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Quantum metric spaces of quantum maps

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Abstract

We show that any quantum family of quantum maps from a noncommutative space to a compact quantum metric space has a canonical quantum pseudo-metric structure. Here by a 'compact quantum metric space' we mean a unital C*-algebra together with a Lipschitz seminorm, in the sense of Rieffel, which induces the weak* topology on the state space of the C*-algebra. Our main result generalizes a classical result to noncommutative world.

1. Introduction

One of the basic ideas of *Noncommutative Geometry* is that any unital C*-algebra A can be considered as the algebra of *continuous functions* on a (symbolic) *compact quantum (noncommutative) space* $\mathfrak{Q}A$. From this point of view, any unital *-homomorphism $\Phi : B \to A$ between unital C*-algebras can be interpreted as a *quantum map* $\mathfrak{Q}\Phi$ from $\mathfrak{Q}A$ into $\mathfrak{Q}B$. There are many notions in Topology and Geometry that can be translate into NC language. The notion of *quantum family of (quantum) maps*, defined by Woronowicz [16] and Sołtan [15] (see also [10, 11, 12]), conclude from the following fact: "Every map f from X to the set of all maps from Y to Z (or in other word, any family of maps from Y to Z parameterized by f with parameters x in X) can be considered as a map $\tilde{f} : X \times Y \to Z$ defined by $\tilde{f}(x,y) = f(x)(y)$." A translation of this to noncommutative language is as follows.

Definition 1.1. *([10, 11, 12, 15, 16] Let B, C be unital C*-algebras. A quantum family of morphisms from B to C (or, a quantum family of maps from $\mathfrak{Q}C$ to $\mathfrak{Q}B$) is a pair (A, Φ) consisting of a unital C*-algebra A and a unital *-homomorphism $\Phi : B \to C \otimes A$, where $\otimes$ denotes the spatial tensor product of C*-algebras.*

Another concept that can be translate from Geometry into NC Geometry, is *distance* or *metric*. Marc Rieffel, by using the notion of *order unite spaces*, has developed the notion of *quantum metric space* in a series of papers [5, 6, 7, 8, 9]. For two other different notions of quantum metric see [3, 13, 14]. Here, we deals with special examples of Rieffel's quantum metric spaces, stated in the C*-algebraic formalism. The aim of this note is to show that any quantum family of maps from a quantum space to a compact quantum metric space has a canonical quantum pseudo-metric structure. We are motivated by the following trivial fact: Let (Z,d) be a metric space and $f : X \times Y \to Z$ be a family of maps from Y to Z, then X has a pseudo-metric ρ defined by

$$\rho(x,x') = \sup_{y \in Y} d(f(x,y), f(x',y)).$$

In Section 2 we introduce the notion of *compact quantum pseudo-metric space*. In Section 3 we define a natural compact quantum pseudo-metric space structure on any quantum family of maps from a quantum space to a compact quantum metric space. In Section 4 we examine our definition in the classical case.

2. Compact quantum pseudo-metric spaces

By a pseudo-metric d on a set X we mean a positive valued function on $X \times X$ which is symmetric, satisfies triangle inequality, and $d(x,x) = 0$ for every $x \in X$. For any topological space X with topology τ (resp. pseudo-metric space (X,d)) $\mathbf{C}(X,\tau)$ (resp. $\mathbf{C}(X,d)$) denotes the

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C*-algebra of all continuous bounded complex valued maps on X with the uniform norm. For a pseudo-metric d, τ_d denotes the topology induced by d. Let (X,d) be a pseudo-metric space. For every $f \in \mathbf{C}(X,d)$, the Lipschitz semi norm $\|f\|_d$ is defined by

$$\|f\|_d = \sup\{\frac{|f(x)-f(x')|}{d(x,x')} : x,x' \in X, d(x,x') \neq 0\}.$$

Also, the Lipschitz algebra of (X,d) is defined by,

$$\mathbf{Lip}(X,d) = \{f \in \mathbf{C}(X,d) : \|f\|_d < \infty\}.$$

We need the following simple lemma.

Lemma 2.1. *Let (X,d) be a pseudo-metric space and a be a complex valued map on X. Then $a \in \mathbf{Lip}(X,d)$ and $\|a\|_d \leq 1$ if and only if $|a(x)-a(x')| \leq d(x,x')$ for every $x,x' \in X$. In particular, if $b \in \mathbf{C}(X,d)$, then $\|b\|_d = 0$ if and only if b is a constant map.*

Proof. Let $a \in \mathbf{Lip}(X,d)$ and $\|a\|_d \leq 1$. Suppose that $x,x' \in X$. If $d(x,x') = 0$, then $a(x) = a(x')$, since a is continuous with τ_d. If $d(x,x') \neq 0$, then $1 \geq \|a\|_d \geq \frac{|a(x)-a(x')|}{d(x,x')}$, and thus $|a(x)-a(x')| \leq d(x,x')$. The other direction is trivial. □

For any C*-algebra $\mathfrak{A}$, $S(\mathfrak{A})$ denotes the state space of $\mathfrak{A}$ with w* topology. If $\mathfrak{A}$ is unital, $1_{\mathfrak{A}}$ denotes the unit element of $\mathfrak{A}$.
Let $\mathscr{A}$ be a self adjoint linear subspace of the C*-algebra $\mathfrak{A}$, and let $L : \mathscr{A} \to [0,\infty)$ be a semi norm on $\mathscr{A}$. Connes has pointed out [1], [2], that one can define a pseudo-metric ρ_L on $S(\mathfrak{A})$ by

$$\rho_L(\mu,\nu) = \sup\{|\mu(a)-\nu(a)| : a \in \mathscr{A}, L(a) \leq 1\} \qquad (\mu,\nu \in S(\mathfrak{A})). \tag{2.1}$$

Note that ρ_L can take values $+\infty$ and 0 for different states of $\mathfrak{A}$. Conversely, let d be a pseudo-metric on $S(A)$ (such that the topology induced by d on $S(\mathfrak{A})$ is not necessarily w* topology). Define a semi norm $L_d : \mathfrak{A} \to [0,+\infty]$ by

$$L_d(a) = \sup\{\frac{|\mu(a)-\nu(a)|}{d(\mu,\nu)} : \mu,\nu \in S(\mathfrak{A}), d(\mu,\nu) \neq 0\} \qquad (a \in \mathfrak{A}).$$

Note that $L_d(a) = L_d(a^*)$ for every $a \in \mathfrak{A}$.
Let (X,d) be a compact metric space. Consider the Lipschitz semi norm

$$\|\cdot\|_d : \mathbf{Lip}(X,d) \subset \mathbf{C}(X,d) \to [0,+\infty).$$

Then it is easily checked that the semi norm $\rho_{\|\cdot\|_d}$ on the state space of $\mathbf{C}(X,d)$ is a metric, called Monge-Kantorovich metric [4]. It is well known that the topology induced by $\rho_{\|\cdot\|_d}$, is the w* topology, and for every $x,y \in X$, $d(x,y) = \rho_{\|\cdot\|_d}(\delta_x,\delta_y)$, where $\delta : X \to \mathbf{C}(X,d)^*$ is the point mass measure map.

Proposition 2.2. *Let (X,τ) be a compact Hausdorff space and d be a pseudo-metric on X such that the topology induced by d on X is weaker than τ, i.e. $\tau_d \subset \tau$. Consider the Lipschitz semi norm $\|\cdot\|_d : \mathbf{Lip}(X,d) \subset \mathbf{C}(X,\tau) \to [0,+\infty)$ and let $\rho = \rho_{\|\cdot\|_d}$. Then the following are satisfied.*

i) $d(x,y) = \rho(\delta_x,\delta_y)$, for every $x,y \in X$.
ii) $L_\rho = \|\cdot\|_d$ on $\mathbf{C}(X,d) \subset \mathbf{C}(X,\tau)$.
iii) Let $a \in \mathbf{C}(X,\tau)$, then $a \in \mathbf{C}(X,d)$ if and only if the map $\nu \longmapsto \nu(a)$ on $S(\mathbf{C}(X,\tau))$ is continuous with ρ.
iv) the topology induced by ρ on $S(\mathbf{C}(X,\tau))$ is weaker than the w topology.*

Proof. i) Let x,y be in X. Suppose that $a \in \mathbf{Lip}(X,d)$ and $\|a\|_d \leq 1$. Then by Lemma 2.1, $|\delta_x(a)-\delta_y(a)| = |a(x)-a(y)| \leq d(x,y)$, and thus by definition of ρ, we have $\rho(\delta_x,\delta_y) \leq d(x,y)$. Conversely, let $a_x \in \mathbf{C}(X,d)$ be defined by $a_x(z) = d(x,z)$ $(z \in X)$; then for every $x',y' \in X$, $|a_x(x')-a_x(y')| = |d(x,x')-d(x,y')| \leq d(x',y')$, and thus by lemma 2.1, $a \in \mathbf{Lip}(X,d)$ and $\|a\|_d \leq 1$. Now, we have

$$\rho(\delta_x,\delta_y) \geq |\delta_x(a_x)-\delta_y(a_x)| = |a_x(x)-a_x(y)| = d(x,y).$$

ii) By i) and definitions of L_ρ and $\|\cdot\|_d$, it is clear that $\|\cdot\|_d \leq L_\rho$ on $\mathbf{C}(X,\tau)$.
Let $a \in \mathbf{C}(X,d)$. If $\|a\|_d = 0$, then by Lemma 2.1, a is a constant map and thus $L_\rho(a) = 0$. If $\|a\|_d = \infty$ then $L_\rho(a) = \infty$ since $\|a\|_d \leq L_\rho(a)$. Thus suppose that $0 < \|a\| < \infty$. Then for every $\mu,\nu \in S(\mathbf{C}(X,\tau))$, we have

$$\rho(\mu,\nu) \geq |\mu(\frac{a}{\|a\|_d}) - \nu(\frac{a}{\|a\|_d})| = \frac{|\mu(a)-\nu(a)|}{\|a\|_d}$$

and thus if $\rho(\mu,\nu) \neq 0$ then $\|a\|_d \geq \frac{|\mu(a)-\nu(a)|}{\rho(\mu,\nu)}$. Therefore,

$$\|a\|_d \geq \sup\{\frac{|\mu(a)-\nu(a)|}{\rho(\mu,\nu)} : \quad \mu,\nu \in S(\mathbf{C}(X,\tau)), \rho(\mu,\nu) \neq 0\} = L_\rho(a).$$

iii) The ‘if’ part is an immediate consequence of i). For the other direction, we need some notations: Let $\sim$ be the equivalence relation on X defined by $x \sim x' \Leftrightarrow d(x,x') = 0$. Let $Y = X/\sim$ and let $\hat{}: X \to Y$ be the canonical projection. Then $\hat{d}$, defined by $\hat{d}(\hat{x}_1,\hat{x}_2) = d(x_1,x_2)$, is a well defined metric on Y, and $\hat{}$ is an isometry between (X,d) and $(Y,\hat{d})$. Thus the C*-algebras $\mathbf{C}(X,d)$ and $\mathbf{C}(Y,\hat{d})$, and the Lipschitz algebras $(\mathbf{Lip}(X,d),\|\cdot\|_d)$ and $(\mathbf{Lip}(Y,\hat{d}),\|\cdot\|_{\hat{d}})$ are isometric isomorph. In particular, the topology induced by ρ on $S(\mathbf{C}(X,d))$ is the w* topology, since as mentioned above the Monge-Kantorovich metric $\rho_{\|\cdot\|_{\hat{d}}}$ induces the w* topology on $S(\mathbf{C}(Y,\hat{d}))$. Consider the canonical embedding $\Phi : \mathbf{C}(X,d) \to \mathbf{C}(X,\tau)$. For every $\nu,\nu' \in S(\mathbf{C}(X,\tau))$, $\nu \circ \Phi$ and $\nu' \circ \Phi$ are in $S(\mathbf{C}(X,d))$ and

$$\rho(\nu,\nu') = \rho(\nu \circ \Phi, \nu' \circ \Phi). \tag{2.2}$$

Now, let $a \in \mathbf{C}(X,d)$ and $\nu_i \to \nu$ be a convergent net in $S(\mathbf{C}(X,\tau))$ with ρ. Then $\nu_i \circ \Phi \to \nu \circ \Phi$ is a convergent net in $S(\mathbf{C}(X,d))$ with ρ, and since the topology induced by ρ agrees with the w* topology on $S(\mathbf{C}(X,d))$, we have

$$\nu_i(a) = \nu_i \circ \Phi(a) \to \nu \circ \Phi(a) = \nu(a).$$

Thus we get the desired result.
iv) Let $\nu_i \to \nu$ be a convergent net in $S(\mathbf{C}(X,\tau))$ with w* topology. Thus as in the proof of iii), $\nu_i \circ \Phi \to \nu \circ \Phi$ with ρ, and by (2.2), $\nu_i \to \nu$ in $S(\mathbf{C}(X,\tau))$ with the topology induced by ρ. This completes the proof of iv). □

Definition 2.3. *By a compact quantum pseudo-metric space (QSM space, for short) we mean a triple $(\mathfrak{A}, \mathscr{A}, L)$, where $\mathfrak{A}$ is a unital C*-algebra, $\mathscr{A}$ is a self adjoint linear subspace of $\mathfrak{A}$ with $1_{\mathfrak{A}} \in \mathscr{A}$, and $L : \mathscr{A} \to [0,+\infty)$ is a semi norm such that*

(a) $L(a) = L(a^*)$ for every $a \in \mathscr{A}$,
(b) for every $a \in \mathscr{A}$, $L(a) = 0$ if and only if $a \in \mathbb{C}1_{\mathfrak{A}}$, and
(c) the topology induced by the pseudo-metric ρ_L on $S(\mathfrak{A})$ is weaker than the w topology.*

As an immediate corollary of the definition, for any compact quantum pseudo-metric space $(\mathfrak{A}, \mathscr{A}, L)$, the topology induced by ρ_L on $S(\mathfrak{A})$ is compact and in particular the diameter of $S(\mathfrak{A})$ under ρ_L is finite.

Proposition 2.4. *Let $(\mathfrak{A}, \mathscr{A}, L)$ be a QSM space. Then, for every $a \in \mathscr{A}$, the map $\mu \longmapsto \mu(a)$ on $S(\mathfrak{A})$ is continuous with topology induced by ρ_L.*

Proof. Straightforward. □

Definition 2.5. *A QSM space $(\mathfrak{A}, \mathscr{A}, L)$ is called a compact quantum metric space (QM space, for short) if $\mathscr{A}$ is a dense subspace of $\mathfrak{A}$.*

Let $(\mathfrak{A}, \mathscr{A}, L)$ be a QM space and μ, ν be two different states of $\mathfrak{A}$. Then since $\mathscr{A}$ is dense in $\mathfrak{A}$, there is $a \in \mathscr{A}$ such that $\mu(a) \neq \nu(a)$. Thus (by (2.1)) ρ_L is a metric on $S(\mathfrak{A})$. It is an elementary result in Topology that any Hausdorff topology τ weaker than a compact Hausdorff topology τ' on a set X, is equal to the same topology τ'. Using this, we conclude that the topology induced by ρ_L on $S(\mathfrak{A})$ is the w* topology.

Example 2.6. *Let (X,d) be a compact metric space. Then*

$$(\mathbf{C}(X,d), \mathbf{Lip}(X,d), \|\cdot\|_d)$$

is a compact quantum metric space.

Example 2.7. *Let (X,τ) be a compact Hausdorff space and let d be a pseudo-metric on X such that $\tau_d \subset \tau$. Then Proposition 2.2 and Lemma 2.1, show*

$$(\mathbf{C}(X,\tau), \mathbf{Lip}(X,d), \|\cdot\|_d)$$

is a compact quantum pseudo-metric space.

Remark 2.8. *Let $(\mathfrak{A}, \mathscr{A}, L)$ be a QM space and $A \subset \mathscr{A}$ be the linear subspace of all self-adjoint elements of $\mathscr{A}$. Then A is an order unite space and $(A, L|_A)$ is a compact quantum metric space in the sense of Rieffel's definition [7].*

Lemma 2.9. *Let $\mathfrak{A}$ be a C*-algebra with the C*-norm $\|\cdot\|$, $\mathscr{A}$ be a self adjoint linear subspace of $\mathfrak{A}$ containing $1_{\mathfrak{A}}$ and $L : \mathscr{A} \to [0,+\infty)$ be a semi norm such that for every $a \in \mathscr{A}$, $L(a) = 0$ if and only if $a \in \mathbb{C}1_{\mathfrak{A}}$. Let $\tilde{L}$ and $\|\cdot\|^{\sim}$ denote the quotient norm of L and $\|\cdot\|$ on $\frac{\mathscr{A}}{\mathbb{C}1_{\mathfrak{A}}}$ and $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$, respectively. Suppose that the image of $\{a \in \mathscr{A} : L(a) \leq 1\}$ in $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$ is totally bounded for $\|\cdot\|^{\sim}$. Then the topology induced by ρ_L on $S(\mathfrak{A})$ is weaker than the w* topology.*

Proof. See Theorem 1.8 of [5]. □

Example 2.10. *Let $\mathfrak{A}$ be a finite dimensional C*-algebra and N be a Banach space norm on $\mathfrak{A}$ such that $N(a) = N(a^*)$ for every $a \in \mathfrak{A}$. Let the semi norm $N_0 : \mathfrak{A} \to [0,\infty)$ be defined by*

$$N_0 = \inf\{N(a + \lambda 1_{\mathfrak{A}}) : \quad \lambda \in \mathbb{C}\}.$$

Since $\mathfrak{A}$ is finite dimensional, the C-norm of $\mathfrak{A}$ and N are equivalent. Thus the image K of $\{a \in \mathfrak{A} : N_0(a) \leq 1\}$ is closed and bounded in $\frac{\mathfrak{A}}{\mathbb{C}1_{\mathfrak{A}}}$. Again, since $\mathfrak{A}$ is finite dimensional, K is compact and thus totally bounded for the quotient norm of the C*-norm. Thus by Lemma 2.9, $(\mathfrak{A}, \mathfrak{A}, N_0)$ is a QM space.*

Example 2.11. *Let G be a compact Hausdorff group with identity element e. Let ℓ be a length function on G, i.e. ℓ is a continuous non negative real valued function on G such that*

(i) $\ell(gg') \leq \ell(g) + \ell(g')$, for every $g, g' \in G$,
(ii) $\ell(g) = \ell(g^{-1})$ for every $g \in G$, and
(iii) $\ell(g) = 0$ if and only if $g = e$.

Let $\mathfrak{A}$ be a unital C-algebra with a strongly continuous action $\cdot : G \times \mathfrak{A} \to \mathfrak{A}$ of G by automorphisms of $\mathfrak{A}$, i.e.*

*(a) for every $g \in G$ the map $a \longmapsto g \cdot a$ is a *-automorphism of $\mathfrak{A}$,*
(b) $e \cdot a = a$ for every $a \in \mathfrak{A}$,
(c) $g \cdot (g' \cdot a) = (gg') \cdot a$, for every $g, g' \in G, a \in A$, and
(d) if $g_i \to g$ is a convergent net in G and $a \in \mathfrak{A}$, then $g_i \cdot a \to g \cdot a$ with the C-norm of $\mathfrak{A}$.*

Define a semi norm L on $\mathfrak{A}$ by

$$L(a)=\sup\{\frac{\|g\cdot a-a\|}{\ell(g)}:\quad g\in G, g\neq e\}\qquad (a\in\mathfrak{A}).$$

*Let $\mathscr{A}=\{a\in\mathfrak{A}: L(a)<+\infty\}$. Then by Proposition 2.2 of [5], $\mathscr{A}$ is a dense *-subalgebra of $\mathfrak{A}$. Now, suppose that the action of G is ergodic, i.e. if $a\in\mathfrak{A}$ and for every $g\in G$, $g\cdot a=a$, then $a\in\mathbb{C}1_{\mathfrak{A}}$. Then it is trivial that $L(a)=0$ if and only if $a\in\mathbb{C}1_{\mathfrak{A}}$. Rieffel has proved [5, Theorem 2.3], that the topology induced by ρ_L on $S(\mathfrak{A})$ agrees with the w* topology. Thus $(\mathfrak{A},\mathscr{A},L)$ is a QM space.*

For some other examples that completely match our notion of QM space, see [5]. As we will see in the next section, using quantum family of morphisms we can construct many QSM spaces from a QSM space.

3. The main definition

We need the following simple topological lemma.

Lemma 3.1. *Let Y be a compact space, X be an arbitrary space and (Z,ρ) be a pseudo-metric space. Also, let $\mathbf{C}(Y,Z)$ be the space of all continuous maps from Y to Z, with the pseudo-metric $\hat{\rho}$ defined by*

$$\hat{\rho}(f,g)=\sup\{\rho(f(y),g(y)):\quad y\in Y\}\qquad (f,g\in\mathbf{C}(Y,Z)).$$

Suppose that $F:Y\times X\to Z$ is a continuous map. Then the map $\tilde{F}:X\to\mathbf{C}(Y,Z)$, defined by $\tilde{F}(x)(y)=F(y,x)$ is continuous.

Proof. Let $x_0\in X$ and $\varepsilon>0$ be arbitrary. Since F is continuous, for every $y\in Y$, there are open sets U_y,V_y in X and Y respectively, such that $(y,x_0)\in V_y\times U_y$ and $\rho(F(y,x_0),F(y',x))<\varepsilon/2$ for every $(y',x)\in V_y\times U_y$. Since Y is compact, there are $y_1,\cdots,y_n\in Y$ such that $Y=\cup_{i=1}^n V_{y_i}$. Let W be the open set $\cap_{i=1}^n U_{y_i}$. Let $x\in W$ and $y\in Y$ be arbitrary. Then for some i $(i=1,\cdots,n)$, y belongs to V_{y_i} and we have,

$$\rho(F(y,x),F(y,x_0))\leq\rho(F(y,x),F(y_i,x_0))+\rho(F(y_i,x_0),F(y,x_0))<\varepsilon.$$

Thus we have $\hat{\rho}(\tilde{F}(x),\tilde{F}(x_0))<\varepsilon$ for every $x\in W$. The proof is complete. □

Let $(\mathfrak{A},\mathscr{A},L)$ be a QSM space, $\mathfrak{B}$ be a unital C*-algebra, and $(\mathfrak{C},\Phi)$ be a quantum family of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$, $\Phi:\mathfrak{A}\to\mathfrak{B}\otimes\mathfrak{C}$. Let d be a pseudo-metric on $S(\mathfrak{C})$, defined by

$$d(\nu,\nu')=\sup\{\rho_L((\mu\otimes\nu)\Phi,(\mu\otimes\nu')\Phi):\quad \mu\in S(\mathfrak{B})\}\qquad (\nu,\nu'\in S(\mathfrak{C})).$$

Proposition 3.2. *With the above assumptions, let $\mathscr{C}$ be the linear space of all $c\in\mathfrak{C}$ such that the map $\nu\longmapsto\nu(c)$ on $S(\mathfrak{C})$ is continuous with the topology induced by d, and $L_d(c)<\infty$. Then the following are satisfied.*

- *i) $\mathscr{C}$ is a self adjoint linear subspace of $\mathfrak{C}$ and $1_{\mathfrak{C}}\in\mathscr{C}$.*
- *ii) For every $c\in\mathscr{C}$, $L_d(c)=0$ if and only if $c\in\mathbb{C}1_{\mathfrak{C}}$.*
- *iii) The topology induced by d on $S(\mathfrak{C})$ is weaker than the w* topology.*
- *iv) With the restriction of the domain of L_d to $\mathscr{C}$, $\rho_{L_d}\leq d$.*
- *v) The topology induced by ρ_{L_d} on $S(\mathfrak{C})$ is weaker than the w* topology.*

Proof. i) is easily checked.
ii) Let c be in $\mathscr{C}$ and $L_d(c)=0$. By Lemma 2.1, the map $\nu\longmapsto\nu(c)$ on $S(\mathfrak{C})$ is constant, and thus $c\in\mathbb{C}1_{\mathfrak{C}}$.
iii) Apply Lemma 3.1, with $X=S(\mathfrak{C})$, $Y=S(\mathfrak{B})$, $Z=S(\mathfrak{A})$, $\rho=\rho_L$ and $F:Y\times X\to Z$ defined by

$$F(\mu,\nu)=(\mu\otimes\nu)\Phi\qquad (\mu\in Y,\nu\in X).$$

We get $\tilde{F}:X\to\mathbf{C}(Y,Z)$ is continuous with the metric $\hat{\rho}$ on $\mathbf{C}(Y,Z)$. On the other hand, for every ν,ν' we have $d(\nu,\nu')=\hat{\rho}(\tilde{F}(\nu),\tilde{F}(\nu'))$. Thus, if $\nu_i\to\nu$ is a convergent net in X with w* topology, then

$$d(\nu_i,\nu)=\hat{\rho}(\tilde{F}(\nu_i),\tilde{F}(\nu))\to 0.$$

This implies that the topology induced by d is weaker than the w* topology.
iv) Let ν,ν' be in $S(\mathfrak{C})$. If $d(\nu,\nu')=0$ then for every $c\in\mathscr{C}$, $\nu(c)=\nu'(c)$ (since the map $\mu\longmapsto\mu(c)$ is continuous with d) and thus by the definition of ρ_{L_d}, $\rho_{L_d}(\nu,\nu')=0$. Thus suppose that $d(\nu,\nu')\neq 0$. Let $c\in\mathscr{C}$ with $L_d(c)\leq 1$. Then $1\geq L_d(c)\geq\frac{|\nu(c)-\nu'(c)|}{d(\nu,\nu')}$, and thus $|\nu(c)-\nu'(c)|\leq d(\nu,\nu')$. Therefore

$$\rho_{L_d}(\nu,\nu')\leq d(\nu,\nu').$$

v) follows directly from iv) and iii). □

Definition 3.3. *With the above assumptions, Proposition 3.2, shows that $(\mathfrak{C},\mathscr{C},L_d)$ is a QSM space that is called QSM space induced by the QSM space $(\mathfrak{A},\mathscr{A},L)$ and quantum family of maps $(\mathfrak{C},\Phi)$.*

Lemma 3.4. *With the above assumptions, let $a\in\mathscr{A}$ and let $\mu\in S(\mathfrak{B})$. Then $c=(\mu\otimes id_{\mathfrak{C}})\Phi(a)$ is in $\mathscr{C}$, and $L_d(c)\leq L(a)$.*

Proof. We first show that $L_d(c) \leq L(a)(<\infty)$. If $L(a)=0$ then $a\in\mathbb{C}1_{\mathfrak{A}}$ and thus $c\in\mathbb{C}1_{\mathfrak{C}}$ and $L_d(c)=0$. Suppose that $L(a)\neq 0$. We prove that for every $\nu,\nu'\in S(\mathfrak{C})$ with $d(\nu,\nu')\neq 0$,

$$\frac{|\nu(c)-\nu'(c)|}{d(\nu,\nu')} \leq L(a). \tag{3.1}$$

Let $\nu,\nu'\in S(\mathfrak{C})$ be such that $d(\nu,\nu')\neq 0$. If $|\nu(c)-\nu'(c)|=0$, then (3.1) is satisfied. Suppose that

$$|\nu(c)-\nu'(c)| = |(\mu\otimes\nu)\Phi(a)-(\mu\otimes\nu')\Phi(a)|\neq 0.$$

By the definition of d, we have $d(\nu,\nu')\geq \rho_L((\mu\otimes\nu)\Phi,(\mu\otimes\nu')\Phi)$. On the other hand, by the definition of ρ_L,

$$\begin{aligned}\rho_L((\mu\otimes\nu)\Phi,(\mu\otimes\nu')\Phi) &\geq |(\mu\otimes\nu)\Phi(\frac{a}{L(a)})-(\mu\otimes\nu')\Phi(\frac{a}{L(a)})|\\ &= \frac{|(\mu\otimes\nu)\Phi(a)-(\mu\otimes\nu')\Phi(a)|}{L(a)}.\end{aligned}$$

Thus, (3.1) is satisfied and $L_d(c)\leq L(a)$.
Now, we show that the map $\nu\longmapsto\nu(c)$ on $S(\mathfrak{C})$ is continuous with τ_d. Let $\nu_n\to\nu$ be a convergent sequence in $S(\mathfrak{C})$ with the metric d. Thus, by the definition of d, we have

$$\rho_L((\mu\otimes\nu_n)\Phi,(\mu\otimes\nu)\Phi)\to 0.$$

Therefore, by Proposition 2.4,

$$\nu_n(c)=(\mu\otimes\nu_n)\Phi(a)\to(\mu\otimes\nu)\Phi(a)=\nu(c).$$

□

Proposition 3.5. *With the above assumptions, suppose that $(\mathfrak{A},\mathscr{A},L)$ is a QM space and the linear span of*

$$G=\{(\mu\otimes id_{\mathfrak{C}})\Phi(a):\quad \mu\in S(\mathfrak{B}),a\in\mathfrak{A}\}$$

is dense in $\mathfrak{C}$ (for example Φ is surjective). Then $(\mathfrak{C},\mathscr{C},L_d)$ is a QM space.

Proof. Since $\mathscr{A}$ is dense in $\mathfrak{A}$ and the linear span of G is dense in $\mathfrak{C}$, we have

$$G_0=\{(\mu\otimes id_{\mathfrak{C}})\Phi(a):\quad \mu\in S(\mathfrak{B}),a\in\mathscr{A}\}$$

is dense in $\mathfrak{C}$. On the other hand, by Lemma 3.4, $G_0\subset\mathscr{C}$. Thus $\mathscr{C}$ is dense in $\mathfrak{C}$ and $(\mathfrak{C},\mathscr{C},L_d)$ is a QM space. □

Example 3.6. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be unital C*-algebras. Suppose that $\mathfrak{A}\otimes\mathfrak{C}$ has a QSM structure. Consider *-homomorphisms*

$$id:\mathfrak{A}\otimes\mathfrak{C}\to\mathfrak{A}\otimes\mathfrak{C}\quad\text{and}\quad F:\mathfrak{A}\otimes\mathfrak{C}\to\mathfrak{C}\otimes\mathfrak{A},$$

where F is the flip map, i.e. $F(a\otimes c)=c\otimes a$ for $a\in\mathfrak{A},c\in\mathfrak{C}$. Then

$$(\mathfrak{C},id_{\mathfrak{A}\otimes\mathfrak{C}})\qquad\text{and}\qquad(\mathfrak{A},F)$$

are quantum families of morphisms. Thus $\mathfrak{A}$ and $\mathfrak{C}$ have naturally QSM structures. Also, by Proposition 3.5, if $\mathfrak{A}\otimes\mathfrak{C}$ has a QM structure then so are $\mathfrak{A}$ and $\mathfrak{C}$.

Example 3.7. *Let $\mathfrak{A}$ be a unital C*-algebra and suppose that $\mathfrak{A}$ has a QSM structure. Let $\Phi:\mathfrak{A}\to\mathfrak{B}$ be a unital *-homomorphism. Then $(\mathfrak{B},\Phi)$ can be considered as a quantum family of morphisms from $\mathfrak{A}$ to $\mathbb{C}$. Thus $\mathfrak{B}$ naturally has a QSM structure. Also, if Φ is surjective and $\mathfrak{A}$ has a QM structure, then by Proposition 3.5, $\mathfrak{B}$ has a QM structure.*

4. The commutative case

In this last section we study induced metric structures on ordinary families of maps.

Lemma 4.1. *Let (X,τ) be a compact Hausdorff space and let d be a pseudo-metric on $S(\mathbf{C}(X,\tau))$ such that τ_d is weaker than the w* topology. Let $\mathscr{C}$ be the space of all $c\in\mathbf{C}(X,\tau)$ such that the map $\nu\longmapsto\nu(c)$ is continuous on $S(\mathbf{C}(X,\tau))$ and $L_d(c)<\infty$. Consider the semi norm $L_d:\mathscr{C}\to[0,+\infty)$. Then for every $x,x'\in X$, $d(\delta_x,\delta_{x'})=\rho_{L_d}(\delta_x,\delta_{x'})$.*

(We remark that Lemma 4.1 is different from part i) of Proposition 2.2.)

Proof. Let x,x' be in X. By the definition of ρ_{L_d}, we have

$$\rho_{L_d}(\delta_x,\delta_{x'})=\sup\{|a(x)-a(x')|:\quad a\in\mathscr{C},L_d(a)\leq 1\}. \tag{4.1}$$

Let $a\in\mathscr{C}$ and $L_d(a)\leq 1$. If $d(\delta_x,\delta_{x'})=0$, then $a(x)=a(x')$ since the map $\delta_x\longmapsto\delta_x(a)=a(x)$ is continuous with d, thus (4.1) implies that

$$\rho_{L_d}(\delta_x,\delta_{x'})=d(\delta_x,\delta_{x'})=0.$$

Now, suppose that $d(\delta_x,\delta_{x'})\neq 0$. Since $1=L_d(a)\geq\frac{|a(x)-a(x')|}{d(\delta_x,\delta_{x'})}$, we have $d(\delta_x,\delta_{x'})\geq|a(x)-a(x')|$, thus (4.1) implies that $\rho_{L_d}(\delta_x,\delta_{x'})\leq d(\delta_x,\delta_{x'})$. Now, define a map b_x on X by $b_x(y)=d(\delta_x,\delta_y)$. Then $b_x\in\mathscr{C}$ and $L_d(b_x)\leq 1$. Thus

$$\rho_{L_d}(\delta_x,\delta_{x'})\geq|b_x(x)-b_x(x')|=d(\delta_x,\delta_{x'}).$$

This completes the proof. □

Theorem 4.2. *Let (X,τ), (Y,τ'), (Z,τ'') be compact Hausdorff spaces and let d_0 be a pseudo-metric on X such that $\tau_{d_0}\subset\tau$. Let*

$$F: Y\times Z\to X$$

be a continuous map with τ,τ',τ'', and define a pseudo-metric d_1 on Z by

$$d_1(z,z')=\sup_{y\in Y} d_0(F(y,z),F(y,z')).$$

With the canonical identification $\mathbf{C}(Y\times Z,\tau'\times\tau'')\cong\mathbf{C}(Y,\tau')\otimes\mathbf{C}(Z,\tau'')$ let

$$\hat{F}:\mathbf{C}(X,\tau)\to\mathbf{C}(Y,\tau')\otimes\mathbf{C}(Z,\tau'')$$

be defined by $\hat{F}(a)=aF$, for $a\in\mathbf{C}(X,\tau)$. Let

$$(\mathbf{C}(Z,\tau''),\mathscr{C},N)$$

be the QSM space induced by QSM space $(\mathbf{C}(X,\tau),\mathbf{Lip}(X,d_0),\|\cdot\|_{d_0})$ and quantum family of morphisms $(\mathbf{C}(Z,\tau''),\hat{F})$. Then the following are satisfied.

i) $d_1(z,z')=\rho_N(\delta_z,\delta_{z'})$ *for every* $z,z'\in Z$.
ii) $\mathscr{C}\subset\mathbf{Lip}(Z,d_1)$.
iii) $\|\cdot\|_{d_1}\leq N$.

Proof. i) Let $L=\|\cdot\|_{d_0}$. Let us recall the definition of $(\mathbf{C}(Z,\tau''),\mathscr{C},N)$. Let d be the pseudo-metric on $S(\mathbf{C}(Z,\tau''))$ defined by

$$d(\nu,\nu')=\sup\{\rho_L((\mu\otimes\nu)\hat{F},(\mu\otimes\nu')\hat{F}):\quad \mu\in S(\mathbf{C}(Y,\tau'))\}.$$

Then $N=L_d$ and $\mathscr{C}$ is the space of all $c\in\mathbf{C}(Z,\tau'')$ such that the map $\nu\longmapsto\nu(c)$ on $S(\mathbf{C}(Z,\tau''))$ is continuous with d and $N(c)<\infty$. By Lemma 4.1, we have,

$$d(\delta_z,\delta_{z'})=\rho_N(\delta_z,\delta_{z'}),\tag{4.2}$$

for every $z,z'\in Z$. Now, we explain the relation between d_1 and d.
Let $z,z'\in Z$ and $y\in Y$. Then

$$(\delta_y\otimes\delta_z)\hat{F}=\delta_{F(y,z)}\qquad\text{and}\qquad(\delta_y\otimes\delta_{z'})\hat{F}=\delta_{F(y,z')}.$$

On the other hand, by Proposition 2.2, for every $x,x'\in X$, $d_0(x,x')=\rho_L(\delta_x,\delta_{x'})$. Thus

$$\rho_L((\delta_y\otimes\delta_z)\hat{F},(\delta_y\otimes\delta_{z'})\hat{F})=d_0(F(y,z),F(y,z')).$$

This formula together with the definitions of d and d_1, show that

$$d_1(z,z')\leq d(\delta_z,\delta_{z'}).\tag{4.3}$$

Let $\mu\in S(\mathbf{C}(Y,\tau'))$ be arbitrary. We consider μ as a probability Borel regular measure on (Y,τ'). Then for every $a\in\mathbf{Lip}(X,d_0)$ with $\|a\|_{d_0}\leq 1$, we have,

$$\begin{aligned}|(\mu\otimes\delta_z)\hat{F}(a)-(\mu\otimes\delta_{z'})\hat{F}(a)|&=|\int_Y(aF(y,z)-aF(y,z'))d_\mu(y)|\\&\leq\int_Y|a(F(y,z))-a(F(y,z'))|d_\mu(y).\end{aligned}\tag{4.4}$$

For every $y\in Y$, by Lemma 2.1,

$$|a(F(y,z))-a(F(y,z'))|\leq d_0(F(y,z),F(y,z')).$$

Therefore, we have

$$|a(F(y,z))-a(F(y,z'))|\leq d_1(z,z').\tag{4.5}$$

(4.5) and (4.4) implies that

$$|(\mu\otimes\delta_z)\hat{F}(a)-(\mu\otimes\delta_{z'})\hat{F}(a)|\leq d_1(z,z').$$

Therefore, by the definition of d,

$$d(\delta_z,\delta_{z'})\leq d_1(z,z').\tag{4.6}$$

Now, by (4.6) and (4.3), $d(\delta_z,\delta_{z'})=d_1(z,z')$, and thus by (4.2),

$$d_1(\delta_z,\delta_{z'})=\rho_N(\delta_z,\delta_{z'})$$

for every $z,z'\in Z$, and i) is satisfied. ii) and iii) are immediate consequence of i) and definitions of $\mathscr{C}$, $\|\cdot\|_{d_1}$ and N. □

5. Conclusion

In this note, we introduced the new concept of *compact quantum pseudo-metric space* as a generalization of the concept of *compact quantum metric space*. The C*-algebraic examples of the latter concept, which has been introduced by Rieffel, are very restricted. But, by using the concept of *quantum family of maps*, it was denoted that the source of examples for (C*-algebraic) quantum pseudo-metric spaces are very wider than those for (C*-algebraic) quantum metric spaces.

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