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Upper and Lower Continuity of Fuzzy Soft Multifunctions

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Abstract – In this paper, we define the upper and lower inverse of a fuzzy soft multifunction and prove some basic identities. Then by using these ideas we introduced the concept of fuzzy soft continuity and obtain many interesting properties of upper and lower fuzzy soft continuous multifunctions.

Keywords – Fuzzy soft sets, Fuzzy soft multifunction, Fuzzy soft continuity.

1 Introduction

Engineering, physics, computer sciences, economics, social sciences, medical sciences and many other diverse fields deal with the uncertain data that may not be successfully modeled by the classical mathematics. There are some mathematical tools for dealing with uncertainties; two of them are fuzzy set theory, developed by Zadeh [24], and soft set theory, introduced by Molodtsov [19], that are related to this work. At present, works on the soft set theory and its applications are progressing rapidly. Maji et al [14] defined operations of soft sets to make a detailed theoretical study on the soft sets. By using these definitions, soft set theory has been applied in several directions, such as topology [5, 17, 22, 23, 25], various algebraic structures [2, 3, 7, 11], operations research [4, 9, 10] especially decision-making [6, 8, 13, 15, 20]. In recent times, researchers have contributed a lot towards fuzzification of soft set theory. Maji et al. [16] introduced the concept of fuzzy soft set and some properties regarding fuzzy soft union, intersection, complement of a fuzzy soft set, De Morgan Law etc. These results were further revised and improved by Ahmad and Kharal [1]. Tanay and Kandemir [23] introduced the definition of fuzzy soft topology over a subset of the initial universe set. Later, Roy and Samanta [21] gave the definition of fuzzy soft topology over the initial universe set. There are various types of functions which play an important role in the classical theory of set topology. A great deal of works on such functions has been extended to the setting of multifunctions. A multifunction is a set-valued function. The theory of multifunctions was first codified by Berge [26]. In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory, can be found for example, in economic theory, noncooparative games, artificial intelligence, medicine,

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information sciences and decision theory. Papageorgiou [27], Allbrycht and Maltoka [28] , Beg [29], Heilpein [30] and Butnairu [31] have started the study of fuzzy multifunctions and obtained several fixed point theorems for fuzzy mappings.

In this paper our purpose is two fold. First, we define upper and lower inverse of a fuzzy soft multifunction and study their various properties. Next, we use these ideas to introduce upper fuzzy soft continuous multifunctions and lower soft continuous multifunctions. Moreover, we obtain some characterizations and several properties concerning such multifunctions.

2 Preliminary

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes, characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by $P(X)$.

Definition 2.1. [24] A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \to [0,1]$ whose value $\mu_A(x)$ represents the "grade of membership" of x in A for $x \in X$.

Let I^X denotes the family of all fuzzy sets on X. If $A, B \in I^X$, then some basic set operations for fuzzy sets are given by Zadeh as follows:

(1) $A \leq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$, for all $x \in X$.
(2) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$, for all $x \in X$.
(3) $C = A \vee B \Leftrightarrow \mu_C(x) = \mu_A(x) \vee \mu_B(x)$, for all $x \in X$.
(4) $D = A \wedge B \Leftrightarrow \mu_D(x) = \mu_A(x) \wedge \mu_B(x)$, for all $x \in X$.
(5) $E = A^C \Leftrightarrow \mu_E(x) = 1 - \mu_A(x)$, for all $x \in X$.

A fuzzy point in X, whose value is α $(0 < \alpha \leq 1)$ at the support $x \in X$, is denoted by x_α [24]. A fuzzy point $x_\alpha \in A$, where A is a fuzzy set in X iff $\alpha \leq \mu_A(x)$ [24]. The class of all fuzzy points will be denoted by $S(X)$.

Definition 2.2. [18] For two fuzzy sets A and B in X, we write AqB to mean that A is quasi-coincident with B, i.e., there exists at least one point $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$. If A is not quasi-coincident with B, then we write $A\overline{q}B$.

Definition 2.3. [19] Let X be the initial universe set and E be the set of parameters. A pair (F, A) is called a soft set over X where F is a mapping given by $F : A \longrightarrow P(X)$ and $A \subseteq E$.

In the other words, the soft set is a parametrized family of subsets of the set X. Every set $F(e)$, for every $e \in A$, from this family may be considered as the set of e-elements of the soft set (F, A).

Definition 2.4. [16] Let $A \subseteq E$. A pair (f, A) is called a fuzzy soft set over X if $f : A \longrightarrow I^X$ is a function.

We will use $FS(X, E)$ instead of the family of all fuzzy soft sets over X.

Roy and Samanta [21] did some modifications in above definition analogously ideas made for soft sets.

Definition 2.5. [21] Let $A \subseteq E$. A fuzzy soft set f_A over universe X is mapping from the parameter set E to I^X, i.e., $f_A : E \longrightarrow I^X$, where $f_A(e) \neq 0_X$ if $e \in A \subset E$ and $f_A(e) = 0_X$ if $e \notin A$, where 0_X denotes empty fuzzy set on X.

Definition 2.6. [21] The fuzzy soft set $f_\emptyset \in FS(X, E)$ is called null fuzzy soft set, denoted by $\widetilde{0}_E$, if for all $e \in E$, $f_\emptyset(e) = 0_X$.

Definition 2.7. [21] Let $f_E \in FS(X, E)$. The fuzzy soft set f_E is called universal fuzzy soft set, denoted by $\widetilde{1}_E$, if for all $e \in E$, $f_E(e) = 1_X$ where $1_X(x) = 1$ for all $x \in X$.

Definition 2.8. [21] Let $f_A, g_B \in FS(X, E)$. f_A is called a fuzzy soft subset of g_B if $f_A(e) \leq g_B(e)$ for every $e \in E$ and we write $f_A \sqsubseteq g_B$.

Definition 2.9. [21] Let $f_A, g_B \in FS(X,E)$. f_A and g_B are said to be equal, denoted by $f_A = g_B$ if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

Definition 2.10. [21] Let $f_A, g_B \in FS(X,E)$. Then the union of f_A and g_B is also a fuzzy soft set h_C, defined by $h_C(e) = f_A(e) \vee g_B(e)$ for all $e \in E$, where $C = A \cup B$. Here we write $h_C = f_A \sqcup g_B$.

Definition 2.11. [21] Let $f_A, g_B \in FS(X,E)$. Then the intersection of f_A and g_B is also a fuzzy soft set h_C, defined by $h_C(e) = f_A(e) \wedge g_B(e)$ for all $e \in E$, where $C = A \cap B$. Here we write $h_C = f_A \sqcap g_B$.

Definition 2.12. [23] Let $f_A \in FS(X,E)$. The complement of f_A, denoted by f_A^c, is a fuzzy soft set defined by $f_A^c(e) = 1 - f_A(e)$ for every $e \in E$.

Let us call f_A^c to be fuzzy soft complement function of f_A. Clearly $(f_A^c)^c = f_A$, $(\widetilde{1}_E)^c = \widetilde{0}_E$ and $(\widetilde{0}_E)^c = \widetilde{1}_E$.

Definition 2.13. [12] Let $FS(X,E)$ and $FS(Y,K)$ be the families of all fuzzy soft sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be two functions. Then f_{up} is called a fuzzy soft mapping from X to Y and denoted by $f_{up} : FS(X,E) \to FS(Y,K)$.

(1) Let $f_A \in FS(X,E)$, then the image of f_A under the fuzzy soft mapping f_{up} is the fuzzy soft set over Y defined by $f_{up}(f_A)$, where

$$f_{up}(f_A)(k)(y) = \begin{cases} \vee_{x \in u^{-1}(y)} (\vee_{e \in p^{-1}(k) \cap A} f_A(e))(x) & \text{if } u^{-1}(y) \neq \varnothing,\ p^{-1}(k) \cap A \neq \varnothing; \\ 0_Y & \text{otherwise.} \end{cases}$$

(2) Let $g_B \in FS(Y,K)$, then the preimage of g_B under the fuzzy soft mapping f_{up} is the fuzzy soft set over X defined by $f_{up}^{-1}(g_B)$, where

$$f_{up}^{-1}(g_B)(e)(x) = \begin{cases} g_B(p(e))(u(x)) & \text{for } p(e) \in B; \\ 0_X & \text{otherwise.} \end{cases}$$

If u and p are injective then the fuzzy soft mapping f_{up} is said to be injective. If u and p are surjective then the fuzzy soft mapping f_{up} is said to be surjective. The fuzzy soft mapping f_{up} is called constant, if u and p are constant.

Theorem 2.14. [12] Let $f_A, \in FS(X,E)$, $\{f_{A_i}\}_{i \in J} \subset FS(X,E)$ and $g_B \in FS(Y,K)$, $\{g_{B_i}\}_{i \in J} \subset FS(Y,K)$, where J is an index set.

(1) If $(f_{A_1}) \sqsubseteq (f_{A_2})$, then $f_{up}(f_{A_1}) \sqsubseteq f_{up}(f_{A_2})$.
(2) If $(g_{B_1}) \sqsubseteq (g_{B_2})$, then $f_{up}^{-1}(g_{B_1}) \sqsubseteq f_{up}^{-1}(g_{B_2})$.
(3) $f_{up}(\sqcup_{i \in J}(f_{A_i})) = \sqcup_{i \in J} f_{up}(f_{A_i})$.
(4) $f_{up}(\sqcap_{i \in J}(f_{A_i})) \sqsubseteq \sqcap_{i \in J} f_{up}(f_{A_i})$.
(5) $f_{up}^{-1}(\sqcup_{i \in J}(g_{B_i})) = \sqcup_{i \in J} f_{up}^{-1}(g_{B_i})$.
(6) $f_{up}^{-1}(\sqcap_{i \in J}(g_{B_i})) = \sqcap_{i \in J} f_{up}^{-1}(g_{B_i})$.
(7) $f_{up}^{-1}(\widetilde{1}_K) = \widetilde{1}_E$ and $f_{up}^{-1}(\widetilde{0}_K) = \widetilde{0}_E$.
(8) $f_{up}(\widetilde{0}_E) = \widetilde{0}_K$ and $f_{up}(\widetilde{1}_E) \sqsubseteq \widetilde{1}_K$.

Theorem 2.15. [32] Let $f_A, \in FS(X,E)$, $\{f_{A_i}\}_{i \in J} \subset FS(X,E)$ and $g_B \in FS(Y,K)$, $\{g_{B_i}\}_{i \in J} \subset FS(Y,K)$, where J is an index set.

(1) $f_{up}(\sqcap_{i \in J}(f_{A_i})) = \sqcap_{i \in J} f_{up}(f_{A_i})$ if f_{up} is injective.
(2) $f_{up}(\widetilde{1}_E) = \widetilde{1}_K$ if f_{up} is surjective.
(3) $f_{up}^{-1}(f_A^c) = (f_{up}^{-1}(f_A))^c$.

2.1 Soft quasi-coincidence

Definition 2.16. [32] The fuzzy soft set $f_A \in FS(X,E)$ is called fuzzy soft point if $A = \{e\} \subseteq E$ and $f_A(e)$ is a fuzzy point in X, i.e., there exists $x \in X$ such that $f_A(e)(x) = \alpha$ $(0 < \alpha \leq 1)$ and $f_A(e)(y) = 0$ for all $y \in X - \{x\}$. We denote this fuzzy soft point $f_A = e_x^\alpha = \{(e, x_\alpha)\}$.

Definition 2.17. [32] Let e_x^α, $f_A \in FS(X,E)$. We say that $e_x^\alpha \widetilde{\in} f_A$ read as e_x^α belongs to the fuzzy soft set f_A if for the element $e \in A$, $\alpha \leq f_A(e)(x)$.

Proposition 2.18. [32] Every non null fuzzy soft set f_A can be expresssed as the union of all the fuzzy soft points which belong to f_A.

Definition 2.19. [32] Let $x_\alpha \in S(X)$ and $f_A \in FS(X,E)$. We say that $x_\alpha \in f_A$ read as x_α belongs to the fuzzy soft set f_A whenever $x_\alpha \in f_A(e)$, i.e., $\alpha \leq f_A(e)(x)$ for all $e \in A$.

Definition 2.20. [32] Let f_A, $g_B \in FS(X,E)$. f_A is said to be soft quasi-coincident with g_B, denoted by $f_A q g_B$, if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$.

If f_A is not soft quasi-coincident with g_B, then we write $f_A \overline{q} g_B$.

Definition 2.21. [32] Let $x_\alpha \in S(X)$ and $f_A \in FS(X,E)$. x_α is said to be soft quasi-coincident with f_A, denoted by $x_\alpha q f_A$, if and only if there exists an $e \in E$ such that $\alpha + f_A(e)(x) > 1$.

Proposition 2.22. [32] Let f_A, $g_B \in FS(X,E)$, then the followings are true.
(1) $f_A \sqsubseteq g_B \Leftrightarrow f_A \overline{q} g_B^c$.
(2) $f_A q g_B \Rightarrow f_A \sqcap g_B \neq \widetilde{0}_E$.
(3) $x_\alpha \overline{q} f_A \Leftrightarrow x_\alpha \in f_A^c$.
(4) $f_A \overline{q} f_A^c$.
(5) $f_A \sqsubseteq g_B \Rightarrow x_\alpha q f_A$ implies $x_\alpha q g_B$.
(6) $f_A q g_B \Leftrightarrow$there exists an $e_x^\alpha \widetilde{\in} f_A$ such that $e_x^\alpha q g_B$.
(7) $e_x^\alpha \overline{q} f_A \Leftrightarrow e_x^\alpha \widetilde{\in} f_A^c$.
(8) $f_A \sqsubseteq g_B \Leftrightarrow$ If $e_x^\alpha q f_A$, then $e_x^\alpha q g_B$ for all $e_x^\alpha \in FS(X,E)$.

Definition 2.23. (see [23] , [21]) A fuzzy soft topological space is a pair (X,τ) where X is a nonempty set and τ is a family of fuzzy soft sets over X satisfying the following properties:
(1) $\widetilde{0}_E, \widetilde{1}_E \in \tau$
(2) If f_A, $g_B \in \tau$, then $f_A \sqcap g_B \in \tau$
(3) If $f_{Ai} \in \tau$, $\forall i \in J$, then $\sqcup_{i\in J} f_{Ai} \in \tau$.
Then τ is called a topology of fuzzy soft sets on X. Every member of τ is called fuzzy soft open. g_B is called fuzzy soft closed in (X,τ) if $(g_B)^c \in \tau$.

Theorem 2.24. [32] Let (X,τ) be a fuzzy soft topological space and τ' denotes the collection of all fuzzy soft closed sets. Then
(1) $\widetilde{0}_E, \widetilde{1}_E \in \tau'$
(2) If f_A, $g_B \in \tau'$, then $f_A \sqcup g_B \in \tau'$
(3) If $f_{Ai} \in \tau'$, $\forall i \in J$, then $\sqcap_{i\in J} f_{Ai} \in \tau'$.

Definition 2.25. [23] Let (X,τ) be a fuzzy soft topological space and $f_A \in FS(X,E)$. The fuzzy soft closure of f_A denoted by $cl\,(f_A)$ is the intersection of all fuzzy soft closed supersets of f_A.

Clearly, $cl\,(f_A)$is the smallest fuzzy soft closed set over X which contains f_A.

Definition 2.26. [23] Let (X,τ) be a fuzzy soft topological space and $f_A \in FS(X,E)$. The fuzzy soft interior of f_A denoted by f_A° is the union of all fuzzy soft open subsets of f_A.

Clearly, f_A° is the largest fuzzy soft open set over X which contained in f_A.

Theorem 2.27. [32] Let (X,τ) be a fuzzy soft topological space and $f_A, g_B \in FS(X,E)$. Then,
(1) $(\overline{f_A})^c \sqsubseteq (f_A^c)^\circ$.
(2) $(f_A^\circ)^c \sqsubseteq \overline{(f_A^c)}$.

Definition 2.28. [32] A fuzzy soft set f_A in $FS(X,E)$ is called Q-neighborhood (briefly, Q-nbd) of g_B if and only if there exists a fuzzy soft open set h_C in τ such that $g_B q h_C \sqsubseteq f_A$.

Theorem 2.29. [32] Let e_x^α, $f_A \in FS(X,E)$. Then $e_x^\alpha \widetilde{\in} \overline{f_A}$ if and only if each Q-nbd of e_x^α is soft quasi-coincident with f_A.

Definition 2.30. [23] Let (X,τ) be a fuzzy soft topological space and β be a subfamily of τ. If every element of τ can be written as the arbitrary fuzzy soft union of some elements of β, then β is called a fuzzy soft basis for the fuzzy soft topology τ.

Proposition 2.31. [32] Let (X,τ) be a fuzzy soft topological space and β is subfamily of τ. β is a base for τ if and only if for each e_x^α in $FS(X,E)$ and for each fuzzy soft open Q-nbd f_A of e_x^α, there exists a $g_B \in \beta$ such that $e_x^\alpha q g_B \sqsubseteq f_A$.

Definition 2.32. [23] A fuzzy soft set g_B in a fuzzy soft topological space (X,τ) is called a fuzzy soft neighborhood (briefly: nbd) of the fuzzy soft set f_A if there exists a fuzzy soft open set h_C such that $f_A \sqsubseteq h_C \sqsubseteq g_B$.

Theorem 2.33. [32] g_B is fuzzy soft open if and only if for each fuzzy soft set f_A contained in g_B, g_B is a fuzzy soft neighborhood of f_A.

Definition 2.34. [32] Let (X,τ_1) and (Y,τ_2) be two fuzzy soft topological spaces. A fuzzy soft mapping $f_{up}:(X,\tau_1)\to(Y,\tau_2)$ is called fuzzy soft continuous if $f_{up}^{-1}(g_B)\in\tau_1$ for all $g_B\in\tau_2$.

Theorem 2.35. [32] Let (X,τ_1) and (Y,τ_2) be fuzzy soft topological spaces. For a function f_{up} : $FS(X,E)\longrightarrow FS(Y,K)$, the following statements are equivalent:
(a) f_{up} is fuzzy soft continuous;
(b) for each fuzzy soft set f_A in $FS(X,E)$, the inverse image of every nbd of $f_{up}(f_A)$ is a nbd of f_A;
(c) for each soft set f_A in $FS(X,E)$ and each nbd h_C of $f_{up}(f_A)$, there is a nbd g_B of f_A such that $f_{up}(g_B)\sqsubseteq h_C$.

Theorem 2.36. [32] A mapping $f_{up}:(X,E)\to(Y,K)$ is fuzzy soft continuous if and only if corresponding fuzzy soft open Q-nbd g_B of k_y^α in $FS(Y,K)$ there exists a fuzzy soft open Q-nbd f_A of e_x^α in $FS(X,E)$ such that $f_{up}(f_A)\sqsubseteq g_B$, where $f_{up}(e_x^\alpha)=k_y^\alpha$.

Theorem 2.37. [32] Let (X,τ_1) and (Y,τ_2) be two fuzzy soft topological spaces and $f_{up}:FS(X,E)\to FS(Y,K)$ be a fuzzy soft mapping. Then the followings are equivalent:
(1) f_{up} is continuous;
(2) $f_{up}^{-1}(h_C)\sqsubseteq(\ f_{up}^{-1}(h_C))^\circ$, $\forall h_C\in\tau_2$;
(3) $f_{up}(cl\,(f_A))\sqsubseteq cl\,(f_{up}(f_A))$, $\forall f_A\in FS(X,E)$;
(4) $cl\left(f_{up}^{-1}(g_B)\right)\sqsubseteq f_{up}^{-1}(cl\,(g_B))$, $\forall g_B\in FS(Y,K)$;
(5) $f_{up}^{-1}(g_B^\circ)\sqsubseteq(f_{up}^{-1}(g_B))^\circ$, $\forall g_B\in FS(Y,K)$.

3 Continuity of Fuzzy Soft Multifunctions

Let Y be an initial universe set and E be the non-empty set of parameters.

Definition 3.1. A soft multifunction F from an ordinary topological space (X,τ) into a fuzzy soft topological space (Y,σ,E) assings to each x in X a soft set $F(x)$ over Y. A fuzzy soft multifunction will be denoted by $F:(X,\tau)\to(Y,\sigma,E)$. F is said to be onto if for each fuzzy soft set g_B over Y, there exists a point $x\in X$ such that $F(x)=g_B$.

Definition 3.2. For a fuzzy soft multifunction $F:(X,\tau)\to(Y,\sigma,E)$, the upper inverse $F^+(g_B)$ and the lower inverse $F^-(g_B)$ of a fuzzy soft set g_B over Y are defined as follows: $F^+(g_B)=\{x\in X: F(x)\tilde{\sqsubseteq}g_B\}$ and $F^-(g_B)=\{x\in X: F(x)\tilde{\sqcap}g_B\neq\tilde{\Phi}\}$. Moreover, for a subset M of X, $F(M)=\tilde{\sqcup}\{F(x): x\in X\}$.

Definition 3.3. [27] Let (X,τ) be an ordinary topological space and (Y,ϑ) be a fuzzy topological space. $F:(X,\tau)\to(Y,\vartheta)$ is called a fuzzy multifunction iff for every $x\in X$, $F(x)$ is a fuzzy set in Y.

Remark 3.4. Since every fuzzy set is a soft set, then every fuzzy multifunction is a soft multifunction.

Proposition 3.5. Let M be a subset of X. Then the follows are true for a fuzzy soft multifunction $F:(X,\tau)\to(Y,\sigma,E)$;
(a) $M\subset F^+(F\,(M))$. If F is onto $M=F^+(F\,(M))$.
(b) $M\subset F^-(F\,(M))$. If F is onto $M=F^-(F\,(M))$.

Proof. (a) Let $x \in M$. Then $F(x)\widetilde{\subseteq}F(M) = \widetilde{\sqcup}\{F(x) : x \in M\}$ and so $x \in F^+(F(M))$. Hence, $M \subset F^+(F(M))$.
(b) The proof is similar to (a). □

Proposition 3.6. Let g_B be a fuzzy soft set over Y. Then the followings are true for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$:
(a) $F^+((g_B)^c) = X - F^-(g_B)$
(b) $F^-((g_B)^c) = X - F^+(g_B)$.

Proof. (a) If $x \in X - F^-(g_B)$ then $x \notin F^-(g_B)$ which implies $F(x)\widetilde{\sqcap}(g_B) = \widetilde{\Phi}$ and therefore $F(x)\widetilde{\subseteq}(g_B)^c$. Thus $x \in F^+((g_B)^c)$ and $X - F^-(g_B)\widetilde{\subseteq}F^+((g_B)^c)$.

Conversely, if $x \in F^+((g_B)^c)$ then $F(x)\widetilde{\subseteq}(g_B)^c$ which implies $F(x)\widetilde{\sqcap}(g_B) = \widetilde{\Phi}$ and therefore $x \notin F^-(g_B)$. Thus $x \in X - F^-(g_B)$ and $F^+((g_B)^c)\widetilde{\subseteq}X - F^-(g_B)$.

(b) If $x \in X - F^+(g_B)$ then $x \notin F^+(g_B)$ which implies $F(x)\widetilde{\nsubseteq}(g_B)$ and therefore $F(x)\widetilde{\sqcap}((g_B)^c) \neq \widetilde{\Phi}$. Thus $x \in F^-((g_B)^c)$ and $X - F^+(g_B)\widetilde{\subseteq}F^-((g_B)^c)$.

Conversely, if $x \in F^-((g_B)^c)$ then $F(x)\widetilde{\sqcap}((g_B)^c) \neq \widetilde{\Phi}$ which implies $F(x) \widetilde{\nsubseteq} (g_B)$ and therefore $x \notin F^+(g_B)$. Thus $x \in X - F^+(g_B)$ and $F^-((g_B)^c)\widetilde{\subseteq}X - F^+(g_B)$. □

Proposition 3.7. Let (g_{Bi}) be fuzzy soft sets over Y for each $i \in I$. Then the follows are true for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$;
(a) $F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi}) = \underset{i\in I}{\sqcup}(F^-(g_{Bi}))$.
(b) $F^+(\underset{i\in I}{\widetilde{\sqcap}} g_{Bi}) = \underset{i\in I}{\sqcap}(F^-(g_{Bi}))$.

Proof. (a) For every $x \in F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi})$, $F(x)\widetilde{\sqcap}(\underset{i\in I}{\sqcup} g_{Bi}) \neq \widetilde{\Phi}$. There exists $i \in I$ such that $F(x)\widetilde{\sqcap}(g_{Bi}) \neq \widetilde{\Phi}$. For the same $i \in I$, $x \in F^-(g_{Bi})$. Therefore $x \in \underset{i\in I}{\sqcup}(F^-(g_{Bi})$. Thus $F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi}) \sqsubseteq \underset{i\in I}{\sqcup}(F^-(g_{Bi})$.
Conversely, for every $x \in \underset{i\in I}{\sqcup}(F^-(g_{Bi})$, there exists $i \in I$ such that $x \in F^-(g_{Bi})$. For the same $i \in I$, $F(x)\widetilde{\sqcap}(g_{Bi}) \neq \Phi$. Therefore, $F(x)\widetilde{\sqcap}(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi}) \neq \phi$ and $x \in F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi})$. Thus $\underset{i\in I}{\sqcup}(F^-(g_{Bi}) \sqsubseteq F^-(\underset{i\in I}{\widetilde{\sqcup}} g_{Bi})$.
(b)The proof is similar of (a). □

Definition 3.8. Let (X, τ) be an ordinary topological space and (Y, σ, E) be a fuzzy soft topological space. Then a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$ is said to be;

(a) upper fuzzy soft continuous (briefly: u.fuzzy soft c.) at a point $x \in X$ if for each fuzzy soft open g_B such that $F(x)\widetilde{\subseteq}(g_B)$, there exists an open neighborhood $P(x)$ of x such that $F(z)\widetilde{\subseteq}g_B$ for all $z \in P(x)$.

(b) lower fuzzy soft continuous (briefly: l. fuzzy soft c.) at a point $x \in X$ if for each fuzzy soft open g_B such that $F(x)\widetilde{\sqcap}g_B \neq \widetilde{\Phi}$, there exists an open neighborhood $P(x)$ of x such that $F(z)\widetilde{\sqcap}g_B \neq \widetilde{\Phi}$ for all $z \in P(x)$.

(c) upper(lower) fuzzy soft continuous if F has this property at every point of X.

Proposition 3.9. A fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$ is upper fuzzy soft continuous if and only if for all fuzzy soft open set g_B over Y, $F^+(g_B)$ is open in X.

Proof. First suppose that F is upper fuzzy soft continuous. Let g_B is be fuzzy soft open set over Y and $x \in F^+(g_B)$. Then from Definition 29, we know that there exists an open neighborhood $P(x)$ of x such that for all $z \in P(x)$, $F(z)\widetilde{\subseteq}F^+(g_B)$ which means that $F^+(g_B)$ is open as claimed. The direction is just the definition of upper fuzzy soft continuity of F. □

Proposition 3.10. $F : (X, \tau) \to (Y, \sigma, E)$ is lower fuzzy soft continuous multifunction if and only if for every fuzzy soft open set g_B over Y, $F^-(g_B)$ is open set in X.

Proof. First assume that F is lower fuzzy soft continuous. Let g_B fuzzy soft open over Y and $x \in F^-(g_B)$. Then there is an open neighborhood $P(x)$ of x such that $F(z)\widetilde{\sqcap}\, g_B \neq \widetilde{\Phi}$ for all $z \in P(x)$. So $P(x) \subseteq F^-(g_B)$ which implies that $F^-(g_B)$ is open in X. Now suppose that $F^-(g_B)$ is open. Let $x \in F^-(g_B)$.Then $F^-(g_B)$ is an open neighborhood of x and for all $z \in F^-(g_B)$ we have $F(z)\widetilde{\sqcap}\, g_B \neq \widetilde{\Phi}$. So, F is lower fuzzy soft continuous. □

Theorem 3.11. The followings are equivalent for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$;
(a) F is upper fuzzy soft continuous
(b) for each fuzzy soft closed set g_B over Y, $F^-(g_B)$ is closed in X.
(c) for each fuzzy soft set g_B over Y, $cl(F^-(g_B) \subseteq F^-(cl(g_B))$.
(d) for each fuzzy soft set g_B over Y, $F^+(Int(g_B)) \subseteq Int(F^+(g_B)$.

Proof. (a)$\Longrightarrow$(b) Let g_B be a closed fuzzy soft over Y. Then proposition1 implies $(g_B)^c$ is fuzzy soft open and $F^+((g_B)^c) = X - F^-(g_B)$, then since $F^-(g_B)$ is open and so $F^-(g_B)$ is closed.

(b)$\Longrightarrow$(c) Let g_B be any fuzzy soft set over Y. Then $cl(g_B)$ is fuzzy soft closed set. By (b) $F^-(cl(g_B))$ is closed in X. Hence, $cl(F^-(g_B) \subseteq F^-(cl(g_B))$ and since $F^-(g_B) \subseteq F^-(cl(g_B))$. Thus, $cl(F^-(g_B)) \subseteq F^-(cl(g_B))$.

(c)$\Longrightarrow$(d) Let g_B be any fuzzy soft set over Y. By (c), $cl(F^-((g_B)^c) \subseteq F^-(cl(g_B)^c)$, $X - F^-((int(g_B)^c) \subseteq int(X - F^-((int(g_B)^c))$, $X - X - F^+(int(g_B)) \subseteq intF^+(g_B)$.

(d)$\Longrightarrow$(a) Let g_B be any fuzzy soft set over Y. By (d), $F^+(int(g_B)) = F^+(g_B) \subseteq int(F^+(g_B)$ and so $F^+(g_B)$ is open in X. They by proposition (1), F is upper fuzzy soft continuous. □

Theorem 3.12. The following are equivalent for a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$;
(a) F is lower fuzzy soft continuous.
(b) for each fuzzy soft closed set g_B over Y, $F^+(g_B)$ is closed in X.
(c) for each fuzzy soft set g_B over Y, $cl(F^+(g_B) \subseteq F^+(cl(g_B))$.
(d) for each fuzzy soft set g_B over Y, $F^-(int(g_B) \subseteq int(F^-(g_B)$.

Proof. It is similar the proof of Theorem 4. □

Definition 3.13. For a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$, the graph fuzzy soft multifunction $G_F : X \to X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$, for every $x \in X$.

Lemma 3.14. For a fuzzy soft multifunction $F : (X, \tau) \to (Y, \sigma, E)$, the followings are hold:
(a) $G_F^+(M \times h_B) = M \cap F^+(h_B)$
(b) $G_F^-(M \times h_B) = M \cap F^-(h_B)$

Proof. (a)Let M be any subset of X and let h_B be any fuzzy soft set over Y. Let $x \in G_F^+(M \times h_B)$. Then $G_F(x)\widetilde{\subseteq}(M \times h_B)$ that is $(\{x\} \times F(x))\widetilde{\subseteq}M \times h_B$. Therefore, we have $x \in M$ and $F(x)\widetilde{\subseteq}h_B$. Hence $x \in M \cap F^+(h_B)$.
Conversely, let $x \in M \cap F^+(h_B)$. Then $x \in M$ and $x \in F^+(h_B)$. Thus $x \in M$ and $F(x)\widetilde{\subseteq}h_B$ that is $G_F(x)\widetilde{\subseteq}(M \times h_B)$. Therefore $x \in G_F^+(M \times h_B)$.
(b)Let M be any subset of X and let h_B be any fuzzy soft set over Y. Let $x \in G_F^-(M \times h_B)$. Then $\widetilde{\Phi} \neq G_F(x)\widetilde{\sqcap}(M \times h_B) = (\{x\} \times F(x))\widetilde{\sqcap}(M \times h_B) = (\{x\} \cap M) \times (F(x)\widetilde{\sqcap}h_B)$. Therefore, we have $x \in M$ and $F(x)\widetilde{\sqcap}h_B \neq \widetilde{\Phi}$. Hence $x \in M \cap F^-(h_B)$.
Conversely, let $x \in M \cap F^-(h_B)$. Then $x \in M$ and $x \in F^-(h_B)$. Thus $x \in M$ and $F(x)\widetilde{\sqcap}h_B \neq \widetilde{\Phi}$ that is $G_F(x)\widetilde{\sqcap}(M \times h_B) \neq \widetilde{\Phi}$. Therefore $x \in G_F^-(M \times h_B)$. □

Theorem 3.15. Let $F : (X, \tau) \to (Y, \sigma, E)$ be a fuzzy soft multifunction. If the graph fuzzy soft function of F is lower (upper) fuzzy soft continuous, then F is lower (upper) fuzzy soft continuous.

Proof. For a subset V of X and h_B a fuzzy soft set over Y, we take $(V \times h_B)(z, y) = \begin{cases} \Phi, & if\ z \notin V \\ h_B(y), & if\ z \in V \end{cases}$
Let $x \in X$ and let h_B be fuzzy soft open set such that $x \in F^-(h_B)$. Then we obtain that $x \in G_F^-(X \times h_B)$ and $X \times h_B$ is a fuzzy soft set over Y. Since fuzzy soft graph multifuntion G_F is lower fuzzy soft continuons, it follows that there exists an open set P containg x such that $P \subseteq G_F^-(X \times h_B)$. From here, we obtain that $P \subseteq F^-(h_B)$. Thus, F is lower fuzzy soft continuous.
The proof of the upper fuzzy soft continuity of F is similar to the above. □

Theorem 3.16. Let $F : (X, \tau) \to (Y, \sigma, E)$ be a fuzzy soft multifunction and M be an open set of X. Then the restriction $F|_M$ is upper fuzzy soft continuous if F is upper fuzzy soft continuous.

Proof. Let h_B be any fuzzy soft open set over Y such that $(F|_M)(x)\tilde{\subseteq} h_B$. Since F is upper fuzzy soft continuous and $F(x) = (F_{|M})(x)\tilde{\subseteq} h_B$, there exists open set $U \subseteq X$ containing x such that $F(z)\tilde{\subseteq} h_B$ for all $z \in U$. Put $U_1 = U \cap M$ then we have U_1 is open set in M containing x and $F(U_1) = (F|_M)(U_1)\tilde{\subseteq} h_B$. This shows that $F|_M$ is upper fuzzy soft continuous. □

Theorem 3.17. Let $F : (X, \tau) \to (Y, \sigma, E)$ be a fuzzy soft multifunction and M be an open set of X. Then F is lower fuzzy soft continuous if and only if the restriction $F|_M$ is lower fuzzy soft continuous.

Proof. Let h_B be any fuzzy soft open set over Y such that $(F|_M)(x)\tilde{\sqcap} h_B \neq \tilde{\Phi}$. Since $F(x) = (F|_M)(x)$, then $F(x)\tilde{\sqcap} h_B \neq \tilde{\Phi}$. Also since F is lower fuzzy soft continuous there exists an open set $U \subseteq X$ containing x such that $F(z)\tilde{\sqcap} h_B \neq \tilde{\Phi}$ for all $z \in U$. Put $U_1 = U \cap M$ then we have U_1 is open set in M containing x and $F(U_1)\tilde{\sqcap} h_B \neq \tilde{\Phi}$.Therefore $(F|_M)(U_1)\tilde{\sqcap} h_B \neq \tilde{\Phi}$.This shows that $F|_M$ is lower fuzzy soft continuous. □

Remark 3.18. Let $F : (X, \tau) \to (Y, \sigma, E)$ be a fuzzy soft multifunction and $\{M_i : i \in I\}$ be an open cover set of X. The followings are hold :

(a) F is lower fuzzy soft continuous if and only if the restriction $F|_{M_i}$ is lower fuzzy soft continuous for every $i \in I$.

(b) F is upper fuzzy soft continuous if and only if the restriction $F|_{M_i}$ is upper fuzzy soft continuous for every $i \in I$.

Definition 3.19. Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and let $G : (Y, \sigma) \to (Z, \vartheta, E)$ be a fuzzy soft multifunction. Then the fuzzy soft multifunction $G \circ F : (X, \tau) \to (Z, \vartheta, E)$ is defined by $(G \circ F)(x) = G(F(x))$.

Proposition 3.20. Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and let $G : (Y, \sigma) \to (Z, \vartheta, E)$ be a fuzzy soft multifunction. Then we have

(a) $(G \circ F)^+(h_B) = F^+(G^+(h_B))$

(b) $(G \circ F)^-(h_B) = F^-(G^-(h_B))$

Proof. Clear from the Definitions 17 and 20. □

Definition 3.21. [1] Let (X, τ) and (Y, σ) be two ordinary topological spaces. Then a multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be

(a) upper semi continuous if for each open V in Y, $F^+(V)$ is an open set in X.

(b) lower semi continuous if for each soft open V in Y, $F^-(V)$ is an open set in X.

Theorem 3.22. Let $F : (X, \tau) \to (Y, \sigma)$ be a multifunction and let $G : (Y, \sigma) \to (Z, \vartheta, E)$ be a fuzzy soft multifunction. If F is upper semi continuous and G is upper fuzzy soft continuous then $G \circ F$ is upper fuzzy soft continuous.

Proof. Let h_B be any fuzzy soft open subset of Z. Since G is upper fuzzy soft continuous then $G^+(h_B)$ is open in Y. Since F is upper semi continuous then $F^+(G^+(h_B)) = (G \circ F)^+(h_B)$ is open in X. Therefore $G \circ F$ is upper fuzzy soft continuous. □

Definition 3.23. A family Ψ of fuzzy soft sets is a cover of a soft set h_B if $h_B \tilde{\subset} \tilde{\bigcup} \{h_{B_i} : h_{B_i} \in \Psi, i \in I\}$. It is fuzzy soft open cover if each of Ψ is a fuzzy soft open set. A subcover of Ψ is a subfamily of Ψ which is also cover.

Definition 3.24. A fuzzy fuzzy soft topological space (Y, σ, E) is fuzzy soft compact if each fuzzy soft open cover of $\tilde{Y}$ has a finite subcover.

Theorem 3.25. The image of a fuzzy soft compact set under upper fuzzy soft continuous multifunction is fuzzy soft compact.

Proof. Let $F : (X, \tau) \to (Y, \sigma, E)$ be an onto fuzzy soft multifunction and let $\Psi = \{h_{B_i} : i \in I\}$ be a cover of $\tilde{Y}$ by fuzzy soft open sets. Then since F is upper fuzzy soft continuous, the family of all open sets of the form $F^+(h_{B_i})$, for $h_{B_i} \in \Psi$ is an open cover of X which has a finite subcover. However since F is surjective, then it is eaisly seen that $F(F^+(h_{B_i})) = h_{B_i}$ for any fuzzy soft set h_{B_i} over Y. There the family of image members of subcover is a finite subfamily of Ψ which covers $\breve{Y}$. Conseqventhy (Y, σ, E) is fuzzy soft compact. □

4 Conclusion

In the present work, we have continued to study the properties of fuzzy soft topological spaces. We introduce soft quasi-coincidence and have established several interesting properties. We hope that the findings in this paper will help researcher enhance and promote the further study on fuzzy soft topology to carry out a general framework for their applications in practical life.

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