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THE HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR p-CONVEX FUNCTIONS IN HILBERT SPACE

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Abstract − In this paper, we introduce operator p-convex functions and establish some Hermite-Hadamard type inequalities in which some operator p-convex functions of positive operators in Hilbert spaces are involved.

Keywords − *The Hermite-Hadamard inequality, p-convex functions, operator p-convex functions, selfadjoint operator, inner product space, Hilbert space.*

1 Introduction

The following inequality holds for any convex function f define on $\mathbb{R}$ and $a, b \in \mathbb{R}$, with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_0^1 f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

both inequalities hold in the reversed direction if f is concave.

The inequality (1) is known in the literature as the Hermite-Hadamard's inequality. The Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

In this paper, Firstly we defined for bounded positive selfadjoint operator p-convex functions in Hilbert space, secondly established some new theorems for them and finally Hermite-Hadamard type inequalities for product two bounded positive selfadjoint operators p-convex set up in Hilbert space.

In the paper [1] Dragomir et al. consider $P(I)$. This class is defined in the following way.

Definition 1.1. [1] We say that $f : I \to \mathbb{R}$ is a P-function, or that f belongs to the class $P(I)$, if f is a non-negative function and for all $x, y \in I, \alpha \in [0, 1]$, we have

$$f\big(\alpha x + (1-\alpha)y\big) \leq f(x) + f(y).$$

For some results about the class $P(I)$ see, e.g., [2] and [3].

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2 Preliminary

First, we review the operator order in $B(H)$ and the continuous functional calculus for a bounded selfadjoint operator. For selfadjoint operators $A, B \in B(H)$ we write, for every $x \in H$

$$A \leq B(\text{or } B \geq A) \text{ if } \langle Ax, x\rangle \leq \langle Bx, x\rangle (\text{or } \langle Bx, x\rangle \geq \langle Ax, x\rangle)$$

we call it the operator order.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H, \langle ., .\rangle)$ and $C(Sp(A))$ the C^* -algebra of all continuous complex-valued functions on the spectrum A. The Gelfand map establishes a $*$-isometrically isomorphism Φ between $C(Sp(A))$ and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows [6].

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

i. $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

ii. $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

iii. $\|\Phi(f)\| = \|f\| := sup_{t\in Sp(A)}|f(t)|$;

iv. $\Phi(f_0) = 1$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$

If f is a continuous complex-valued functions on $C(Sp(A))$, the element $\Phi(f)$ of $C^*(A)$ is denoted by $f(A)$, and we call it the continuous functional calculus for a bounded selfadjoint operator A.

If A is bounded selfadjoint operator and f is real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq f(B)$ in the operator order $B(H)$.

A real valued continuous function f on an interval I is said to be operator convex (operator concave) if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda \in [0, 1]$ and for every bounded self-adjoint operator A and B in $B(H)$ whose spectra are contained in I.

3 Operator p-convex Functions in Hilbert Space

The following definition and function class are firstly defined by Seren Salaş.

Definition 3.1. Let I be interval in $\mathbb{R}$ and K be a convex subset of $B(H)^+$. A continuous function $f : I \to \mathbb{R}$ is said to be operator p-convex on I, operators in K if

$$f\big(\alpha A + (1-\alpha)B\big) \leq f(A) + f(B) \tag{2}$$

in the operator order in $B(H)$, for all $\alpha \in [0, 1]$ and for every positive operators A and B in K whose spectra are contained in I .

In the other words, if f is an operator p-convex on I, we denote by $f \in S_pO$.

Lemma 3.2. If f belongs to S_pO for operators in K, then $f(A)$ is positive for every $A \in K$.

Proof. For $A \in K$, we have

$$f(A) = f\left(\frac{A}{2} + \frac{A}{2}\right) \leq f(A) + f(A) = 2f(A).$$

This implies that $f(A) \geq 0$.

Moslehian and Najafi [4] proved the following theorem for positive operators as follows :

Theorem 3.3. [4] Let $A, B \in B(H)^+$. Then $AB+BA$ is positive if and only if $f(A+B) \leq f(A)+f(B)$ for all non-negative operator functions f on $[0, \infty)$.

Dragomir in [5] has proved a Hermite-Hadamard type inequality for operator convex function as follows:

Theorem 3.4. [5] Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for all selfadjoint operators A and B with spectra in I we have the inequality

$$\begin{aligned}\left(f\left(\frac{A+B}{2}\right)\leq\right) \quad & \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right]\\ \leq\ & \int_0^1 f\Big(\Big(1-t)A+tB\Big)\Big)dt\\ \leq\ & \frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right]\left(\leq\left(\frac{f(A)+f(B)}{2}\right)\right].\end{aligned}$$

Let X be a vector space, $x, y \in X, x \neq y$. Define the segment

$$[x, y] := (1-t)x + ty; t \in [0, 1].$$

We consider the function $f : [x, y] :\to \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \to \mathbb{R}$$

$$g(x, y)(t) := f((1-t)x + ty), t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \in X$, we have the Hermite-Hadamard integral inequality

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f((1-t)x + ty)dt \leq \frac{f(x)+f(y)}{2}$$

which can be derived from the classical Hermite-Hadamard inequality for the convex $g(x, y) : [0, 1] \to \mathbb{R}$.

Lemma 3.5. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function on the interval I. Then for every two positive operators $A, B \in K \subseteq B(H)^+$ with spectra in I the function $f \in S_pO$ for operators in

$$[A, B] := (1-t)A + tB; t \in [0, 1]$$

if and only if the function $\varphi_{x,A,B} : [0, 1] \to \mathbb{R}$ defined by

$$\varphi_{x,A,B} := \big\langle f((1-t)A + tB)x, x\big\rangle$$

is operator p-convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$.

Proof. Since $f \in S_pO$ operator in $[A, B]$, then for any $t_1, t_2 \in [0, 1]$ and $\alpha \in [0, 1]$ we have

$$\begin{aligned}\varphi_{x,A,B}(\alpha t_1 + (1-\alpha)t_2) &= \big\langle f((1-(\alpha t_1 + (1-\alpha)t_2)A + (\alpha t_1 + (1-\alpha)t_2)B)x, x\big\rangle\\ &= \big\langle f(\alpha[(1-t_1)A + t_1 B] + (1-\alpha)[(1-t_2)A + t_2 B])x, x\big\rangle\\ &\leq \big\langle f((1-t_1)A + t_1 B)x, x\big\rangle + f((1-t_2)A + t_2 B)x, x\big\rangle\\ &\leq \varphi_{x,A,B}(t_1) + \varphi_{x,A,B}(t_2)\end{aligned}$$

Theorem 3.6. Let $f \in S_pO$ on the interval $I \subseteq [0, \infty)$ for operators $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality

$$\frac{1}{2}f\left(\frac{A+B}{2}\right) \leq \int_0^1 f\big(tA + (1-t)B\big)dt \leq \big[f(A) + (B)\big] \tag{3}$$

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$\big\langle((1-t)A + tB)x, x\big\rangle = (1-t)\big\langle Ax, x\big\rangle + t\big\langle Bx, x\big\rangle \in I, \tag{4}$$

Since $\big\langle Ax, x\big\rangle \in Sp(A) \subseteq I$ and $\big\langle Bx, x\big\rangle \in Sp(B) \subseteq I$.

Continuity of f and 4 imply that the operator-valued integral $\int_0^1 f\big(tA + (1-t)B\big)dt$ exists.

Since f is operator p-convex, therefore for t in $[0, 1]$, and $A, B \in K$ we have

$$f\big(tA + (1-t)B\big)dt \le f(A) + f(B) \tag{5}$$

Integrating both sides of 5 over $[0, 1]$ we get the following inequality

$$\int_0^1 f\big(tA + (1-t)B\big)dt \le f(A) + f(B)$$

To prove the first inequality of 3, we observe that

$$f\left(\frac{A+B}{2}\right) \le f\big(tA + (1-t)B\big) + f\big((1-t)A + tB\big) \tag{6}$$

Integrating the inequality 6 over $t \in [0, 1]$ and taking into account that

$$\int_0^1 f\big(tA + (1-t)B\big)dt = \int_0^1 f\big((1-t)A + tB\big)dt$$

then we deduce the first part of 3.

4 The Hermite-Hadamard Type Inequality for the Product Two Operators p-convex Functions

Let $f, g \in S_pO$ on the interval in I. Then for all positive operators A and B on a Hilbert space H with spectra in I, we define real functions $M(A, B)$ and $N(A, B)$ on H by

$$M(A, B)(x) = \langle f(A)x, x\rangle\langle g(A)x, x\rangle + \langle f(B)x, x\rangle\langle g(B)x, x\rangle \ (x \in H),$$
$$N(A, B)(x) = \langle f(A)x, x\rangle\langle g(B)x, x\rangle + \langle f(B)x, x\rangle\langle g(A)x, x\rangle \ (x \in H).$$

Theorem 4.1. Let $f, g \in S_pO$ be on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality

$$\int_0^1 \langle f\big(tA + (1-t)B\big)x, x\rangle\langle g\big(tA + (1-t)B\big)x, x\rangle dt$$
$$\le M(A, B) + N(A, B)$$

hold for any $x \in H$ with $\|x\| = 1$.

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$\big\langle(A + B)x, x\big\rangle = \big\langle Ax, x\big\rangle + \big\langle Bx, x\big\rangle \in I, \tag{7}$$

since $\big\langle Ax, x\big\rangle \in Sp(A) \subseteq I$ and $\big\langle Bx, x\big\rangle \in Sp(B) \subseteq I$.

Continuity of f, g and 7 imply that the operator-valued integrals

$$\int_0^1 f\big(tA + (1-t)B\big)dt, \ \int_0^1 g\big(tA + (1-t)B\big)dt \text{ and } \int_0^1 (fg)\big(tA + (1-t)B\big)dt$$

exist.

Since $f, g \in S_pO$, therefore for t in $[0,1]$ and $x \in H$ we have

$$\langle f\big(tA + (1-t)B\big)x, x\rangle \leq \langle f(A) + f(B)x, x\rangle \tag{8}$$

$$\langle g\big(tA + (1-t)B\big)x, x\rangle \leq \langle g(A) + g(B)x, x\rangle. \tag{9}$$

From 8 and 9, we obtain

$$\begin{aligned} \langle f\big(tA + (1-t)B\big)x, x\rangle\langle g\big(tA + (1-t)B\big)x, x\rangle \leq\ & \langle f(A)x, x\rangle\langle g(A)x, x\rangle \\ & + \langle f(A)x, x\rangle\langle g(B)x, x\rangle \\ & + \langle f(B)x, x\rangle\langle g(A)x, x\rangle \\ & + \langle f(B)x, x\rangle\langle g(B)x, x\rangle \end{aligned} \tag{10}$$

Integrating both sides of 10 over $[0,1]$, we get the required inequality 7.

Theorem 4.2. Let f, g belong to S_pO on the interval I for operators in $K \subseteq B(H)^+$. Then for all positive operators A and B in K with spectra in I, we have the inequality

$$\frac{1}{2}\left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle\left\langle g\left(\frac{A+B}{2}\right)x, x\right\rangle \tag{11}$$

$$\begin{aligned} \leq\ & \int_0^1 \langle f\big(tA + (1-t)B\big)x, x\rangle\langle g\big(tA + (1-t)B\big)x, x\rangle dt \\ & + M(A, B) + N(A, B) \end{aligned} \tag{12}$$

hold for any $x \in H$ with $\|x\| = 1$.

Proof. Since $f, g \in S_pO$, therefore for any $t \in I$ and any $x \in H$ with $\|x\| = 1$, we observe that

$$\begin{aligned} & \left\langle f\left(\frac{A+B}{2}\right)x, x\right\rangle\left\langle g\left(\frac{A+B}{2}\right)x, x\right\rangle \\ \leq\ & \left\langle f\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right)x, x\right\rangle \\ & \times\left\langle g\left(\frac{tA + (1-t)B}{2} + \frac{(1-t)A + tB}{2}\right)x, x\right\rangle \end{aligned}$$

$$\begin{aligned} \leq\ & \Big\{\langle f\big(tA + (1-t)B\big)\rangle + \langle f\big((1-t)A + tB\big)\rangle \\ & \times\langle g\big(tA + (1-t)B\big)\rangle + \langle g\big((1-t)A + tB\big)\rangle\Big\} \end{aligned}$$

$$\begin{aligned} \leq\ & \Big\{\Big[\langle f\big(tA + (1-t)B\big)x, x\rangle\langle g\big(tA + (1-t)B\big)x, x\rangle\Big] \\ & + \Big[\langle f\big((1-t)A + tB\big)x, x\rangle\langle g\big((1-t)A + tB\big)x, x\rangle\Big] \\ & + \Big[\langle f(A)x, x\rangle + \langle f(B)x, x\rangle\Big] \times \Big[\langle g(A)x, x\rangle + \langle g(B)x, x\rangle\Big] \\ & + \Big[\langle f(A)x, x\rangle + \langle f(B)x, x\rangle\Big] \times \Big[\langle g(A)x, x\rangle + \langle g(B)x, x\rangle\Big]\Big\} \end{aligned}$$

$$
\begin{aligned}
= & \Big\{ \Big[\langle f\big(tA+(1-t)B\big)x,x\rangle g\big(tA+(1-t)B\big)x,x\rangle \Big] \\
& + \Big[\langle f\big((1-t)A+tB\big)x,x\rangle \langle g\big((1-t)A+tB\big)x,x\rangle \Big] \\
& + 2\Big[\langle f(A)x,x\rangle \langle g(A)x,x\rangle \Big] + 2\Big[\langle f(B)x,x\rangle \langle g(B)x,x\rangle \Big] \\
& + 2\Big[\langle f(A)x,x\rangle \langle g(B)x,x\rangle \Big] + 2\Big[\langle f(B)x,x\rangle \langle g(A)x,x\rangle \Big] \Big\}
\end{aligned}
$$

By integration over $[0,1]$, we obtain

$$
\begin{aligned}
& \left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle \left\langle g\left(\frac{A+B}{2}\right)x,x\right\rangle \\
\leq & \int_0^1 \Big[\langle f\big((1-t)A+tB\big)x,x\rangle \langle g\big(tA+(1-t)B\big)x,x\rangle \\
& + \langle f\big(tA+(1-t)B\big)x,x\rangle \langle g\big((1-t)A+tB\big)x,x\rangle \Big] dt \\
& + 2M(A,B) + 2N(A,B)
\end{aligned}
$$

This implies the inequality 11.

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