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COMMON FIXED POINT THEOREMS FOR f-CONTRACTION MAPPINGS IN TVS-VALUED CONE METRIC SPACE

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Abstract – We generalize the result of Abbas and Rhoades [1] and obtained some common fixed point results for two Banach pair of mappings which satisfies f-contraction condition on Topological vector space Valued Cone metric space (TVS-CMS) without the notion of normality condition of the cone.

***Keywords* –** *Fixed* point, TVS-CMS, f-contraction

1 Introduction.

Haung and Zhang [5] generalized the concept of metric Space, replacing the set of real numbers by an ordered Banach space, and obtained some fixed point theorems of contractive mappings on complete cone metric space with the assumption of the normality of the cone. Subsequently , various authors have generalized the result of Haung and Zhang and have studied fixed point theorems for normal and non-normal cones. In 2009, Beg et al [2] and in 2010 Du [4] generalized cone metric spaces to topological vector space valued cone metric spaces (TVS-CMS). In this approach ordered topological vector spaces are used as the co domain of the metric, instead of Banach spaces. While Beg et al used Hausdorff TVS, Du used locally convex Hausdorff TVS. However, a result in [10] shows that if the underlying cone of an ordered TVS is solid and normal it must be an ordered normed space. So, proper generalizations from Banach space valued cone metric space to TVS-CMS can be obtained only in the case of non normal cones. In [8] many authors have proved some common fixed point theorems for a Banach pair of mappings satisfying $f-$

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Hardy- Rogers type contraction condition in cone metric space. Also Morales and Rojas [6-7] have extended the notion of $f-$contraction mappings to Cone metric space by proving fixed point theorems for $f-$Kannan, $f-$Zamfirescu, $f-$weakly contraction mappings. In this paper, we recall the definition which was introduced and called Banach operator of type k by Subrahmanyam [9] . Chen and Li [3] also proved some best approximation result using common fixed point theorems for $f-$non expensive mappings. Here we generalize the contraction mappings which were given in [1] and obtained some common fixed point results for two Banach pair of mappings which satisfies f-contraction condition on TVS-CMS without the notion of normality condition of the cone.

2 Preliminaries

Definition 2.1. A vector space V over a field K ($\mathbb{R}$ or $\mathbb{C}$) is said to be TVS over K if it is furnished with a topology τ such that the vector space operation are continuous. i.e,

The addition operation $(x, y) \to x + y$ as a function from $V \times V \to V$ is continuous.

The scalar multiplication operation $(a, x) \to a.x$ as a function from $K \times V \to V$ is continuous.

In this case, one says that τ is a vector topology or a linear topology on the vector space V, or that τ is compatible with the linear structure of V.

Definition 2.2 [2-4] Let X be a non empty set. A mapping $d : X \times X \to E$ satisfying
1. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y.$
2. $d(x, y) = d(y, x)$, and
3. $d(x, y) \leq d(x, z) + d(z, y)\ \ \forall\, x, y, z \in X$

Is called a TVS-valued cone metric on X. The pair (X, d) is called TVS-CMS.

Definition 2.3. Let (X, d) be a TVS-CMS, and let $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then

i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$.

ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

iii) (X, d) is a complete TVS-CMS if every Cauchy sequence is convergent.

Definition 2.4. A self mapping f of a metric space (X, d) is called contraction if for any fixed constant r, $0 \leq r < 1$ and for all $x, y \in X$,

$$d(fx, fy) \leq r\, d(x, y)$$

Definition 2.5. [11] Let $\left(X,d\right)$ be a metric space and $T, f: X \to X$ be two function. A mapping T is said to be $f-$contraction if there exist $0 \le r < 1$ and for all $x, y \in X$

$$d\left(fTx, fTy\right) \le d\left(fx, fy\right).$$

Example 2.6. Let $X = [0,\infty)$ be with the usual metric. Let define two mappings $T, S: X \to X$ by,

$$Tx = \alpha x, \alpha > 1$$

$$Sx = \frac{\beta}{x^2}, \beta \in \mathbb{R}.$$

It is clear that , T is not a contraction but it is $f-$contraction since,

$$d(STx, STy) = \left|\frac{\beta}{\alpha^2 x^2} - \frac{\beta}{\alpha^2 y^2}\right| = \frac{1}{\alpha^2}\left|Sx - Sy\right|.$$

Definition 2.7. [9] Let f be a self mapping of a normed space X. Then f is called a Banach operator of type k if,

$$\left\| f^2 x - fx \right\| \le k \left\| fx - x \right\|$$

for some $k \ge 0$ and all $x \in X$.

Definition 2.8. [3] Let f and g be self mappings of a non empty subset M of a normed linear space X. Then $\left(f, g\right)$ is a Banach operator pair, if any one of the following conditions is satisfied.

1. $f\left[F(g)\right] \subseteq F(g)$.
2. $gfx = fx \;\; \forall \;\; x \in F(g)$
3. $fgx = gfx \;\;\; \forall \; x \in F(g)$
4. $\left\| fgx - gx \right\| \le r \left\| gx - x \right\|$ for some $r \ge 0$.

3 The results

Theorem 3.1. Let f, S and T be continuous self mappings of a complete TVS-cone metric space (X, d). Assume that f is injective mapping. If the mapping f, T and S satisfy,

$$d(fSx, fTy) \le \alpha d(fx, fy) + \beta[d(fx, fSx) + d(fy, fTy)] + \gamma[d(fx, fTy) + d(fy, fSx)] \qquad (1)$$

$\forall \; x, y \in X$ and $\alpha, \beta, \gamma \ge 0, \;\; \alpha + 2\beta + 2\gamma < 1,$ then T and S have a unique common fixed point in X . Moreover, if (f,T) and (f,S) are Banach pairs, then f,T and S have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary element and define the sequences $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ $\forall n \geq 0$ then by using (1) and triangle inequality,

$$d(fx_{2n+1}, fx_{2n}) = d(fSx_{2n}, fTx_{2n-1})$$

$$\leq \alpha d(fx_{2n}, fx_{2n-1}) + \beta[d(fx_{2n}, fSx_{2n}) + d(fx_{2n-1}, fTx_{2n-1})]$$
$$+\gamma[d(fx_{2n}, fTx_{2n-1}) + d(fTx_{2n-1}, fSx_{2n})]$$

$$\leq \alpha d(fx_{2n}, fx_{2n-1}) + \beta[d(fx_{2n}, fx_{2n+1}) + d(fx_{2n-1}, fx_{2n})]$$
$$+\gamma[d(fx_{2n}, fx_{2n}) + d(fx_{2n}, fx_{2n+1})]$$

$$\leq \alpha d(fx_{2n}, fx_{2n-1}) + \beta[d(fx_{2n-1}, fx_{2n}) + d(fx_{2n}, fx_{2n-1})]$$
$$+\gamma[d(fx_{2n}, fx_{2n-1}) + d(fx_{2n}, fx_{2n+1})]$$

$$d(fx_{2n+1}, fx_{2n}) = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} d(fx_{2n}, fx_{2n-1})$$

Similarly,

$$d(fx_{2n+3}, fx_{2n+2}) = d(fSx_{2n+2}, fTx_{2n+1})$$

$$\leq \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta[d(fx_{2n+2}, fSx_{2n+2}) + d(fx_{2n+1}, fTx_{2n+1})]$$
$$+\gamma[d(fx_{2n+2}, fTx_{2n+1}) + d(fTx_{2n+1}, fSx_{2n+2})]$$

$$\leq \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta[d(fx_{2n+2}, fx_{2n+3}) + d(fx_{2n+1}, fx_{2n+2})]$$
$$+\gamma[d(fx_{2n+2}, fx_{2n+2}) + d(fx_{2n+2}, fx_{2n+3})]$$

$$\leq \alpha d(fx_{2n+2}, fx_{2n+1}) + \beta[d(fx_{2n+2}, fx_{2n+3}) + d(fx_{2n+1}, fx_{2n+2})]$$
$$+\gamma[d(fx_{2n+2}, fx_{2n+1}) + d(fx_{2n+2}, fx_{2n+3})]$$

$$d(fx_{2n+3}, fx_{2n+2}) = \frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} d(fx_{2n+2}, fx_{2n+1})$$

Thus

$$d(fx_{n+1}, fx_n) \leq \lambda d(fx_n, fx_{n-1}) \leq \cdots\cdots\cdots \leq \lambda^n d(fx_1, fx_0)$$

for all $n \geq 0$ where,

$$\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)} < 1$$

Now for $n \geq m$ we get

$$d(fx_n, fx_m) \leq d(fx_n, fx_{n-1}) + d(fx_{n-1}, fx_{n-2}) + \cdots\cdots + d(fx_{m+1}, fx_m)$$

$$\leq (\lambda^{n-1} + \lambda^{n-2} + \cdots\cdots + \lambda^m) d(fx_1, fx_0)$$
$$\leq \frac{\lambda^m}{1-\lambda} d(fx_1, fx_0)$$

Let $0 \ll c$ be given. Choose $\delta > 0$ such that

$$c + N_\delta(0) \subseteq K \quad \text{,where}$$

$$N_\delta(0) = \{y \in E : \|y\| < \delta\}$$

Also, choose a natural number N_1 such that $\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \in N_\delta(0)$, for all $m \geq N_1$.

$$\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \ll c \text{ for all } m \geq N_1$$

Thus

$$d(fx_n, fx_m) \leq \frac{\lambda^m}{1-\lambda} d(fx_1, fx_0)$$

and

$$\frac{\lambda^m}{1-\lambda} d(fx_1, fx_0) \ll c$$

for all $m > n$. Then we get $d(fx_n, fx_m) \ll c \;\; \forall n > m$. Therefore, $\{fx_n\}$ is a Cauchy sequence in (X, d). As X is complete, there exist $q \in X$ such that, $\lim_{n\to\infty} fx_n = q$. Since f is subsequentially convergent, $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that

$\lim_{n\to\infty} x_m = u$. As f is continuous, $\lim_{m\to\infty} fx_m = fu$. By the uniqueness of limit, $q = fu$
.

Since T and S is continuous, $\lim_{m\to\infty} Tx_m = Tu$ and $\lim_{n\to\infty} Sx_m = Su$.

Again Since f is continuous, $\lim_{m\to\infty} fTx_m = fTu$ and $\lim_{m\to\infty} fSx_m = fSu$. Therefore, if m is

odd, then $\lim_{n\to\infty} fTx_{2n+1} = fTu$.

Choose a natural number N_2 such that,

$$d(fx_{2n+1}, fu) \ll \left[\frac{c}{2}\left(\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma}\right)\right] \quad \text{for all } n \geq N_2. \text{ Now consider,}$$

$$d(fu, fTu) \leq d(fu, fx_{2n+1}) + d(fx_{2n+1}, fTu)$$

$$= d(fu, fSx_{2n}) + d(fSx_{2n}, fTu)$$

$$\leq d(fu, fSx_{2n}) + \alpha d(fx_{2n}\, fu) + \beta(d(fx_{2n}, fSx_{2n}) + d(fu, fTu))$$

$$+\gamma(d(fx_{2n}, fTu) + d(fu, fSx_{2n}))$$

$$\leq d(fu, fx_{2n+1}) + \alpha d(fx_{2n}\, fu) + \beta(d(fx_{2n}, fx_{2n+1}) + d(fu, fTu))$$

$$+\gamma(d(fx_{2n}, fu) + d(fu, fTu) + d(fu, fx_{2n+1}))$$

$$\leq d(fu, fx_{2n+1}) + \alpha d(fx_{2n}\, fu) + \beta(d(fu, fx_{2n+1}) + d(fu, fTu))$$

$$+\gamma(d(fx_{2n}, fu) + d(fu, fTu) + d(fu, fx_{2n+1}))$$

$$= (1+\beta+\gamma)d(fu, fx_{2n+1}) + (\alpha+\gamma)d(fx_{2n}, fu) + (\beta+\gamma)d(fu, fTu)$$

So,

$$d(fu, fTu) \leq \left[\frac{1+\beta+\gamma}{1-\beta-\gamma}\right]d(fu, fx_{2n+1}) + \left[\frac{\alpha+\gamma}{1-\beta-\gamma}\right]d(fx_{2n}, fu) \ll c$$

for all $n \geq N_2$. Therefore, $d(fu, fTu) \ll \frac{c}{i}$ for all $i \geq 1$. Hence,

$$\frac{c}{i} - d(fu, fTu) \in K \ \forall\, i \geq 1$$

Since K is closed, $-d(fu, fTu) \in K$ and so $d(fu, fTu) = 0$. Hence, $fu = fTu$. As f is injective, $u = Tu$. Thus u is fixed point of T. Similarly, again by using (1) and triangular inequality , the rest can be proved if m is even, then we have,

$$\lim_{n\to\infty} fSx_{2n} = fSu\,.$$

For the uniqueness, suppose that v is another common fixed point of T and S. So,

$d(fu, fv) = d(fSu, fTv)$

$$\leq \alpha d(fu, fv)+\beta(d(fu, fSv)+d(fv, fTv))+\gamma(d(fu, fTv)+d(fv, fSu))$$

$$d(fu, fv)\leq(\alpha+2\gamma)d(fu, fv)$$

Since $(\alpha+2\gamma)<1$, $d(fu, fv)=0$ which implies that $fu=fv$. So $u=v$ is unique common fixed point of S and T.

Corollary 3.2. Let f and S be continuous self mappings of a complete TVS-cone metric space (X,d). Assume that f is injective mapping. If the mapping f and S satisfy,

$$d(fSx, fSy)\leq \alpha d(fx, fy)+\beta[d(fx, fSx)+d(fy, fSy)]+\gamma[d(fx, fSy)+d(fy, fSx)]$$

$\forall\ x,y\in X$ and $\alpha,\beta,\gamma\geq 0$, $\alpha+2\beta+2\gamma<1$, then S has a unique fixed point in X.

Corollary 3.3. Let f and S be continuous self mappings of a complete TVS-cone metric space (X,d). If the mapping f and S satisfy,

$$d(fSx, fSy)\leq \alpha d(fx, fy)+\beta[d(fx, fSx)+d(fy, fSy)]$$

$\forall\ x,y\in X$ and $\alpha,\beta\geq 0$, $\alpha+2\beta<1$. Then S has a unique fixed point in X.

Corollary 3.4. Let f and S be continuous self mappings of a complete TVS-cone metric space (X,d). If the mapping f and S satisfy,

$$d(fSx, fSy)\leq \beta(d(fx, fSx)+d(fy, fSy))$$

$\forall\ x,y\in X$ and $\beta\in[0,\frac{1}{2})$. Then S has a unique fixed point in X.

Corollary 3.5. Let f and S be continuous self mappings of a complete TVS-cone metric space (X,d). If the mapping f and S satisfy,

$$d(fSx, fSy)\leq \gamma(d(fx, fSy)+d(fy, fSx))$$

$\forall\ x,y\in X$ and $\gamma\in[0,\frac{1}{2})$. Then S has a unique fixed point in X.

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