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A NOTE ON RELATION BETWEEN POINT-LINE DISPLACEMENT AND EQUIFORM TRANSFORMATION

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Abstract − The present paper studies the relation between the point-line displacement and the equiform transformation in Euclidean 3-space $\mathbb{R}^3$. A point-line can be transformed into another point-line via an equiform transformation. Observing that a point-line is nothing but a line element when its reference point is the origin of the coordinate system, we show that this transformation can also be performed by using dual quaternions.

Keywords − *Line geometry, Line element, Dual quaternion, Equiform kinematics.*

1 Introduction

In kinematics, a point-line is represented by an oriented (directed) line and an incident point on this line. The point-line in kinematics has many implementation areas in manufacturing. Zhang and Ting [8] examine the point-line positions and displacement with the help of dual quaternion algebra. On the other hand, Odehnal, Pottmann and Wallner [1] investigate Plücker coordinates of the line elements in Euclidean three-space $\mathbb{R}^3$. Also, the relation between the point-line displacement and the equiform transformation in Minkowski 3-space is studied in [7].

Our interest in this paper is to investigate the relation between point-line representations and equiform kinematics in Euclidean 3-space $\mathbb{R}^3$. In Section 2, we give dual quaternions and some of their algebraic properties. Then in Section 3, we give the point-line operator, the equiform transformation and the Plücker coordinates of line elements in Euclidean 3-space $\mathbb{R}^3$. We examined the similarity between a point-line and a line element. Finally, we introduce the point-line operator which transforms one point-line to another.

2 Preliminaries

In this section, we give some definitions and fundamental facts about Euclidean three-space $\mathbb{R}^3$, that will be used through the paper.

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2.1 Some Properties of Euclidean 3-space $\mathbb{R}^3$

Theorem 2.1. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be two vectors in Euclidean three-space $\mathbb{R}^3$. Then,
i. $\vec{u} \times (\vec{v} \times \vec{w}) = \langle \vec{u}, \vec{w} \rangle \vec{v} - \langle \vec{u}, \vec{v} \rangle \vec{w}$,
ii. $\langle \vec{u} \times \vec{v}, \vec{u} \times \vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle - \langle \vec{u}, \vec{v} \rangle^2$,
where $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1) \end{aligned}$$

is the *vector product* in $\mathbb{R}^3$.

Let $\mathbb{R}^m_n$ be the set of matrices of m rows and n columns.

Definition 2.2. Let $A = [a_{ij}] \in \mathbb{R}^m_n$ and $B = [b_{jk}] \in \mathbb{R}^n_p$. Matrix multiplication is defined as

$$AB = \left[\sum_{j=1}^{n} a_{ij} b_{jk} \right]. \tag{1}$$

Note that AB is an $m \times p$ matrix.

Definition 2.3. An $n \times n$ identity matrix with respect to matrix multiplication, denoted by I_n, is given by

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}. \tag{2}$$

Note that for every $A \in \mathbb{R}^n_n$, $I_n A = A I_n = A$.

Definition 2.4. A matrix $A \in \mathbb{R}^n_n$ is called invertible if there exists an $n \times n$ matrix B such that $AB = BA = I_n$. Then B is called the inverse of A and is denoted by A^{-1}.

Definition 2.5. The transpose of a matrix $A = [a_{ij}] \in \mathbb{R}^m_n$ is denoted by A^T and defined as $A^T = [a_{ji}] \in \mathbb{R}^n_m$.

Definition 2.6. A matrix $A \in \mathbb{R}^n_n$ is called orthogonal matrix if $A^{-1} = A^T$.

2.2 Dual Quaternions

In analogy with the complex numbers, W. K. Clifford, defined [2] the dual numbers and showed that they form an algebra. As the dual numbers are defined by

$$\begin{aligned} D &= \{A = a + \varepsilon a^* \mid a, a^* \in R\} && (3) \\ &= \{A = (a, a^*) \mid a, a^* \in R\}, && (4) \end{aligned}$$

where ε is the dual symbol subjected to the rules

$$\varepsilon \neq 0,\ 0\varepsilon = \varepsilon 0 = 0,\ 1\varepsilon = \varepsilon 1 = \varepsilon,\ \varepsilon^2 = 0.$$

The set D of dual numbers is a commutative ring with the operations $(+)$ and $(\cdot)$.
The algebra

$$H = \{q = q_0 + q_1 \vec{e}_1 + q_2 \vec{e}_2 + q_3 \vec{e}_3 \mid q_0,\ q_1,\ q_2,\ q_3 \in R\}$$

of quaternions is defined as the four-dimensional vector space over R having basis $\{1, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ with the following properties:

$$\begin{aligned} &1)\ (\vec{e}_1)^2 = (\vec{e}_2)^2 = (\vec{e}_3)^2 = 1, \\ &2)\ \vec{e}_1\vec{e}_2 = -\vec{e}_2\vec{e}_1 = \vec{e}_3,\ \vec{e}_2\vec{e}_3 = -\vec{e}_3\vec{e}_2 = \vec{e}_1,\ \vec{e}_3\vec{e}_1 = -\vec{e}_1\vec{e}_3 = \vec{e}_2. \end{aligned} \tag{5}$$

It is clear that H is an associative and not commutative algebra and 1 is the identity element of H. H is called quaternion algebra (see [4] for quaternions).

Similarly, as a consequence of this definition, a dual quaternion Q can also be written as

$$Q = q + \varepsilon q^*,$$

where q and q^* are quaternions.

A dual quaternion

$$Q = q + \varepsilon q^*$$

is characterized by the following properties in [4]:

Scalar and vector parts of a dual quaternion $Q = A_0 + A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$ are denoted by $S_Q = A_0$ and $\vec{V}_Q = A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3$, respectively. The basis $\{1,\ \vec{e}_1,\ \vec{e}_2,\ \vec{e}_3\}$ have the same multiplication properties of basis elements in real quaternions.

Two dual quaternions Q and P obey the following multiplication rule,

$$QP = (qp) + \varepsilon\,(qp^* + pq^*)$$

where $P = p + \varepsilon p^*$, p and p^* are quaternions.

Scalar product of quaternions Q and P is given by

$$\begin{aligned}\langle Q,\ P\rangle &= \langle P,\ Q\rangle \\ &= \langle q,\ p\rangle + \varepsilon\,(\langle q,\ p^*\rangle + \langle q^*,\ p\rangle)\,. \end{aligned} \tag{6}$$

3 Point-line Displacement with Equiform Transformations of $\mathbb{R}^3$

In [1], a point-line is represented by an oriented (directed) line and an incident point on this line. Moreover, an oriented (directed) line can be represented with a unit line vector or signed Plücker coordinates. Thus, we can say the point-line representation can be built up as a dual vector or signed Plücker coordinates.

Let L be an oriented (directed) line and P be a reference point in Euclidean three-space $\mathbb{R}^3$. If we take N as the foot of the perpendicular from P to the directed line L and E is an incident on this directed line L, then the distance h from N to E depends on the location of E and the oriented (directed) line L, (see Fig. 1).

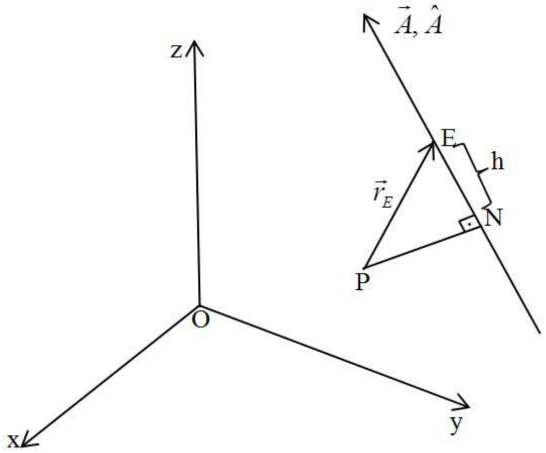

Figure 1. Point-line representation (7)

The oriented (directed) line L passing through points E and N can be represented by a unit dual vector.

Let $\vec{A}=\vec{a}+\varepsilon\overrightarrow{a}_0$ be a unit dual vector satisfying $\langle\vec{a},\ \vec{a}\rangle=1$ and $\langle\vec{a},\overrightarrow{a}_0\rangle=0$ where the vector $\overrightarrow{a}$ denotes the unit vector along the oriented line, and the vector $\overrightarrow{a}_0$ is the moment vector of the oriented line with respect to the origin of reference frame $O-xyz$.
A point-line can be represented by multiplication of a dual number $exp(\varepsilon h)=1+\varepsilon h$, and $\vec{A}$, namely

$$\begin{aligned}\hat{A}&=exp(\varepsilon h)\vec{A}\\&=\left\|\hat{A}\right\|\vec{A}\\&=\vec{a}+\varepsilon\overrightarrow{a}'_0,\end{aligned}\tag{8}$$

where $\overrightarrow{a}'_0=\overrightarrow{a}_0+h\vec{a}$ and $\hat{A}$ is a dual vector with dual length $exp(\varepsilon h)$.
When we have the point-line coordinates, the incident offset, the directed line, and the incident can be determined easily. Then,

$$\vec{A}=\vec{a}+\varepsilon\left(\overrightarrow{a}'_0-h\vec{a}\right),\tag{9}$$

and

$$h=g(\vec{a},\overrightarrow{a}'_0).\tag{10}$$

Here, the value of h changes related to the reference point. Without losing generality, if we assume that the reference point is the origin of the coordinate system, we can write the position vector of the incident E as

$$\vec{r}_E=\overrightarrow{PN}+\overrightarrow{NE},$$

where $\overrightarrow{a}_0=\overrightarrow{PN}\times\vec{a}$ and $\overrightarrow{NE}=h\vec{a}$. Therefore, from Theorem 2.1 and $\overrightarrow{a}'_0=\overrightarrow{a}_0+h\vec{a}$, the position vector $\vec{r}_E$ of the incident E is

$$\begin{aligned}\vec{r}_E&=\vec{a}\times\overrightarrow{a}_0+h\vec{a}\\&=\vec{a}\times\overrightarrow{a}'_0+\langle\vec{a},\overrightarrow{a}'_0\rangle\,\vec{a},\end{aligned}$$

where $\times$ is the cross-product.

3.1 Equiform Transformations

This section describes equiform transformations, which means affine transformations whose linear part is composed from an orthogonal transformation and a homothetical transformation in Euclidean three-space $\mathbb{R}^3$.
Such an equiform transformation maps points $x\in\mathbb{R}^3$ by using

$$\begin{aligned}&\varphi:\mathbb{R}^3\longrightarrow\mathbb{R}^3\\&\quad x\longrightarrow\varphi(x)=y(t)=\alpha(t)D(t)x+b(t),\end{aligned}\tag{11}$$

where $D\in O\left(3\right),\ b\in\mathbb{R}^3$ and α is a homothetic scale. D, α and b are differentiable functions of class C^∞ of a parameter t.
The velocity $\dot{y}(t)$ has the form

$$v\left(y\right)=\dot{D}D^Ty+\frac{\dot{\alpha}}{\alpha}y-\dot{D}D^Tb-\frac{\dot{\alpha}}{\alpha}b+\dot{b},\tag{12}$$

where $v\left(y\right)=\dot{y}(t)=\frac{dy}{dt}$.
Since D is orthogonal, the matrix $\dot{D}D^T:=C^\times$ is skew-symmetric and the product $C^\times x$ can be written in the form $c\times x$ in Euclidean three-space $\mathbb{R}^3$:

$$v\left(y\right)=c\times y+\gamma y+\bar{c},\tag{13}$$

where $\gamma=\frac{\dot{\alpha}}{\alpha}$ and $\bar{c}=\dot{D}D^Tb-\frac{\dot{\alpha}}{\alpha}b+\dot{b}$.
Any triple $(c,\bar{c},\gamma)\in\mathbb{R}^7$ defines a uniform equiform motion in Euclidean three-space $\mathbb{R}^3$, uniquely *[1]*.

3.2 Plücker Coordinates of Line Elements

Let L be an oriented (directed) line in Euclidean three-space $\mathbb{R}^3$ passing through a point $\vec{x}$. In order to assign coordinates to the line element $(L, \vec{x})$, we use the familiar definition of Plücker coordinates. The triple $(\vec{a}, \overrightarrow{a}_0, h) \in \mathbb{R}^7$ is called the Plücker coordinates of the line element $(L, \vec{x})$ in $\mathbb{R}^3_1$, if $\vec{a} \neq \vec{0}$ is parallel to L, then $\overrightarrow{a}_0 = \vec{x} \times \vec{a}$, $h = \langle \vec{x}, \vec{a} \rangle$. It is easy to show that

$$\vec{x} = N(\vec{a}, \overrightarrow{a}_0) + h\vec{a}, \tag{14}$$

where $N(\vec{a}, \overrightarrow{a}_0) = \vec{a} \times \overrightarrow{a}_0$.
The point $N(\vec{a}, \overrightarrow{a}_0)$ is the foot point of the origin on the line L. We know that Plücker coordinates satisfy $\langle \vec{a}, \overrightarrow{a}_0 \rangle = 0$, and $\vec{a} \neq \vec{0}$ occurs as coordinates of lines in $\mathbb{R}^3$. Therefore, from (14) we obtain the equation

$$\vec{x} = \vec{a} \times \overrightarrow{a}_0 + h\vec{a}, \tag{15}$$

where $h = \langle \vec{x}, \vec{a} \rangle$ and $\vec{a}$ is a unit parallel vector to the line L.

If the corresponding line has an orientation, then a line element becomes oriented. The equiform transformation (11) transforms the line element $(\vec{a}, \overrightarrow{a}_0, h_1)$ into $(\vec{u}, \vec{u}_0, h_2)$ with $\vec{x}' = \alpha R\vec{x} + \vec{b}$, $\vec{u} = R\vec{a}$, $\vec{u}_0 = \vec{x}' \times \vec{u}$, $h_2 = \langle \vec{x}', \vec{u} \rangle$. In block matrix form, this transformation reads

$$\begin{bmatrix} \vec{u} \\ \vec{u}_0 \\ h_2 \end{bmatrix} = \begin{bmatrix} D & 0 & 0 \\ D^{\times} D & \alpha D & 0 \\ \vec{b}^T D & 0^T & \alpha \end{bmatrix} \begin{bmatrix} \vec{a} \\ \overrightarrow{a}_0 \\ h_1 \end{bmatrix}, \tag{16}$$

where $D \in O(3)$, $b \in \mathbb{R}^3$, α is a homothetic scale D, $D^{\times}\vec{x} = \vec{b} \times \vec{x}$, $\vec{A} = \vec{a} + \varepsilon \overrightarrow{a}_0$, $\langle \vec{a}, \vec{a} \rangle = 1$, $\langle \vec{a}, \overrightarrow{a}_0 \rangle = 0$ and $\vec{U} = \overrightarrow{u} + \varepsilon \overrightarrow{u}_0$, $\langle \overrightarrow{u}, \overrightarrow{u} \rangle = 1$, $\langle \overrightarrow{u}, \overrightarrow{u}_0 \rangle = 0$, ([1]).
Using the correspondence between line elements and point-lines we observe the following:

Conclusion 3.1. Let $\hat{A} = \left\| \hat{A} \right\| \vec{A}$ and $\hat{U} = \left\| \hat{U} \right\| \vec{U}$ be two point-lines. When the reference point is chosen as the origin of the coordinate system for a point-line, the transformations (16) transform the point-line $\hat{A}$ to the point-line $\hat{U}$ if $\vec{A}$ is a unit dual quaternion vector.

We can obtain the oriented (directed) line elements in the equation (16) by using dual quaternions. Moreover, we also can transform a point-line to another point-line by using dual quaternions with the following theorem.

Theorem 3.2. A dual quaternion Q transforms a given point-line to another given point-line and is defined by

$$Q = \frac{1}{\left\| \hat{A} \right\|^2} \left(\left\langle \hat{A}, \hat{U} \right\rangle + \left(\hat{A} \times \hat{U} \right) \right), \tag{17}$$

where $\hat{A}$ and $\hat{U}$ denoted two point-lines, $\times$ is cross product and the Q is called the point-line operator which acts on point-lines.

Proof. Let $\hat{A}$ and $\hat{U}$ be two point-lines defined by $\hat{A} = \left\| \hat{A} \right\| \vec{A}$ and
$\hat{U} = \left\| \hat{U} \right\| \vec{U}$. Here, from the Eq. (8) $\vec{A}$ and $\vec{U}$ are unit dual vectors, dual length $\left\| \hat{A} \right\| = \exp \varepsilon(h_1)$ of $\hat{A}$ and dual length $\left\| \hat{U} \right\| = \exp \varepsilon(h_2)$ of $\hat{U}$.
If we apply quaternion multiplication to the Eq. (17) with $\hat{A}$ from right-side, then we have

$$Q\hat{A} = \frac{1}{\left\| \hat{A} \right\|^2} \left[\left\langle \hat{A}, \hat{U} \right\rangle \hat{A} + (\hat{A} \times \hat{U}) \times \hat{A} \right]$$

and from Theorem 2.1 we have

$$Q\hat{A} = \frac{1}{\left\| \hat{A} \right\|^2} \left[\left\langle \hat{A}, \hat{U} \right\rangle \hat{A} + \left\langle \hat{A}, \hat{A} \right\rangle \hat{U} - \left\langle \hat{A}, \hat{U} \right\rangle \hat{A} \right]$$

and from $\left\langle \hat{A}, \hat{A} \right\rangle = 1$

$$Q\hat{A} = \hat{U}.$$

Also, since

$$\begin{aligned} \hat{A} &= \left\| \hat{A} \right\| \vec{A}, \\ \hat{U} &= \left\| \hat{U} \right\| \vec{U}, \end{aligned}$$

Eq. (17) can be modified

$$Q = \frac{\left\| \hat{U} \right\|}{\left\| \hat{A} \right\|} (\left\langle \vec{A}, \vec{U} \right\rangle + \left(\vec{A} \times \vec{U} \right)),$$

and from the Eq. (8) since $\left\| \hat{A} \right\| = \exp \varepsilon(h_1)$ and $\left\| \hat{U} \right\| = \exp \varepsilon(h_2)$, the last equation can be rewritten as

$$Q = \{\exp [\varepsilon(h_2 - h_1)]\} Q_0,$$

where $\frac{\|\hat{U}\|}{\|\hat{A}\|} = \exp [\varepsilon(h_2 - h_1)]$ is dual length of Q and $Q_0 = \left\langle \vec{A}, \vec{U} \right\rangle + \left(\vec{A} \times \vec{U} \right)$.
Because $\left\langle \vec{A}, \vec{U} \right\rangle$ is the scalar part of Q_0 and $\left(\vec{A} \times \vec{U} \right)$ is the vector part of Q_0, then Q is a dual quaternion.

Example 3.3. Let $\hat{A} = \left(0,\ \frac{1}{2},\ \frac{\sqrt{3}}{2},\ \frac{\sqrt{3}}{2},\ 0,\ 0,\ \frac{1}{2}\right)$
and $\hat{U} = \left(0,\ 0,\ 1,\ -\frac{3}{2},\ -1,\ 0,\ -\frac{\sqrt{3}}{2}\right)$ be two point-lines in $\mathbb{R}^7$. Since from the Eq. (8)

$$\hat{A} = \underbrace{\left(1 + \frac{\varepsilon}{2}\right)}_{\|\hat{A}\|} \underbrace{\left[\left(0,\ \frac{1}{2},\ \frac{\sqrt{3}}{2}\right) + \varepsilon \left(\frac{\sqrt{3}}{2},\ 0,\ 0\right) \right]}_{\vec{A}}$$

and

$$\hat{U} = \underbrace{\left(1 - \frac{\sqrt{3}}{2}\varepsilon\right)}_{\|\hat{U}\|} \underbrace{\left[(0,\ 0,\ 1) + \varepsilon \left(-\frac{3}{2},\ -1,\ 0\right) \right]}_{\vec{U}},$$

from the Eq. (17) it can be written

$$Q = \left(1 - \frac{\sqrt{3}+1}{2}\varepsilon\right) (\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\varepsilon\right) + \left(\left(\frac{1}{2},\ 0,\ 0\right) + \varepsilon \left(\frac{\sqrt{3}}{2},\ -\frac{5}{4}\sqrt{3},\ \frac{3}{4}\right)\right)).$$

If we apply quaternion multiplication to Q with $\hat{A}$ from right-side, then we have

$$\begin{aligned} Q\hat{A} &= \left(1 - \frac{\sqrt{3}}{2}\varepsilon\right) \left[(0,\ 0,\ 1) + \varepsilon \left(-\frac{3}{2},\ -1,\ 0\right) \right] \\ &= \hat{U}. \end{aligned}$$

4 Conclusion

In this study, we used a block matrix to transform a given point-line to another given one that is given in [1]. We prove that dual quaternions can be used to map a given point-line to another given one. Since it is compact, free of redundancies and easier to compute compared to the matrix given in the Eq. (16), this approach has some advantages.

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