, A) \leq h(\phi, C) + H(A|C_n).$$

*Proof.* See Proposition 4.6 and 4.9. □

**Definition 4.11.** The dynamical systems  $(D_1, s_1, \phi_1)$  and  $(D_2, s_2, \phi_2)$  are said to be isomorphic if there exists a bijective map  $\psi : D_1 \rightarrow D_2$  satisfying the following condition for each  $a, b \in D_1$ :

1. If  $a \leq b'$ , then  $\psi(a) \leq \psi(b)'$  and  $\psi(a \oplus b) = \psi(a) \oplus \psi(b)$ ;
2.  $\psi(a \odot b) = \psi(a) \odot \psi(b)$ ;
3.  $s_1(a) = s_2(\psi(a))$ ;
4.  $\psi(\phi_1(a)) = \phi_2(\psi(a))$ .

**Proposition 4.12.** *If  $(D_1, s_1, \phi_1)$  and  $(D_2, s_2, \phi_2)$  are isomorphic dynamical systems, then  $h(\phi_1) = h(\phi_2)$ , i.e., the entropy of their dynamical systems is an isomorphism invariant.*

*Proof.* Let  $(D_1, s_1, \phi_1)$  and  $(D_2, s_2, \phi_2)$  are isomorphic and  $\psi : D_1 \longrightarrow D_2$  be the mapping representing the isomorphism of the dynamical systems. Let  $A = \{a_1, a_2, \dots, a_n\}$  be a finite partition of  $D_1$ , then  $\psi(A)$  is the finite partition of  $D_2$ . Now

$$\begin{aligned} H(\psi(A)) &= \sum_{i=1}^n f(s_2(\psi(a_i))) \\ &= \sum_{i=1}^n f(s_1(a_i)) = H(A). \end{aligned}$$

Thus,

$$\begin{aligned} h(\phi_2, \psi(A)) &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi_2^i(\psi(A))\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \psi(\phi_1^i(A))\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\psi\left(\bigvee_{i=0}^{n-1} \phi_1^i(A)\right)\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi_1^i(A)\right) = h(\phi_1, A). \end{aligned}$$

□

## 5. Conclusion

For the classification of the dynamical systems based on isomorphism, isomorphism invariants play an important role. In the classical setting “entropy” fits in this role. The first step in the evolution of entropy theory is to define the entropy of a partition; the definition of the entropy of a state-preserving transformation is based on that of the entropy of a partition; and finally, the entropy of a dynamical system is introduced. In this paper, we have introduced entropy of dynamical systems on a  $D$ -poset. Then we have proved some properties for entropy and entropy of dynamical systems.

## References

- [1] D. Dubois and H. Prade, A review of fuzzy set aggregation connectives, *Information Science* **36** (1985), 85–121.

- [2] A. Dvurecenskij and S. Pulmannova, Difference posets, effects, and quantum measurements, *International Journal of Theoretical Physics* **33** (1994), 819–850.
- [3] D. Dumitrescu, Fuzzy partitions with the connectives  $T_\infty$ ,  $S_\infty$ , *Fuzzy Sets and Systems* **47** (1992), 193–195.
- [4] M. Ebrahimi and U. Mohamadi,  $m$ -Generators of fuzzy dynamical systems, *Çankaya University Journal of Science and Engineering* **9** (2012), 167–182.
- [5] D. J. Foulis and M. K. Bennett, Effect algebras and unsharp quantum logics, *Foundations of Physics* **24** (1994), 1331–1352.
- [6] M. Khare and S. Roy, Conditional entropy and the Rokhlin metric on an orthomodular lattice with Bayesian state, *International Journal of Theoretical Physics* **47** (2008), 1386–1396.
- [7] M. Khare and S. Roy, Entropy of quantum dynamical systems and sufficient families in orthomodular lattices with Bayesian state, *Communications in Theoretical Physics* **50** (2008), 551–556.
- [8] F. Kopka and F. Chovanec,  $D$ -posets, *Mathematica Slovaca* **44** (1994), 21–34.
- [9] G. Kalmbach, *Orthomodular Lattices*, Academic Press, London, 1983.
- [10] M. Kalina, V. Olejcek and J. Paseka, Sharply dominating  $MV$ -effect algebras, *International Journal of Theoretical Physics* **50** (2011), 1152–1159.
- [11] D. Markechova, The entropy of complete fuzzy partitions, *Mathematica Slovaca* **43** (1993), 1–10.
- [12] T. Neubrunn and B. Riečan, *Miera a integral*, Veda, Bratislava 1981 (in Slovak).
- [13] J. Petrovicova, On the entropy of partitions in product  $MV$  algebra, *Soft Computing* **4** (2000), 41–44.
- [14] J. Petrovicova, On the entropy of dynamical systems in product  $MV$  algebras, *Fuzzy Sets and Systems* **121** (2001), 347–351.
- [15] P. Ptak and S. Pulmannova, *Orthomodular Structures as Quantum Logics*, VEDA and Kluwer Acad. Publ., Bratislava and Dordrecht 1991.
- [16] B. Riečan, Kolmogorov-Sinaj entropy on  $MV$ -algebras, *International Journal of Theoretical Physics* **44** (2005), 1041–1052.
- [17] B. Riečan and D. Markechova, The entropy of fuzzy dynamical systems, general scheme and generators, *Fuzzy Sets and Systems* **96** (1998), 191–199.
- [18] J. Rybarik, The entropy of partitions on  $MV$ -algebras, *International Journal of Theoretical Physics* **39** (2000), 885–892.
- [19] P. Walters, *An Introduction to Ergodic Theory*, Springer, New York, Berlin 1982.
- [20] J. Wang, J. Wu and M. Cho, Entropy of partitions on sequential effect algebras, *Communications in Theoretical Physics* **53** (2010), 399–402.
- [21] H. Yuan, Entropy of partitions on quantum logic, *Communications in Theoretical Physics* **43** (2005), 437–439.
- [22] Y. Zhao and Z. Ma, Conditional entropy of partitions on quantum logic, *Communications in Theoretical Physics* **48** (2007), 11–13.