A Kantorovich Type Generalization of the Szàsz Operators via Two Variable Hermite Polynomials

Serdal Yazıcı¹, Bayram Çekim²,*

¹Gazi University, Faculty of Science, Department of Mathematics, 06100, Beşevler, Ankara, Turkey.
²Gazi University, Faculty of Science, Department of Mathematics, 06100, Beşevler, Ankara, Turkey.

Abstract

The purpose of this paper is to give the Kantorovich generalization of the operators via two variable Hermite polynomials which are introduced by Krech [1] and to research approximating features with help of the classical modulus of continuity, the class of Lipschitz functions, Voronovskaya type asymptotic formula, second modulus of continuity and Peetre’s K-functional for these operators.

1. INTRODUCTION

Heretofore, many authors has studied on linear positive operators and properties of their approximation, see for example [2, 6-11, 17, 18]. In addition to fact that authors working on the approximation theory with help of linear positive operators have been given linear positive operators via some orthogonal polynomials, see for example [1, 3, 4, 5, 12]. Therefore, we are going to define the Kantorovich type of the operators being made up of one of orthogonal polynomials.

Firstly, we recall \( H_k \) which is two variable Hermite polynomial ( see [13] ) defined by

\[
H_k(n,\alpha) = k! \sum_{s=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{n^{k-2s}\alpha^s}{(k-2s)!s!}.
\]

Furthermore, the generating function of two variable Hermite polynomials is as follows ( see [13] )

\[
\sum_{k=0}^{\infty} H_k(n,\alpha) \frac{k^k}{k!} = e^{nt+\alpha^2}.
\]

Secondly, Krech has presented the Szàsz operators including two variable Hermite polynomials (see [1]) as

*Corresponding author, e-mail: bayramcekim@gazi.edu.tr
\[ G_n^\alpha (f; x) = e^{-(nx + \alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n, \alpha) f \left( \frac{k}{n} \right), \]

where \( n = 1, 2, 3, \ldots, \alpha \geq 0 \) and \( x \in [0, \infty) \).

Now, we introduce a Kantorovich type generalization of \( G_n^\alpha \).

### 2. KANTOROVICH TYPE GENERALIZATION OF OPERATORS \( G_n^\alpha \)

In this section, the Kantorovich type generalization of \( G_n^\alpha \) has been defined by

\[ S_n^\alpha (f; x) := n e^{-(ax + \alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n, \alpha) \int_0^1 f(t) dt, \]

where \( n = 1, 2, 3, \ldots, \alpha \geq 0, \quad x \in [0, \infty) \) and \( f \in C[0, \infty) \) for which the corresponding series is convergent, in here \( C[0, \infty) \) is the space of continuous functions on \([0, \infty)\).

**Lemma 1.** The operators given by (2.1) yield the following equalities.

i. \[ S_n^\alpha (1; x) = 1, \]

ii. \[ S_n^\alpha (t; x) = x + \frac{4ax^2 + 1}{2n}, \]

iii. \[ S_n^\alpha (t^2; x) = x^2 + \frac{4ax^3 + 2x}{n} + \frac{12a^2x^4 + 18ax^2 + 1}{3n^2}, \]

iv. \[ S_n^\alpha (t^3; x) = x^3 + \frac{12ax^4 + 9x^2}{2n} + \frac{24a^2x^6 + 48ax^4 + 7x}{2n^2} + \frac{120a^2x^8 + 32ax^6 + 64ax^4 + 1}{4n^3}, \]

v. \[ S_n^\alpha (t^4; x) = x^4 + \frac{8ax^5 + 8x^3}{n} + \frac{24a^2x^6 + 60ax^4 + 15x^2}{2n^2} + \frac{32ax^7 + 144a^2x^5 + 108ax^3 + 6x}{4n^3} + \frac{840a^2x^9 + 560ax^7 + 80ax^5 + 210ax^3 + 1}{5n^4}. \]

Observe that the operators are well-defined for all test function \( e_i(t) = t^i \) for \( i = 0, 1, 2, 3, 4 \).

**Lemma 2.** The operators given by (2.1) yield the following equalities.

i. \[ \psi_1 = S_n^\alpha ((t - x)^1; x) = \frac{4ax^2 + 1}{2n}, \]

ii. \[ \psi_2 = S_n^\alpha ((t - x)^2; x) = \frac{x + 12ax^4 + 18ax^2 + 1}{3n^2}, \]
Now, we can give Theorem 1 for approximation properties of the operators $S_n^g$ using the well known Korovkin theorem with the help of Lemma 1.

**Theorem 1.** Let the operator defined by $S_n^g$ in (2.1) and $f \in C_B[0,\infty)$. So, $S_n^g(f;x)$ is uniformly convergent to $f(x)$ on $[0,b]$ where $C_B[0,\infty)$ is the space of uniformly continuous and bounded functions on $[0,\infty)$.

**Proof.** Follow the standard procedure in [16], see also [14,15].

### 3. APPROXIMATION PROPERTIES OF OPERATORS $S_n^g$

In this section, we present the rate of convergence of the operators with the help of the usual and second modulus of continuity, Lipschitz class functions, Peetre’s $K$-functional and Voronovskaya type formula.

Firstly, we remind some definitions as follows.

Let $Lip_M(\beta)$ be Lipschitz class of order $\beta$. If $f \in Lip_M(\beta)$, the inequality

$$|f(t) - f(u)| \leq M |t - u|^\beta$$

holds, where $t,u \in [0,\infty)$, $0 < \beta \leq 1$ and $M > 0$. The classical modulus of continuity of $f \in C_B[0,\infty)$ is denoted by

$$\omega(f; \delta) = \sup_{|h| \leq \delta} |f(x+h) - f(x)| : x \in [0,\infty)$$

where $\delta > 0$.

Furthermore, a vector space, $C_B^2[0,\infty) = \{f \in C_B[0,\infty) : f' \in C_B[0,\infty), f'' \in C_B[0,\infty]\}$, is normed space with following norm that

$$\|f\|_{C_B^2[0,\infty)} = \|f\|_{C_B[0,\infty)} + \|f'\|_{C_B[0,\infty)} + \|f''\|_{C_B[0,\infty)}$$

for every $f \in C_B^2[0,\infty)$. We can remind Peetre's $K$-functional of the function $f \in C_B[0,\infty)$ that is as follows

$$K(f; \delta) = \inf_{g \in C_B^2[0,\infty)} \left\{ |f - g|_{C_B[0,\infty)} + \delta \|g\|_{C_B^2[0,\infty)} \right\}$$

(3.4)
for $\delta > 0$. We define the second-order modulus of smoothness of function $f \in C_B[0,\infty)$ by

$$\omega_2(f;\delta) = \sup_{0<h<\delta} \{ |f(x+2h) - 2f(x+h) + f(x)| : x \in [0,\infty) \}$$

(3.5)

for $\delta > 0$. Moreover, we have the inequality that is relation between Peetre’s $K$-functional and $\omega_2$ as following that

$$K(f;\delta) \leq M \left\{ \omega_2(f;\sqrt{\delta}) + \min(1,\delta) \|f\|_{C_B[0,\infty)} \right\}$$

(3.6)

for all $\delta > 0$ and $M$ is positive constant.

**Theorem 2.** The operators $S^\alpha_n$ defined in (2.1) verify the following inequality

$$\left| S^\alpha_n(f;x) - f(x) \right| \leq M\omega(f;\delta_n).$$

(3.7)

where $f \in C_B[0,\infty)$, $x \in [0,b]$, $M$ is a constant and $\delta_n = \frac{1}{\sqrt{n}}$.

**Proof.** We know that modulus of continuity of function $f \in C_B[0,\infty)$ verifies the following inequality

$$|f(t) - f(x)| \leq \omega(f;\delta) \left\{ \left| \frac{t-x}{\delta} \right| + 1 \right\}.$$  

(3.8)

Using (3.8), Cauchy-Schwarz inequality and Lemma 2, we have

$$\left| S^\alpha_n(f;x) - f(x) \right| \leq S^\alpha_n \left( |f(t) - f(x)| ; x \right)$$

$$\leq \omega(f;\delta) \left( 1 + \frac{1}{\delta} S^\alpha_n \left( |f(t) - f(x)| ; x \right) \right)$$

$$\leq \omega(f;\delta) \left( 1 + \frac{1}{\delta} S^\alpha_n \left( |(t-x)^2| ; x \right) \right)$$

$$\leq \omega(f;\delta) \left( 1 + \frac{1}{\delta} \sqrt{\psi_2} \right)$$

$$\leq M\omega(f;\delta_n).$$

where $M = 1 + \sqrt{b + 12\alpha^2b^4 + 18\alpha b^2} + 1$ and $\delta_n = \frac{1}{\sqrt{n}}$.

**Theorem 3.** If $f \in Lip_M(\beta)$, then we have
where $x \in [0, b]$, $M^*$ is constant and $\delta_n = \frac{1}{n}$.

**Proof.** From $f \in \text{Lip}_M(\beta)$ and linearity property of $S_n^\alpha$, we obtain

$$\left| S_n^\alpha (f; x) - f(x) \right| \leq M^* \left( \frac{\beta}{2} \right),$$

where $M^* = M \left( b + 12\alpha^2 b^4 + 18\alpha b^2 + 1 \right)^{\frac{1}{2}}$ and $\delta_n = \frac{1}{n}$.

**Theorem 4.** Let $K$ be Peetre’s $K$-functional. The operators $S_n^\alpha$ defined in (2.1) verify the following inequality

$$\left| S_n^\alpha (f; x) - f(x) \right| \leq 2K \left( f, \delta_n \right),$$

where $f \in C_B[0, \infty), x \in [0, b]$ and $\delta_n = \frac{b}{n} + \frac{4\alpha b^2}{2n} + \frac{1}{3n^2}$.

**Proof.** From the Taylor’s series expansion of the function $g \in C_B^2[0, \infty)$, we have

$$g(t) = g(x) + g'(x)(t-x) + g''(c) \frac{(t-x)^2}{2}, \quad c \in (x, t).$$
When we apply the operators $S^\alpha_n$ to both sides of the aforementioned equality and recall the linearity property of the operators $S^\alpha_n$, we obtain

$$S^\alpha_n (g; x) - g(x) = g(x) S^\alpha_n ((t-x); x) + \frac{g'(c)}{2} S^\alpha_n ((t-x)^2; x).$$

By Lemma 2, we have

$$\left| S^\alpha_n (g; x) - g(x) \right| \leq g'(x) \left| \psi_1 + \frac{g'(c)}{2} \psi_2 \right|.$$ 

$$\leq g'(x) \left( \frac{4\alpha x^2 + 1}{2n} + \frac{g'(c)}{2} \left( \frac{x}{n} + \frac{12\alpha^2 x^4 + 18\alpha x^2 + 1}{3n^2} \right) \right).$$

$$\leq \| g \|_{C^1[0,\infty)} \left( \frac{4\alpha x^2 + 1}{2n} + \frac{g'(c)}{2} \left( \frac{x}{n} + \frac{12\alpha^2 x^4 + 18\alpha x^2 + 1}{3n^2} \right) \right).$$

$$\leq \left( \frac{x}{n} + \frac{4\alpha x^2 + 1}{2n} + \frac{12\alpha^2 x^4 + 18\alpha x^2 + 1}{3n^2} \right) \| g \|_{C^1[0,\infty)}.$$ 

Now, let $f \in C_B[0,\infty)$. We use the above inequality as follow

$$\left| S^\alpha_n (f; x) - f(x) \right| \leq S^\alpha_n (f; x) - S^\alpha_n (g; x) + S^\alpha_n (g; x) - g(x) + g(x) - f(x) \leq S^\alpha_n (f - g; x) + \| f(x) - g(x) \| + \| g'(c) \|.$$

$$\leq 2 \left\| f - g \right\|_{C^0[0,\infty)} + 2 \| g'(c) \| \left( \frac{b}{n} + \frac{4\alpha x^2 + 1}{2n} + \frac{12\alpha^2 b^4 + 18\alpha b^2 + 1}{3n^2} \right).$$

By applying infimum both sides of this inequality for $g \in C_B^2[0,\infty)$, we have

$$\left| S^\alpha_n (f; x) - f(x) \right| \leq 2K (f, \delta_n),$$

where $\delta_n = \frac{b}{n} + \frac{4\alpha x^2 + 1}{2n} + \frac{12\alpha^2 b^4 + 18\alpha b^2 + 1}{3n^2}$. 


Theorem 5. For the operators (2.1), the following inequality holds
\[
\left| S_n^\alpha (f; x) - f(x) \right| \leq 2M \left\{ \varphi_2 \left( f, \sqrt{\lambda_n} \right) + \min (1, \lambda_n) \| f \| \right\},
\] (3.11)
where \( f \in C_B [0, \infty) \), \( x \in [0, b] \), \( M \) is a positive constant that is independent of \( n \) and
\[
\lambda_n = \frac{b}{n} + \frac{4\alpha b^2 + 1}{2n} + \frac{12\alpha^2 b^4 + 18\alpha b^2 + 1}{3n^2}.
\]
Proof. By using Theorem 4, we obtain
\[
\left| S_n^\alpha (f; x) - f(x) \right| \leq 2K(f, \lambda_n),
\] The proof is completed by choosing \( \delta = \lambda_n \) in (3.6).

Theorem 6. Let \( f \in C_B^2 [0, \infty) \) and \( x \in [0, \infty) \) is a fixed point. Then, we have
\[
\lim_{n \to \infty} n \left[ S_n^\alpha (f; x) - f(x) \right] = \frac{1}{2} \left[ 4\alpha x^2 + 1 \right] f'(x) + \lambda f'(x) + \lambda f(x) .
\] (3.12)
Proof. By Taylor formula for the function \( f \), we get
\[
f(t) = f(x) + f'(x)(t - x) + f''(x) \frac{(t - x)^2}{2} + (t - x)^2 \mu(t, x),
\] where \( \mu(t, x) \in C_B [0, \infty) \) and \( \lim_{t \to x} \mu(t, x) = 0 \).

When we apply the operators \( S_n^\alpha \) to both sides of the aforementioned equality and recall the linearity property of the operators \( S_n^\alpha \), we obtain
\[
S_n^\alpha (f; x) - f(x) = f'(x)S_n^\alpha ((t - x); x) + \frac{f''(x)}{2} S_n^\alpha ((t - x)^2; x) + S_n^\alpha \left( (t - x)^2 \mu(t, x); x \right).
\]
By Lemma 2, we get
\[
S_n^\alpha (f; x) - f(x) = f'(x)\psi_1 + \frac{f''(x)}{2} \psi_2 + S_n^\alpha \left( (t - x)^2 \mu(t, x); x \right). \] (3.13)
By Cauchy-Schwarz inequality, we have
\[
nS_n^\alpha \left( (t - x)^2 \mu(t, x); x \right) \leq 2 \left[n^2 S_n^\alpha \left( (t - x)^2; x \right) \right]^{\frac{1}{2}} \left[ S_n^\alpha \left( \mu^2(t, x); x \right) \right]^{\frac{1}{2}}.
\]
It is clear that \( \mu^2(t, x) = 0 \) and \( \mu^2(t, x) \) is bounded. Then, we get
\[
\lim_{n \to \infty} S_n^\alpha \left( \mu^2(t, x); x \right) = \mu^2(x, x) = 0 .
\]
So, we obtain
\[
\lim_{n \to \infty} n S^G_n \left( (t-x)^2 \mu(t,x); x \right) = 0.
\] (3.14)

Now, we can write the following equality from (3.13) and (3.14)

\[
\lim_{n \to \infty} n \left[ S^G_n (f;x) - f(x) \right] = \frac{1}{2} \left[ (4ax^2 + 1)f'(x) + xf''(x) \right].
\]

The proof is done.

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REFERENCES


