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SOME NEW INEQUALITIES OF HERMITE-HADAMARD-FEJÉR TYPE FOR $s-$CONVEX FUNCTIONS

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Abstract. In this paper, we establish some new inequalities for differentiable mappings whose derivatives in absolute value are $s-$convex in the second sense. These results are connected with the celebrated Hermite-Hadamard-Fejér type integral inequality.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval I of real numbers, $a, b \in I$ and $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave.

Many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard inequality, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([3]-[7],[11]-[15],[17]) and the references therein.

Definition 1.1. Let real function f be defined on a nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

The class of functions which are s-convex in the second sense has been given as the following (see [9]).

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Definition 1.2. A function $f:[0,\infty)\to\mathbb{R}$ is said to be $s-$convex in the second sense, if

$$f(tx+(1-t)y)\leq t^{s}f(x)+(1-t)^{s}f(y)$$

holds for all $x,y\in[0,\infty)$, $t\in[0,1]$ and for some fixed $s\in(0,1]$.

Some interesting and important inequalities for $s-$convex (in the second sense) functions can be found in [1],[10],[13]-[16]. It can be easily seen that convexity means just $s-$convexity when $s=1$.

In [8], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality:

Theorem 1.1. *Let $f:I\to\mathbb{R}$ be convex on I and let $a,b\in I$ with $a<b$. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_{a}^{b}g(x)dx\leq\int_{a}^{b}f(x)g(x)dx\leq\frac{f(a)+f(b)}{2}\int_{a}^{b}g(x)dx \tag{1.2}$$

holds, where $g:[a,b]\to\mathbb{R}$ is nonnegative and symmetric to $\frac{a+b}{2}$.

If $g=1$, then we are talking about the Hermite-Hadamard inequalities. More about those inequalities can be found in a number of papers and monographs. For recent results and generalizations concerning Fejér inequality (1.2) see ([2],[18]-[24]).

In [1], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for $s-$convex functions in the second sense:

Theorem 1.2. *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L^{1}[a,b]$, then the following inequalities hold:*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(x)dx\leq\frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k=\frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

The main purpose of this paper is to establish new Fejér type inequalities for the class of functions whose derivatives in absolute value at certain powers are $s-$convex in the second sense.

2. MAIN RESULTS

In order to prove our main results, we need the following Lemmas (see [22]):

Lemma 2.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be differentiable on I° and $a,b\in I^{\circ}$ with $a<b$ and let $g:[a,b]\to[0,\infty)$. If $f',g\in L[a,b]$, then the following identity holds:*

$$f\left(\frac{a+b}{2}\right)\int_{a}^{b}g(t)dt-\int_{a}^{b}f(t)g(t)dt=\int_{a}^{b}p(t)f'(t)dt$$

for each $t\in[a,b]$, where

$$p(t)=\begin{cases}\int_{a}^{t}g(s)ds, & t\in\left[a,\frac{a+b}{2}\right)\\[2mm] -\int_{t}^{b}g(s)ds, & t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

Lemma 2.2. *Let $f : I \to \mathbb{R}$ be differentiable on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$. If $f', g \in L[a, b]$, then the following identity holds:*

$$\int_a^b f(u)g(u)du - f(x)\int_a^b g(u)du = (b-a)^2 \int_0^1 k(t)f'(ta+(1-t)b)dt$$

for each $t \in [0, 1]$ and $x, u \in [a, b]$, where

$$k(t) = \begin{cases} \int_0^t g(sa+(1-s)b)ds, & t \in \left[0, \frac{b-x}{b-a}\right) \\ -\int_t^1 g(sa+(1-s)b)ds, & t \in \left[\frac{b-x}{b-a}, 1\right]. \end{cases} \tag{2.1}$$

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$. If $f', g \in L[a, b]$ and $|f'|$ is $s-$convex on $[a, b]$, for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\ \leq\ & \frac{(b-a)^2}{2^{s+2}(s+1)(s+2)} \left\{ \|g\|_{\left[a,\frac{a+b}{2}\right],\infty} \left[(2^{s+2}-(s+3))\left|f'(a)\right| + (s+1)\left|f'(b)\right| \right] \right. \\ &\left. + \|g\|_{\left[\frac{a+b}{2},b\right],\infty} \left[(s+1)\left|f'(a)\right| + (2^{s+2}-(s+3))\left|f'(b)\right| \right] \right\}. \end{aligned}$$

Proof. By Lemma 2.1 and since $|f'|$ is s-convex on $[a, b]$, then we have

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\ \leq\ & \int_a^{\frac{a+b}{2}} \left| \int_a^t g(s)ds \right| \left|f'(t)\right| dt + \int_{\frac{a+b}{2}}^b \left| \int_t^b g(s)ds \right| \left|f'(t)\right| dt \\ \leq\ & \|g\|_{\left[a,\frac{a+b}{2}\right],\infty} \int_a^{\frac{a+b}{2}} (t-a)\left|f'(t)\right| dt + \|g\|_{\left[\frac{a+b}{2},b\right],\infty} \int_{\frac{a+b}{2}}^b (b-t)\left|f'(t)\right| dt \\ \leq\ & \|g\|_{\left[a,\frac{a+b}{2}\right],\infty} \int_a^{\frac{a+b}{2}} (t-a) \left[\left(\frac{b-t}{b-a}\right)^s \left|f'(a)\right| + \left(\frac{t-a}{b-a}\right)^s \left|f'(b)\right| \right] dt \\ & + \|g\|_{\left[\frac{a+b}{2},b\right],\infty} \int_{\frac{a+b}{2}}^b (b-t) \left[\left(\frac{b-t}{b-a}\right)^s \left|f'(a)\right| + \left(\frac{t-a}{b-a}\right)^s \left|f'(b)\right| \right] dt \\ =\ & \frac{(b-a)^2}{2^{s+2}(s+1)(s+2)} \left\{ \|g\|_{\left[a,\frac{a+b}{2}\right],\infty} \left[(2^{s+2}-(s+3))\left|f'(a)\right| + (s+1)\left|f'(b)\right| \right] \right. \\ & \left. + \|g\|_{\left[\frac{a+b}{2},b\right],\infty} \left[(s+1)\left|f'(a)\right| + (2^{s+2}-(s+3))\left|f'(b)\right| \right] \right\} \end{aligned}$$

where use the facts that

$$\begin{aligned} \int_a^{\frac{a+b}{2}} (t-a)\left(\frac{b-t}{b-a}\right)^s dt &= \int_{\frac{a+b}{2}}^b (b-t)\left(\frac{t-a}{b-a}\right)^s dt \\ &= \frac{(b-a)^2\left(2^{s+2}-(s+3)\right)}{2^{s+2}(s+1)(s+2)} \end{aligned}$$

and

$$\int_a^{\frac{a+b}{2}} (t-a)\left(\frac{t-a}{b-a}\right)^s dt = \int_{\frac{a+b}{2}}^b (b-t)\left(\frac{b-t}{b-a}\right)^s dt = \frac{(b-a)^2}{2^{s+2}(s+2)}.$$

which completes the proof. □

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$. If $f', g \in L[a, b]$ and $|f'|^q$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $p > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}} \left\{ \|g\|_{[a,\frac{a+b}{2}],\infty} \left(\frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} \right. \\
&\left. + \|g\|_{[\frac{a+b}{2},b],\infty} \left(\frac{|f'(a)|^q + (2^{s+1}-1)|f'(b)|^q}{2^s(s+1)} \right)^{\frac{1}{q}} \right\} \\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}} \left(\frac{1}{2^s(s+1)} \right)^{\frac{1}{q}} \\
&\times \|g\|_{[a,b],\infty} \left\{ \left[1 + (2^{s+1}-1)^{\frac{1}{q}} \right] (|f'(a)| + |f'(b)|) \right\}
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Suppose that $p > 1$. From Lemma 2.1 and using the Hölder inequality, we obtain

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt - \int_a^b f(t)g(t)dt \right| \\
\leq\ & \int_a^{\frac{a+b}{2}} \left| \int_a^t g(s)ds \right| |f'(t)|\, dt + \int_{\frac{a+b}{2}}^b \left| \int_t^b g(s)ds \right| |f'(t)|\, dt \\
\leq\ & \left(\int_a^{\frac{a+b}{2}} \left| \int_a^t g(s)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q dt \right)^{\frac{1}{q}} \\
&+ \left(\int_{\frac{a+b}{2}}^b \left| \int_t^b g(s)ds \right|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q dt \right)^{\frac{1}{q}} \\
\leq\ & \|g\|_{[a,\frac{a+b}{2}],\infty} \left(\int_a^{\frac{a+b}{2}} |t-a|^p dt \right)^{\frac{1}{p}} \left(\int_a^{\frac{a+b}{2}} |f'(t)|^q dt \right)^{\frac{1}{q}} \\
&+ \|g\|_{[\frac{a+b}{2},b],\infty} \left(\int_{\frac{a+b}{2}}^b |b-t|^p dt \right)^{\frac{1}{p}} \left(\int_{\frac{a+b}{2}}^b |f'(t)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Using the s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b g(t)f(t)dt \right| \\
\leq\ & \|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left[\frac{(b-a)^{p+1}}{2^{p+1}(p+1)}\right]^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}\left[\left(\frac{b-t}{b-a}\right)^s|f'(a)|^q+\left(\frac{t-a}{b-a}\right)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
&+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\left[\frac{(b-a)^{p+1}}{2^{p+1}(p+1)}\right]^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b\left[\left(\frac{b-t}{b-a}\right)^s|f'(a)|^q+\left(\frac{t-a}{b-a}\right)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left\{\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right. \\
&\left.+\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Let $a_1=(2^{s+1}-1)|f'(a)|^q$, $b_1=|f'(b)|^q$, $a_2=|f'(a)|^q$, $b_2=(2^{s+1}-1)|f'(b)|^q$. Here, $0<\frac{1}{q}<1$ for $q>1$. Using the fact that

$$\sum_{k=1}^n(a_k+b_k)^s\leq\sum_{k=1}^n a_k^s+\sum_{k=1}^n b_k^s$$

for $(0\leq s<1)$, $a_1,a_2,...,a_n\geq 0$, $b_1,b_2,...,b_k$; we obtain

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt - \int_a^b g(t)f(t)dt \right| \\
\leq\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}} \\
&\times\|g\|_{[a,b],\infty}\left\{(2^{s+1}-1)^{\frac{1}{q}}|f'(a)|+|f'(b)|+|f'(a)|+(2^{s+1}-1)^{\frac{1}{q}}|f'(b)|\right\} \\
=\ & \frac{(b-a)^2}{4(p+1)^{1/p}}\left(\frac{1}{2^s(s+1)}\right)^{\frac{1}{q}} \\
&\times\|g\|_{[a,b],\infty}\left\{\left[1+(2^{s+1}-1)^{\frac{1}{q}}\right](|f'(a)|+|f'(b)|)\right\}.
\end{aligned}$$

Also

$$\|g\|_{\left[a,\frac{a+b}{2}\right],\infty}\leq\|g\|_{[a,b],\infty}$$

and

$$\|g\|_{\left[\frac{a+b}{2},b\right],\infty}\leq\|g\|_{[a,b],\infty}.$$

This completes the proof. □

Theorem 2.3. *Let $f: I^\circ\subset\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$ and $g:[a,b]\to[0,\infty)$ be differentiable mapping. If $|f'|$ is s-convex on*

$[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:

$$\begin{aligned}
&\left| f(x)\int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \\
\leq\ & \frac{1}{(b-a)^s(s+2)} \\
&\times \left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty} \left[(b-x)^{s+2}\left|f'(a)\right| + \frac{(b-a)^{s+2}+(x-a)^{s+1}[(x-b)(s+1)-(b-a)]}{s+1}\left|f'(b)\right| \right] \right. \\
&\left. + \|g\|_{\left[\frac{b-x}{b-a},1\right],\infty} \left[\frac{(b-a)^{s+2}+(b-x)^{s+1}[(a-x)(s+1)-(b-a)]}{s+1}\left|f'(a)\right| + (x-a)^{s+2}\left|f'(b)\right| \right] \right\}.
\end{aligned}$$

Proof. Let $x \in [a,b]$. Using Lemma 2.2, we obtain

$$\begin{aligned}
&\left| f(x)\int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \\
\leq\ & (b-a)^2 \left\{ \int_0^{\frac{b-x}{b-a}} \left| \int_0^t g(sa+(1-s)b)ds \right| \left|f'(ta+(1-t)b)\right| dt \right. \\
&\left. + \int_{\frac{b-x}{b-a}}^1 \left| \int_t^1 g(sa+(1-s)b)ds \right| \left|f'(ta+(1-t)b)\right| dt \right\} \\
\leq\ & (b-a)^2 \left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty} \int_0^{\frac{b-x}{b-a}} |t| \left|f'(ta+(1-t)b)\right| dt \right. \\
&\left. + \|g\|_{\left[\frac{b-x}{b-a},1\right],\infty} \int_{\frac{b-x}{b-a}}^1 |1-t| \left|f'(ta+(1-t)b)\right| dt \right\}.
\end{aligned}$$

Since $|f'|$ is s-convex on $[a,b]$, we obtain

$$\begin{aligned}
&\left| f(x)\int_a^b g(u)du - \int_a^b f(u)g(u)du \right| \\
\leq\ & (b-a)^2 \left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty} \int_0^{\frac{b-x}{b-a}} t\left[t^s\left|f'(a)\right| + (1-t)^s\left|f'(b)\right|\right] dt \right. \\
&\left. + \|g\|_{\left[\frac{b-x}{b-a},1\right],\infty} \int_{\frac{b-x}{b-a}}^1 (1-t)\left[t^s\left|f'(a)\right| + (1-t)^s\left|f'(b)\right|\right] dt \right\} \\
=\ & \frac{1}{(b-a)^s(s+2)} \\
&\times \left\{ \|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty} \left[(b-x)^{s+2}\left|f'(a)\right| + \frac{(b-a)^{s+2}+(x-a)^{s+1}[(x-b)(s+1)-(b-a)]}{s+1}\left|f'(b)\right| \right] \right. \\
&\left. + \|g\|_{\left[\frac{b-x}{b-a},1\right],\infty} \left[\frac{(b-a)^{s+2}+(b-x)^{s+1}[(a-x)(s+1)-(b-a)]}{s+1}\left|f'(a)\right| + (x-a)^{s+2}\left|f'(b)\right| \right] \right\}.
\end{aligned}$$

This completes the proof. □

Theorem 2.4. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be differentiable mapping. If $|f'|^q$ is s-convex* *on $[a,b]$, for some fixed $s\in(0,1]$ and $p>1$, then the following inequality holds:*

$$\begin{aligned}
&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\
\leq\ &\frac{1}{(b-a)^{\frac{s}{q}}(p+1)^{\frac{1}{p}}}\\
&\times\left\{\|g\|_{[0,\frac{b-x}{b-a}],\infty}\left[\frac{(b-x)^{2q+s}\left|f'(a)\right|^q+(b-x)^{2q-1}[(b-a)^{s+1}-(x-a)^{s+1}]\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}\right.\\
&\left.+\|g\|_{[\frac{b-x}{b-a},1],\infty}\left[\frac{(x-a)^{2q-1}[(b-a)^{s+1}-(b-x)^{s+1}]\left|f'(a)\right|^q+(x-a)^{2q+s}\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.2, Hölder's inequality and the s-convexity of $|f'|^q$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}
&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\
\leq\ &(b-a)^2\\
&\times\left\{\left(\int_0^{\frac{b-x}{b-a}}\left|\int_0^t g(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{b-x}{b-a}}^1\left|\int_t^1 g(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
\leq\ &(b-a)^2\left\{\|g\|_{[0,\frac{b-x}{b-a}],\infty}\left(\int_0^{\frac{b-x}{b-a}}t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.+\|g\|_{[\frac{b-x}{b-a},1],\infty}\left(\int_{\frac{b-x}{b-a}}^1(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}\right\}\\
=\ &\frac{1}{(b-a)^{\frac{s}{q}}(p+1)^{\frac{1}{p}}}\\
&\times\left\{\|g\|_{[0,\frac{b-x}{b-a}],\infty}\left[\frac{(b-x)^{2q+s}\left|f'(a)\right|^q+(b-x)^{2q-1}[(b-a)^{s+1}-(x-a)^{s+1}]\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}\right.\\
&\left.+\|g\|_{[\frac{b-x}{b-a},1],\infty}\left[\frac{(x-a)^{2q-1}[(b-a)^{s+1}-(b-x)^{s+1}]\left|f'(a)\right|^q+(x-a)^{2q+s}\left|f'(b)\right|^q}{s+1}\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. □

Theorem 2.5. *Let $f:I^\circ\subset\mathbb{R}\to\mathbb{R}$ be differentiable mapping on I° and $a,b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be differentiable mapping. If $|f'|^q$ is s-convex*

on $[a,b]$, *for some fixed* $s\in(0,1]$ *and* $p>1$, *then the following inequality holds:*

$$\begin{aligned}&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\ \leq\ &\frac{1}{(p+1)^{\frac{1}{p}}}\left\{\|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}(b-x)^2\left[\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right]^{\frac{1}{q}}\right.\\ &\left.+\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}(x-a)^2\left[\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.2 and using the Hölder inequality, we have

$$\begin{aligned}&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\ \leq\ &(b-a)^2\\ &\times\left\{\left(\int_0^{\frac{b-x}{b-a}}\left|\int_0^t g(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\frac{b-x}{b-a}}^1\left|\int_t^1 g(sa+(1-s)b)ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right\}\\ \leq\ &(b-a)^2\left\{\|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}\left(\int_0^{\frac{b-x}{b-a}}t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{b-x}{b-a}}|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}\left(\int_{\frac{b-x}{b-a}}^1(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{b-x}{b-a}}^1|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Since $|f'|^q$ is s-convex, by (1.3) we have

$$\int_0^{\frac{b-x}{b-a}}|f'(ta+(1-t)b)|^q dt\leq\frac{b-x}{b-a}\left(\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right)$$

and

$$\int_{\frac{b-x}{b-a}}^1|f'(ta+(1-t)b)|^q dt\leq\frac{x-a}{b-a}\left(\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right).$$

Therefore,

$$\begin{aligned}&\left|f(x)\int_a^b g(u)du-\int_a^b f(u)g(u)du\right|\\ \leq\ &\frac{1}{(p+1)^{\frac{1}{p}}}\left\{\|g\|_{\left[0,\frac{b-x}{b-a}\right],\infty}(b-x)^2\left[\frac{|f'(x)|^q+|f'(b)|^q}{s+1}\right]^{\frac{1}{q}}\right.\\ &\left.+\|g\|_{\left[\frac{b-x}{b-a},1\right],\infty}(x-a)^2\left[\frac{|f'(a)|^q+|f'(x)|^q}{s+1}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

This completes the proof. □

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