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Directional Bertrand Curves

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Abstract

It is well known that a characteristic property of the Bertrand curve is the existence of a linear relation between its curvature and torsion. In this paper, we propose a new method for generating Bertrand curves, which avoids the basic restrictions. Our main result is that every space curve is a directional Bertrand curve.

1. INTRODUCTION

A Bertrand curve is a curve in Euclidean 3-space whose principal normal is the principal normal of another curve [14]. More recently, there have been a number of studies in the Computer Aided Geometric Design (CAGD) literature dealing with Bertrand curves [13]. Moreover, the Bertrand curves are also studied even different spaces [10, 12].

The Frenet frame plays an important role in classical differential geometry. Because classical topics, for instance spherical curves, Bertrand curves, involutes and evolutes are investigated by using the Frenet frame [9]. However, the Frenet frame has several disadvantages in applications. For instance, the Frenet frame is undefined wherever the curvature vanishes. Moreover, the main disadvantage of the Frenet frame is that it has undesirable rotation about tangent vector [2]. Therefore, Bishop [1] introduced a new frame along a space curve which is more suitable for applications. But, it is well known that Bishop frame calculations are not an easy task [15] . In order to construct the 3D curve offset, Coquillart [4] introduced the quasi-normal vector of a space curve. In this paper, we extend this concept to Bertrand curves and obtain the differential geometric properties.

The q-frame have many advantages compare to other frames (Frenet, Bishop). For instance, the q-frame can be defined even along a line ($\kappa = 0$). Morever, the q-frame can be calculated easily.

Let $\alpha(s)$ be a curve with arc length parameter s, the q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ along the curve is given by

$$\mathbf{t} = \alpha', \quad \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \quad \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{1}$$

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where $\mathbf{t}$ is the unit tangent vector, $\mathbf{n}_q$ is the quasi-normal vector, $\mathbf{b}_q$ is the quasi-binormal vector and $\mathbf{k}$ is the projection vector. For simplicity, we have choosen the projection vector $\mathbf{k}=(0,0,1)$ in this paper. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in these cases, where $\mathbf{t}$ and $\mathbf{k}$ are paralel, the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k}=(0,1,0)$ or $\mathbf{k}=(0,0,1)$ [17,18].
The variation equations of the q-frame are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} \tag{2}$$

where the q-curvatures are expressed as follows

$$\begin{aligned} k_1 &= \kappa\cos\theta = \langle \mathbf{t}', \mathbf{n}_q \rangle \\ k_2 &= -\kappa\sin\theta = \langle \mathbf{t}', \mathbf{b}_q \rangle \\ k_3 &= d\theta + \tau = -\langle \mathbf{n}_q, \mathbf{b}_q' \rangle \end{aligned} \tag{3}$$

and θ is the Euclidean angle between the principal normal vector $\mathbf{n}$ and the quasi-normal vector $\mathbf{n}_q$.

2. DIRECTIONAL BERTRAND OFFSET CURVES

In this section, we introduce the directional Bertrand curves in Euclidean 3-space. A pair of curves are said to be directional Bertrand mates if there exists a one-to-one correspondence between their points such that both curves have a common principal quasi-normal at their corresponding points.

Proposition 1. Let $\alpha(s)$ be a space curve with arc length parameter s. If the curve $\beta(s_1)$ is directional Bertrand mate of $\alpha(s)$, then we have the following:

(a) The distance between the corresponding points of the directional Bertrand mates is a constant.

(b) The angle between the tangent vectors at the corresponding points of the directional Bertrand mates is a constant, if $k_3=0$.

Proof: (a) Let us denote the q-frames of α and β by $\{\mathbf{t},\mathbf{n}_q,\mathbf{b}_q\}$ and $\{\mathbf{t}^\lambda,\mathbf{n}_q^\lambda,\mathbf{b}_q^\lambda\}$, respectively. Then there exists a relationship between the position vectors of α and β as

$$\beta(s_1)=\alpha(s)+\lambda\mathbf{n}_q(s). \tag{4}$$

By differentiating (4) with respect to s gives

$$\mathbf{t}^\lambda \frac{ds_1}{ds} = \mathbf{t}(1-\lambda k_1)+\lambda'\mathbf{n}_q+\lambda k_3\mathbf{b}_q. \tag{5}$$

On the other hand, since $\mathbf{t}^\lambda$ is orthogonal to $\mathbf{n}_q$, we have

$$\lambda' = 0. \tag{6}$$

which implies that $\lambda = \text{constant},$ thus the distance between the corresponding points of α and β is

$$\|\beta(s_1) - \alpha(s)\| = \lambda,$$

which is a constant. The proof is completed.

(b) From (5) and (6), we get

$$\frac{ds_1}{ds} = \pm\sqrt{(1-\lambda k_1)^2 + \lambda^2 k_3^2}. \tag{7}$$

From (5), we obtain

$$\mathbf{t}^\lambda = \pm \frac{\mathbf{t}(1-\lambda k_1) + \lambda k_3 \mathbf{b}_q}{\sqrt{(1-\lambda k_1)^2 + \lambda^2 k_3^2}}. \tag{8}$$

Let φ denote the angle between the vectors $\mathbf{t}$ and $\mathbf{t}^\lambda$. Thus we can write

$$\left\langle \mathbf{t}^\lambda, \mathbf{t} \right\rangle = \cos\varphi, \tag{9}$$

where

$$\cos\varphi = \pm \frac{1-\lambda k_1}{\sqrt{(1-\lambda k_1)^2 + \lambda^2 k_3^2}}. \tag{10}$$

In the last equation, if $k_3 = 0$, then $\cos\varphi = \pm 1$ which implies that the angle φ between the tangent vectors is a constant.

Theorem 2. Let $\alpha(s)$ be a space curve with arc length parameter s. If the curve $\beta(s_1)$ is directional Bertrand mate of $\alpha(s)$, then the relationship between the q-frames of α and β can be written as

$$\begin{bmatrix} \mathbf{t}^{\lambda} \\ \mathbf{n}_q^{\lambda} \\ \mathbf{b}_q^{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{\pm(1-\lambda k_1)}{\sqrt{(1-\lambda k_1)^2+\lambda^2 k_3^2}} & 0 & \frac{\pm\lambda k_3}{\sqrt{(1-\lambda k_1)^2+\lambda^2 k_3^2}} \\ 0 & \pm 1 & 0 \\ \frac{\mp\lambda k_3}{\sqrt{(1-\lambda k_1)^2+\lambda^2 k_3^2}} & 0 & \frac{\pm(1-\lambda k_1)}{\sqrt{(1-\lambda k_1)^2+\lambda^2 k_3^2}} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{11}$$

Proof: From (1) and (8), we have

$$\mathbf{t}^{\lambda} \wedge \mathbf{k} = \pm \frac{\|\mathbf{t} \wedge \mathbf{k}\|(1-\lambda k_1)\mathbf{n}_q + \lambda k_3 \mathbf{b}_q \wedge \mathbf{k}}{\sqrt{(1-\lambda k_1)^2+\lambda^2 k_3^2}}. \tag{12}$$

On the other hand, using Lagrange's identity and $\mathbf{k} = (0,0,1)$ implies that

$$\|\mathbf{t} \wedge \mathbf{k}\| = \sqrt{1-\langle \mathbf{t}, \mathbf{k} \rangle^2} = \sqrt{1-\mu^2}, \tag{13}$$

where μ is the third component of the tangent vector of the curve $\alpha(s)$. Similarly, a simple calculation implies that

$$\mathbf{b}_q \wedge \mathbf{k} = \mathbf{t} \wedge \mathbf{n}_q \wedge \mathbf{k} = \mu \mathbf{n}_q.$$

Substituting (12) and (13) into (11) gives

$$\mathbf{t}^{\lambda} \wedge \mathbf{k} = \pm \frac{(\sqrt{1-\mu^2}(1-\lambda k_1) + \lambda k_3 \mu)\mathbf{n}_q}{\sqrt{(1-\lambda k_1)^2+\lambda^2 k_3^2}}.$$

Thus, we have the quasi-normal vector $\mathbf{n}_q^{\lambda}$ of β in the following form

$$\mathbf{n}_q^{\lambda} = \frac{\mathbf{t}^{\lambda} \wedge \mathbf{k}}{\|\mathbf{t}^{\lambda} \wedge \mathbf{k}\|} = \pm \mathbf{n}_q,$$
(14)

Note that, the above equation implies that the quasi-normal vectors $\mathbf{n}_q$ and $\mathbf{n}_q^{\lambda}$ of directional Bertrand mates are always linearly dependent.

From (1) and (8), the quasi-binormal of $\beta(s_1)$ is obtained as

$$\mathbf{b}_q^{\lambda} = \pm \frac{\mathbf{b}_q(1-\lambda k_1) - \lambda k_3 \mathbf{t}}{\sqrt{(1-\lambda k_1)^2 + \lambda^2 k_3^2}}.$$

Thus, we state the following main corollary.

Corollary 3. In view of equation (14), every space curve admits infinite directional Bertrand mates.

Theorem 4. If a curve $\alpha(s)$ has a directional Bertrand mate $\beta(s_1)$, then the relations between q-curvatures of the Bertrand mates $\alpha(s)$ and $\beta(s_1)$ can be expressed as

$$k_1^{\lambda} = \pm \frac{(1-\lambda k_1)k_1 - \lambda k_3^2}{(1-\lambda k_1)^2 + \lambda^2 k_3^2},$$

$$k_2^{\lambda} = \pm \frac{\lambda k_3'(1-\lambda k_1) + \lambda^2 k_1' k_3}{(1-\lambda k_1)^2 + \lambda^2 k_3^2} + k_2$$

and

$$k_3^{\lambda} = \pm \frac{k_3}{(1-\lambda k_1)^2 + \lambda^2 k_3^2}$$

where $k_1^{\lambda}, k_2^{\lambda}$ and k_3^{λ} are the q-curvatures of $\beta(s_1)$.

Proof: From (9) and (10), $\mathbf{t}^{\lambda}$ and $\mathbf{b}_q^{\lambda}$ are, respectively, calculated by

$$\mathbf{t}^{\lambda} = \cos\varphi \mathbf{t} + \sin\varphi \mathbf{b}_q, \tag{15}$$

and

$$\mathbf{b}_q^{\lambda} = -\sin\varphi \mathbf{t} + \cos\varphi \mathbf{b}_q, \tag{16}$$

where

$$\sin\varphi = \frac{\pm\lambda k_3}{\sqrt{(1-\lambda k_1)^2 + \lambda^2 k_3^2}}. \tag{17}$$

By differentiating (15) with respect to s_1, then substituting (2) and (7) into the result gives

$$\mathbf{t}^{\lambda\prime} \frac{ds_1}{ds} = \frac{-\sin\varphi(d\varphi + k_2)\mathbf{t} + \mathbf{n}_q(\cos\varphi k_1 - \sin\varphi k_3) + \mathbf{b}_q \cos\varphi(k_2 + d\varphi)}{\sqrt{(1-\lambda k_1)^2 + \lambda^2 k_3^2}}.$$

From (3), we have

$$k_1^{\lambda} = \pm \frac{\cos\varphi k_1 - \sin\varphi k_3}{(1-\lambda k_1)^2 + \lambda^2 k_3^2}. \tag{18}$$

By a similar method, k_2^{λ} and k_3^{λ} can be obtained as

$$k_2^{\lambda} = \pm \frac{\sin\varphi(d\varphi + k_2)\lambda k_3 + \cos\varphi(d\varphi + k_2)(1-\lambda k_1)}{(1-\lambda k_1)^2 + \lambda^2 k_3^2}, \tag{19}$$

and

$$k_3^{\lambda} = \pm \frac{\cos\varphi k_3 + \sin\varphi k_1}{\sqrt{(1-\lambda k_1)^2 + \lambda^2 k_3^2}}. \tag{20}$$

Substituting (10) and (17) into (18) and (20), gives

$$k_1^{\lambda} = \pm \frac{(1-\lambda k_1)k_1 - \lambda^2 k_3^2}{(1-\lambda k_1)^2 + \lambda^2 k_3^2},$$

and

$$k_3^{\lambda} = \pm \frac{k_3}{(1-\lambda k_1)^2 + \lambda^2 k_3^2}.$$

On the other hand, from (10) and (19), we have

$$\tan\varphi = \frac{\lambda k_3}{(1-\lambda k_1)}.$$

By differentiating above equation gives

$$d\varphi = \pm \frac{\lambda k_3'(1-\lambda k_1) + \lambda^2 k_1' k_3}{(1-\lambda k_1)^2 + \lambda^2 k_3^2}. \tag{21}$$

Substituting (10), (17) and (21) into (19) gives

$$k_2^{\lambda} = \pm \frac{\lambda k_3'(1-\lambda k_1) + \lambda^2 k_1' k_3}{(1-\lambda k_1)^2 + \lambda^2 k_3^2} + k_2.$$

This concludes the proof.

3. EXAMPLES

In this section, we have made several examples to show the advantages of this new approach.

Example 1. In this example, we construct the directional Bertrand mate of the line parametrized by $\alpha(t) = (t, t, t)$.

The q-frame of the line is obtained by

$$\mathbf{t} = (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}).$$

By using (1) and $\mathbf{k} = (0,0,1)$ we have the quasi-normal vector as follows

$$\mathbf{n}_q = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0),$$

and

$$\mathbf{b}_q = (\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}).$$

For $\lambda = 1$, the directional Bertrand mate of the line is parametrized by

$$\beta(t) = (t + \frac{\sqrt{2}}{2}, t - \frac{\sqrt{2}}{2}, t).$$

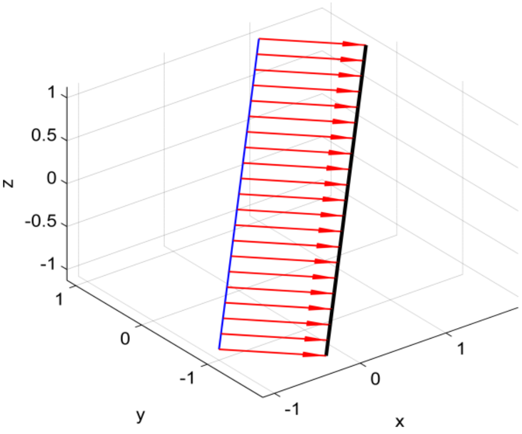

Figure 1. *The curve(blue) with directional Bertrand mate(black). The quasi-normal(red) vectors are shown.*

Figure 1 shows the directional Bertrand mate of the line.

Example 2. Assume that the curve is given by

$$\alpha(t)=(t,t,t^9).$$

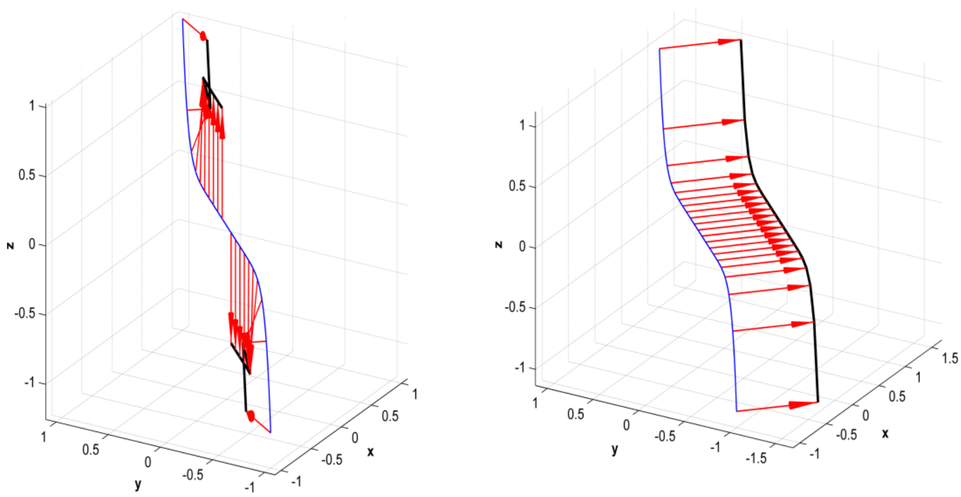

Figure 2. *The curve (blue) and; left: The Bertrand mate (black) with the Frenet normal vectors (red), right: The directional Bertrand mate (black) with the quasi-normal vectors (red).*

It is easy to see that the curvature κ and the torsion τ of the curve are obtained by

$$\kappa=\frac{72\sqrt{2}t^7}{(2+81t^{16})^{\frac{3}{2}}},\quad \tau=0,$$

respectively. Thus this curve is a plane curve. Therefore, this curve admits infinite Bertrand mates.

For $\lambda=1$, the Bertrand mate and the directional Bertrand mate of the curve are shown in Figure 2.

Observe that, there are two issues that we need to discuss. One is that; hence $\tau = 0$, the angle between the rotation minimizing frame (Bishop frame) and Frenet frame is constant, therefore Bishop frame is also not suitable for this example and the other is that; as shown in Figure 2, the Frenet normal vector can not be calculated at the point (0,0,0), therefore, the Bertrand mate of the curve is not accurate. There is no doubt that the best result is obtained by using q-frame.

Example 3. Let us consider the curve parametrized by

$$\alpha(t) = (\frac{(1+t)^{\frac{3}{2}}}{3}, \frac{(1-t)^{\frac{3}{2}}}{3}, \frac{\sqrt{2}t}{2}).$$

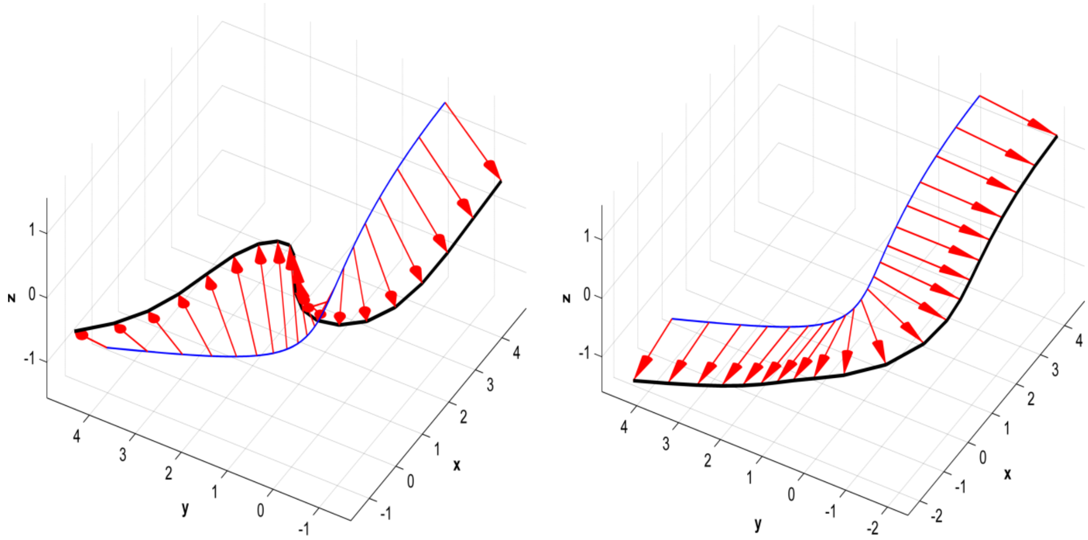

Figure 3. *The curve (blue) and; left: The Bertrand mate (black) with the Frenet normal vectors (red), right: The directional Bertrand mate (black) with the quasi-normalvectors (red).*

It is easy to see that

$$\kappa = -\frac{1}{4}\frac{\sqrt{2}}{\sqrt{1-t^2}}, \text{and } \tau = \frac{1}{4}\frac{\sqrt{2}}{\sqrt{1-t^2}}.$$

Thus, there is linear relation between the curvature and the torsion of the curve, this curve admits infinite Bertrand mates. It should be pointed out, however, that the normal vector of $\alpha(t)$ has undesirable rotation which causes the twist in the Bertrand mate. For $\lambda = 3$, the Bertrand mate and the directional Bertrand mate of the curve $\alpha(t)$ are shown in Figure 3.

4. CONCLUSION

In this paper, we give a new idea to solve the stability problem of parallel plane curves (see Figure 2). This idea can be summarized by the following useful rule:

Assume that we have a plane curve in three dimensional space then, by using the q-frame, we can construct parallel curve in the z-axis direction $\mathbf{k} = (0,0,1)$. The parallel curves constructed by using this new technique can be used in geometric design without requirements of any additional approximation methods.

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