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Fixed point theorems in rational form via Suzuki approaches

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Abstract

In this paper we establish some fixed point theorems by using the new contractive conditions containing various rational forms. The presented results improve and unify several existing results on the topic in the literature.

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1. Introduction and Preliminaries

In this manuscript, inspired by the recent result of Suzuki [12], our goal is to study rational type inequalities that yield the existence and uniqueness of fixed points in metric spaces from the view point of complete metric spaces. Rational type inequalities were initiated by Jaggi [4]. After these first results, many other authors have reported on this topic, see e.g. [1, 2, 5, 6, 7, 8, 9, 10, 11].

Let $\varphi : [0, \infty) \to [0, \infty)$ be a function that satisfies the following conditions

(φ1) $\varphi(t) < t$ for any $t \in (0, \infty)$,

(φ2) For any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon < t < \varepsilon + \delta \text{ implies } \varphi(t) \leq \varepsilon.$$

The collection of all functions φ which satisfies (φ1) and (φ2) will be denoted by Φ.

Remark 1.1. By (φ1), is easy to see that (φ2) is equivalent to the following

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($\varphi 2'$) For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$t < \varepsilon + \delta \text{ implies } \varphi(t) < \varepsilon.$$

Indeed, if $0 < t \leq \varepsilon$ from (φ1), we have, $\varphi(t) < t \leq \varepsilon$.

Very recently, Suzuki [12], proves the following fixed point theorem.

Theorem 1.2. *[12]. Let (X, d) be a complete metric space and a mapping $T : X \to X$. Define a function L from $X \times X$ into $[0, \infty)$ by*

$$L(x, y) = \max\left\{d(x, y), \frac{d(x, Ty) + d(Tx, y)}{2}, d(x, Tx), d(y.Ty)\right\}. \tag{1.1}$$

Assume that there exists a function φ from $[0, \infty)$ into itself satisfying the following:

(φ1) *$\varphi(t) < t$ for any $t \in (0, \infty)$,*

(φ2) *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\epsilon < t < \varepsilon + \delta \text{ implies } \varphi(t) \leq \varepsilon,$$

(φ3) *$d(Tx, Ty) \leq \varphi \circ L(x, y)$.*

Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for all $x \in X$.

Inspired from the interesting result of Suzuki [12], our aim is to obtain some existence and uniqueness results for the certain maps that include rational inequalities in the setting of complete metric space. The main results of this paper cover several existing results reported in this direction.

Throughout the manuscript, we assume $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of positive integers. Further, $\mathbb{R}$ represents the real numbers and $\mathbb{R}_0^+ := [0, \infty)$

2. Main Results

Our first main theorem is the following:

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a continuous mapping. Assume that there exists a function $\varphi \in \Phi$ such that for all $x, y \in X$, with $x \neq y$*

$$d(Tx, Ty) \leq \varphi\left(P(x, y)\right), \tag{2.1}$$

where $P(x, y) = \max\left\{\dfrac{d(x, Tx)d(y, Ty)}{d(x, y)}, d(x, y)\right\}$. Then, T has a unique fixed point u. Moreover for all $x \in X$ the sequence $\{T^n x\}$ converges to u.

Proof. Let $x, y \in X$, with $x \neq y$ satisfying $P(x, y) = 0$. Then, $d(y, Ty) = 0$ or $d(x, Tx) = 0$, so $Tx = x$ or, respectivelly $Ty = y$ this means, clearly, that T has a fixed point. From now, we assume that $P(x, y) > 0$, for any $x \neq y$. Let $x \in X$ satisfy $x \neq Tx$ and $P(x, Tx) > 0$. By (i) and (iii), we have

$$d(Tx, T^2x) \leq \varphi(P(x, Tx)) < P(x, Tx), \tag{2.2}$$

where $P(x, Tx) = \max\left\{\dfrac{d(x, Tx)d(Tx, T^2x)}{d(x, Tx)}, d(x, Tx)\right\} = \max\left\{d(Tx, T^2x), d(x, Tx)\right\}$. Assuming that $d(Tx, T^2x) > d(x, Tx)$ we obtain $d(Tx, T^2x) < d(Tx, T^2x)$, which is a contradiction. Therefore we find that $\max\left\{d(Tx, T^2x), d(x, Tx)\right\} = d(x, Tx)$ and (2.2) yields

$$d(Tx, T^2x) < P(x, Tx) = d(x, Tx). \tag{2.3}$$

Fix $x_0 \in X$ and define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N}$. Suppose that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Then, x_{n_0} is a fixed point of T, that is, $Tx_{n_0} = x_{n_0}$. From now, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$, in other words $d(x_n, x_{n+1}) > 0$. From (2.3), $\{d(x_n, x_{n+1})\}$ converges to some $\varepsilon_1 \geq 0$. We claim that $\varepsilon_1 = 0$. Arguing by contradiction, we assume $\varepsilon_1 > 0$. From $(\varphi 2')$, there is $\delta_1 > 0$ satisfying:

$$t < \varepsilon_1 + \delta_1 \text{ implies } \varphi(t) \leq \varepsilon_1$$

and, we can find $N \in \mathbb{N}$ such that

$$P(x_N, x_{N+1}) = d(x_N, x_{N+1}) < \varepsilon_1 + \delta_1. \tag{2.4}$$

Then, together with $(\varphi 2')$ and (2.1) we get

$$0 < \varepsilon_1 \leq d(x_{N+1}, x_{N+2}) = d(Tx_N, Tx_{N+1}) \leq \varphi(P(x_N, x_{N+1})) \leq \varepsilon_1.$$

By (2.3),

$$\varepsilon_1 \leq d(x_{N+2}, x_{N+3}) < d(x_{N+1}, x_{N+2}) \leq \varepsilon_1,$$

which is a contradiction. Therefore, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. In order to proof that $\{x_n\}$ is a Cauchy sequence, we fix $\varepsilon > 0$. From $(\varphi 2')$, there exists $\delta > 0$ satisfying the following

$$t < \varepsilon + \delta \text{ implies } \varphi(t) \leq \varepsilon$$

(Without loss of generality, we may assume that $\delta < \varepsilon$.) By the convergence of the sequence $d(x_n, x_{n+1})$ to 0, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_{m+1}) < \delta$, for all $m \geq n_0$. We will show by induction that

$$d(x_m, x_{m+k}) < \varepsilon + \delta, \tag{2.5}$$

for $k \in \mathbb{N}$. Obviously, when $k = 1$, inequality (2.5) holds. We assume that (2.5) holds for some $k \in \mathbb{N}$. We shall examine the following two cases.

Case 1. Assume that $d(x_m, x_{m+k}) < \varepsilon < \varepsilon + \delta$. In this case, we have

$$d(x_m, x_{m+k+1}) \leq d(x_m, x_{m+k}) + d(x_{m+k}, x_{m+k+1}) < \delta + \varepsilon. \tag{2.6}$$

Case 2. Suppose that $\varepsilon < d(x_m, x_{m+k}) < \varepsilon + \delta$. Then,

$$d(x_m, x_{m+1}) < \delta < \varepsilon < d(x_m, x_{m+k}). \tag{2.7}$$

On the other hand,

$$P(x_m, x_{m+k}) = \max\left\{\frac{d(x_m, x_{m+1})d(x_{m+k}, x_{m+k+1})}{d(x_m, x_{m+k})}, d(x_m, x_{m+k})\right\}. \tag{2.8}$$

We distinguish two situations.
(a) If $d(x_{m+k}, x_{m+k+1}) < d(x_m, x_{m+k})$ then, taking into account (2.7),

$$P(x_m, x_{m+k}) < \max\{d(x_m, x_{m+1}), d(x_m, x_{m+k})\} = d(x_m, x_{m+k}) < \varepsilon + \delta$$

and using $(\varphi 2')$, we have

$$\begin{aligned} d(x_m, x_{m+k+1}) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+k+1}) = d(x_m, x_{m+1}) + d(Tx_m, Tx_{m+k}) \\ &< \delta + \varphi(P(x_m, x_{m+k})) \leq \delta + \varepsilon. \end{aligned} \tag{2.9}$$

(*b*) If $d(x_{m+k}, x_{m+k+1}) \geq d(x_m, x_{m+k})$, then using the triangle inequality,

$$d(x_m, x_{m+k+1}) \leq d(x_m, x_{m+k}) + d(x_{m+k}, x_{m+k+1}) \leq 2d(x_{m+k}, x_{m+k+1}) < 2\delta < \varepsilon + \delta. \tag{2.10}$$

So, we prove by induction that (2.5) holds, for every $k \in \mathbb{N}$, and since $\varepsilon > 0$ is arbitrarily chosen, we obtain

$$\lim_{n\to\infty} \sup_{m>n} d(x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. Then, using the fact that $x_{n+1} = Tx_n$ and the continuity of T we obtain $u = Tu$, that is, u is a fixed point of T. To show the uniqueness, we assume that v is another fixed point of T, $u \neq v$. Then, we have

$$P(u, v) = \max\left\{\frac{d(u, Tu)d(v, Tv)}{d(u, v)}, d(u, v)\right\} = d(u, v)$$

and hence, from (*iii*) and (*i*)

$$d(u, v) = d(Tu, Tv) \leq \varphi(P(u, v)) < P(u, v) = d(u, v),$$

which implies $d(u, v) = 0$. This proves the uniqueness of the fixed point.

□

Theorem 2.2. *Let (X, d) be a complete metric space and a mapp $T : X \to X$. Assume that there exists a function $\varphi \in \Phi$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \varphi\left(Q(x, y)\right), \tag{2.11}$$

where $Q(x, y) = \max\left\{\frac{[d(x, Tx) + 1]\, d(y, Ty)}{1 + d(x, y)}, d(x, y)\right\}$. *Then, T has a unique fixed point u. Moreover for all $x \in X$ the sequence $\{T^n x\}$ converges to u.*

Proof. Let $x, y \in X$. If $x = y$, then $d(Tx, Ty) = 0 \leq Q(x, y)$. In the other case, if $x \neq y$, then $Q(x, y) > 0$ and from (2.11), respectively ($\varphi 1$) it easy to see that

$$d(Tx, Ty) \leq \varphi(Q(x, y)) < Q(x, y). \tag{2.12}$$

Let $x \in X$ such that $x \neq Tx$. In this case,

$$Q(x, Tx) = \max\left\{\frac{d(Tx, T^2x)\,[1 + d(x, Tx)]}{1 + d(x, Tx)}, d(x, Tx)\right\} = \max\left\{d(Tx, T^2x), d(x, Tx)\right\}$$

and using (2.12) we get

$$d(Tx, T^2x) < \max\left\{d(Tx, T^2x), d(x, Tx)\right\}. \tag{2.13}$$

.

We shall examine two cases. Supposing that $d(Tx, T^2x) > d(x, Tx)$, we obtain $d(Tx, T^2x) < d(Tx, T^2x)$ a contradiction. Therefore, we find that $\max\left\{d(Tx, T^2x), d(x, Tx)\right\} = d(x, Tx)$ and (2.13) yields

$$d(Tx, T^2x) < d(x, Tx), \tag{2.14}$$

for any $x \in X$, $x \neq Tx$, and, obviously,

$$Q(x, Tx) = d(x, Tx). \tag{2.15}$$

In the following, as in the proof of Theorem 2.1, we shall construct an iterative sequence $\{x_n\}$, for an arbitrary initial value $x \in X$:

$$x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}. \tag{2.16}$$

As it is discussed in the proof of Theorem 2.1, we suppose

$$x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}. \tag{2.17}$$

From (2.14) we conclude that $d(x_{n-1}, x_n)$ is a nonincreasing sequence of non-negative real numbers, and hence, it converges to some $\varepsilon \geq 0$.

On the other hand, by $(\varphi 2')$ from remark (1.1) there exists $\delta > 0$ such that

$$t < \varepsilon + \delta \text{ implies } \varphi(t) \leq \varepsilon.$$

Assume that $\varepsilon > 0$. We can choose $N \in \mathbb{N}$ such that, using (2.15)

$$Q(x_N, x_{N+1}) = Q(x_N, Tx_N) = d(x_N, Tx_N) = d(x_N, x_{N+1}) < \varepsilon + \delta$$

and

$$0 < \varepsilon \leq d(x_{N+1}, x_{N+2}) < \varphi(Q(x_N, x_{N+1})) \leq \varepsilon$$

which is a contradiction. Therefore, $\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$.

Let now $\varepsilon_1 > 0$ be fixed. Then, from $(\varphi 2')$ there is $\delta_1 > 0$ such that

$$t < \varepsilon_1 + \delta_1 \text{ implies } \varphi(t) \leq \varepsilon_1$$

We shall prove that $\{x_n\}$ is a Cauchy sequence. For this, let $m \in \mathbb{N}$ large enough to satisfy $d(x_m, x_{m+1}) < \delta_1$. We will show, by induction, that

$$d(x_m, x_{m+k}) < \varepsilon_1 + \delta_1, \tag{2.18}$$

for all $k \in \mathbb{N}$. (Without loss of generality, we assume that $\delta_1 = \delta_1(\varepsilon) < \varepsilon$.) We have already proved for $k = 1$, so, consider the following two situations.

(a) If $d(x_{m+k}, x_{m+k+1}) \leq d(x_m, x_{m+k})$ then

$$\frac{d(x_{m+k}, x_{m+k+1})}{1 + d(x_m, x_{m+k})} \leq d(x_{m+k}, x_{m+k+1}) \text{ respectively } \frac{d(x_{m+k}, x_{m+k+1}) d(x_m, x_{m+1})}{1 + d(x_m, x_{m+k})} < d(x_m, x_{m+1})$$

Hence

$$\begin{aligned} Q(x_m, x_{m+k}) &= \max\left\{\tfrac{d(x_{m+k},x_{m+k+1})[1+d(x_m,x_{m+1})]}{1+d(x_m,x_{m+k})}, d(x_m, x_{m+k})\right\} \\ &\leq \max\{d(x_{m+k}, x_{m+k+1}) + d(x_m, x_{m+1}), d(x_m, x_{m+k})\} < \max\{2\delta_1, \varepsilon_1 + \delta_1\} \\ &< \varepsilon_1 + \delta_1. \end{aligned}$$

So, we have,

$$d(x_m, x_{m+k+1}) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+k+1}) < d(x_m, x_{m+1}) + \varphi(Q(x_m, x_{m+k})) < \varepsilon_1 + \delta_1. \tag{2.19}$$

(b) If $d(x_{m+k}, x_{m+k+1}) > d(x_m, x_{m+k})$ then

$$d(x_m, x_{m+k+1}) \leq d(x_m, x_{m+k}) + d(x_{m+k}, x_{m+k+1}) < 2d(x_{m+k}, x_{m+k+1}) < 2\delta_1 < \varepsilon_1 + \delta_1. \tag{2.20}$$

Thus, by induction, (2.18) holds for every $k \in \mathbb{N}$. Since $\varepsilon_1 > 0$ is arbitrary, we get

$$\lim_{p\to\infty} \sup d(x_m, x_{m+p}) = 0,$$

which implies that $\{x_n\}$ is Cauchy sequence in complete metric space (X, d). Hence, $\{x_n\}$ converges to some $u \in X$. Next, we will prove that u is a fixed point of T. Arguing by contradiction, we suppose that

$Tu \neq u$ or, $d(u, Tu) = \varepsilon_2 > 0$. Since, the sequence $\{x_n\}$ is convergent to u, we can choose $l \in \mathbb{N}$ such that $d(u, x_l) < \frac{\varepsilon_2}{2}$. Also, from $(\varphi 2')$, there exists $\delta_2 = \frac{\varepsilon_2^2}{2}$ satisfying the following

$$t < \frac{\varepsilon_2}{2} + \frac{\varepsilon_2^2}{2} \text{ implies } \varphi(t) \leq \frac{\varepsilon_2}{2}.$$

Since

$$\begin{aligned} 0 < Q(u, x_l) = & \max\left\{\frac{d(x_l, x_{l+1})[1+d(u,Tu)]}{1+d(u,x_l)}, d(u, x_l)\right\} \\ & < \max\left\{d(x_l, x_{l+1})\left[1 + d(u, Tu)\right], d(u, x_l)\right\} \\ & < \max\left\{\frac{\varepsilon_2}{2} + \frac{\varepsilon_2^2}{2}, \frac{\varepsilon_2}{2}\right\} \\ & = \frac{\varepsilon_2}{2} + \frac{\varepsilon_2^2}{2}, \end{aligned} \tag{2.21}$$

from (2.11) together with the triangle inequality and (2.21), we get

$$0 < \varepsilon_2 = d(u, Tu) \leq d(u, x_{l+1}) + d(Tx_l, Tu) < d(u, x_{l+1}) + \varphi\left(Q(u, x_l)\right) < \frac{\varepsilon_2}{2} + \frac{\varepsilon_2}{2} = \varepsilon_2 \tag{2.22}$$

which is a contradiction. We deduce that $d(u, Tu) = 0$, which means that u is a fixed point of T.

To show the uniqueness, we assume that v, is another fixed point of T, with $u \neq v$. Since

$$Q(u, v) = \max\left\{\frac{d(v, Tv)\left[1 + d(u, Tu)\right]}{1 + d(u, v)}, d(u, v)\right\} = d(u, v) > 0$$

from(2.11) it follows that

$$0 < d(u, v) = d(Tu, Tv) < \varphi(Q(u, v)) = \varphi(d(u, v)) < d(u, v)$$

which is a contradiction. This proves the uniqueness of the fixed point and completes the proof of the theorem. □

Example 2.3. Let $X = \{0, 1, 2, 3\}$ and $d(x, y) = |x - y|$ be a metric on X. Assume $T : X \to X$ and $\varphi : [0, \infty) \to [0, \infty)$ be defined by

$$T0 = T1 = T2 = 0, \; T3 = 1$$

and

$$\varphi(t) = \begin{cases} \frac{t}{2}, & \text{if } t \in [0, 2] \\ \frac{2}{t} + 1, & \text{if } t \in (2, \infty) \end{cases}$$

It is easy to see that for any $x, y \in \{0, 1, 2\}$ we get $d(Tx, Ty) = 0$. Therefore relation (2.11) is satisfied. We also notice that $d(3, T3) = 2$ and $d(Tx, T3) = 1$ for any $x \in \{0, 1, 2\}$.For this reason, we have to distinguish the following three cases.

Case (1). If $x = 0$ then $d(0, T0) = 0$, $d(0, 3) = 3$ and

$$Q\left(0, 3\right) = \max\left\{d(0, 3), \frac{d(3, T3)[d(0, T0) + 1]}{d(0, 3) + 1}\right\} = \max\left\{3, \frac{2}{4}\right\} = 3$$

In this case

$$d(T0, T3) = 1 \leq \frac{5}{3} = 1 + \frac{2}{3} = \varphi(Q(0, 3)).$$

Case (2). If $x = 1$ then $d(1, T1) = 1$, $d(1, 3) = 2$ and

$$Q\left(1, 3\right) = \max\left\{d(1, 3), \frac{d(3, T3)[d(1, T1) + 1]}{d(1, 3) + 1}\right\} = \max\left\{2, \frac{4}{3}\right\} = 2.$$

Thus

$$d(T1, T3) = 1 \le 1 = \frac{2}{2} = \varphi(Q(1,3)).$$

Case (3). If $x = 2$ then $d(2, T2) = 2$, $d(2,3) = 1$,

$$Q\,(2,3) = \max\left\{d(2,3), \frac{d(3,T3)[d(2,T2)+1]}{d(2,3)+1}\right\} = \max\left\{1, \frac{6}{2}\right\} = 3$$

and

$$d(T2, T3) = 1 \le \frac{5}{3} = 1 + \frac{2}{3} = \varphi(Q(2,3)).$$

Hence all the conditions of Theorem 2.5are satisfied and T has a unique fixed point, $x = 0$.

Theorem 2.4. *Let (X, d) be a complete metric space and T be self-mapping on X. Suppose that there exists a function $\varphi \in \Phi$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \le \varphi\left(R(x,y)\right), \tag{2.23}$$

where

$$R(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}. \tag{2.24}$$

Suppose also that, either T is continuous or φ is upper semi-continuous. Then, T has a unique fixed point u. Moreover for all $x \in X$ the sequence $\{T^n x\}$ converges to u.

Proof. First, we prove that $R(x,y) = 0$ if and only if $x = y$ is a fixed point of T. In fact, if $x = y$ is a fixed point of T then $Tx = x = y = TY$ and obviously $R(x,y) = 0$. Conversely, if $R(x,y) = 0$ then, using (2.23), $(\varphi 2')$ and (2.24) it is easy to prove that $x = y$ is a fixed point of T. On the other hand,

$$\begin{aligned} R(x,Tx) &= \max\left\{d(x,Tx), d(x,Tx), d(Tx,T^2x), \frac{d(x,Tx)d(Tx,T^2x)}{1+d(x,Tx)}, \frac{d(x,Tx)d(Tx,T^2x)}{1+d(Tx,T^2x)}\right\} \\ &\le \max\left\{d(x,Tx), d(Tx,T^2x\right\}. \end{aligned} \tag{2.25}$$

If $x \ne Tx$, then $R(x,Tx) > 0$ and

$$d(Tx, T^2x) \le \varphi(R(x,Tx) < R(x,Tx) < \max d(x,Tx), d(Tx,T^2x). \tag{2.26}$$

It is easy to see that if $\max d(x,Tx), d(Tx,T^2x) = d(Tx,T^2x)$ then $d(Tx,T^2x) < d(Tx,T^2x)$ which is a contradiction. Thus we conclude that

$$\max d(x,Tx), d(Tx,T^2x) = d(x,Tx).$$

By (2.26), we get that

$$d(Tx, T^2x) < d(x, Tx). \tag{2.27}$$

By the analogous proof in Theorem 2.1 we can construct the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, for which $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}$. Since T satisfies (2.23), the sequence $\{d(x_n, x_{n+1})\}$ converges to some $\varepsilon_1 \ge 0$. We claim that $\varepsilon_1 = 0$. Arguing by contradiction, we assume $\varepsilon_1 > 0$. From $(\varphi 2')$, there is $\delta_1 > 0$ satisfying:

$$t < \varepsilon_1 + \delta_1 \text{ implies } \varphi(t) \le \varepsilon_1$$

and, we can find $N \in \mathbb{N}$ such that

$$R(x_N, x_{N+1}) \le \max\{d(x_N, x_{N+1}), d(x_{N+1}, x_{N+2}\} = d(x_N, x_{N+1}) < \varepsilon_1 + \delta_1. \tag{2.28}$$

Then, together with $(\varphi 2')$ and (2.1) we get

$$0 < \varepsilon_1 \le d(x_{N+1}, x_{N+2}) = d(Tx_N, Tx_{N+1}) \le \varphi(P(x_N, x_{N+1})) \le \varepsilon_1.$$

By (2.27),

$$\varepsilon_1 \leq d(x_{N+2}, x_{N+3}) < d(x_{N+1}, x_{N+2}) \leq \varepsilon_1,$$

which is a contradiction. Therefore, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Next we will show that $\{x_n\}$ is Cauchy sequence. Fix $\varepsilon > 0$. Then, for $(\varphi 2')$, there exists $\delta > 0$ such that

$$t < \varepsilon + \delta \text{ implies } \varphi(t) \leq \varepsilon.$$

Let $m \in \mathbb{N}$ large enough to satisfy $d(x_m, x_{m+1}) < \delta$. We will show, by induction, that

$$d(x_m, x_{m+k}) < \varepsilon + \delta, \tag{2.29}$$

for all $k \in \mathbb{N}$. (Without loss of generality, $\delta = \delta(\varepsilon) < \varepsilon$.) We have already proved for $k = 1$. we assume that (2.27) holds for some $k \in \mathbb{N}$. Then, we have

$$\begin{aligned} R(x_m, x_{m+k}) \quad &= \max\{d(x_m, x_{m+k}), d(x_m, x_{m+1}), d(x_{m+k}, x_{m+k+1}), \\ &\left.\frac{d(x_m, x_{m+1})d(x_{m+k}, x_{m+k+1})}{1 + d(x_m, x_{m+k})}, \frac{d(x_m, x_{m+1})d(x_{m+k}, x_{m+k+1})}{1 + d(x_{m+1}, x_{m+k+1})}\right\}. \end{aligned}$$

The following two situations are distinguished:
(a) If $d(x_{m+k}, x_{m+k+1}) \leq d(x_m, x_{m+k})$ then

$$\frac{d(x_m, x_{m+1})d(x_{m+k}, x_{m+k+1})}{1 + d(x_m, x_{m+k})} \leq d(x_m, x_{m+1}),$$

respectively

$$\frac{d(x_{m+k}, x_{m+k+1})d(x_m, x_{m+1})}{1 + d(x_{m+1}, x_{m+k+1})} \leq d(x_m, x_{m+1}).$$

Hence

$$\begin{aligned} R(x_m, x_{m+k}) \quad &\leq \max\{d(x_m, x_{m+k}), d(x_m, x_{m+1}), d(x_{m+k}, x_{m+k+1})\} \\ &< \max\{\varepsilon + \delta, \delta\} = \varepsilon + \delta \end{aligned}$$

So, we have,

$$d(x_m, x_{m+k+1}) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+k+1}) < d(x_m, x_{m+1}) + \varphi(R(x_m, x_{m+k})) < \varepsilon + \delta. \tag{2.30}$$

(b) If $d(x_{m+k}, x_{m+k+1}) > d(x_m, x_{m+k})$ then

$$d(x_m, x_{m+k+1}) \leq d(x_m, x_{m+k}) + d(x_{m+k}, x_{m+k+1}) < 2d(x_{m+k}, x_{m+k+1}) < 2\delta < \varepsilon + \delta. \tag{2.31}$$

Thus, by induction, (2.29) holds for every $k \in \mathbb{N}$. Since $\varepsilon > 0$ is arbitrary, we get

$$\lim_{p\to\infty} \sup d(x_m, x_{m+p}) = 0,$$

which implies that $\{x_n\}$ is Cauchy sequence in complete metric space (X, d). Hence, $\{x_n\}$ converges to some $u \in X$.

We will show next that the limit u of the sequence x_n is a fixed point of T. First, we suppose that T is continuous. Then

$$\lim_{n\to\infty} d(Tx_n, u) = \lim_{n\to\infty} d(x_{n+1}, u) = 0.$$

We conclude that $Tu = u$, that is, u is a fixed point of T.

Now, we suppose that function φ is upper semi-continuous. Since the sequence $\{x_n\}$ is convergent to u, we can choose $l \in \mathbb{N}$ such that $d(u, x_l) < \epsilon$.

From (2.24) and (2.23) we have

$$R(x_l,u)=\max\left\{d(x_l,u),d(x_l,x_{l+1}),d(u,Tu),\frac{d(x_l,x_{l+1})d(u,Tu)}{1+d(x_l,u)},\frac{d(x_l,x_{l+1})d(u,Tu)}{1+d(x_{l+1},Tu)}\right\} \tag{2.32}$$

and

$$d(x_{l+1},Tu)=d(Tx_l,Tu)\le\varphi(R(x_l,u)). \tag{2.33}$$

Suppose that $Tu\neq u$, so there exists τ such that $d(u,Tu)=\tau>0$. Since $d(x_l,x_{l+1})\to 0$ and $d(x_l,u)\to 0$ we get that $\lim_{l\to\infty}R(x_l,u)=d(u,Tu)$. Letting $l\to\infty$ in (2.33) and using the upper semi-continuity of function φ

$$\tau=d(u,Tu)=\lim_{l\to\infty}d(Tx_l,Tu)\le\limsup_{l\to\infty}\varphi(R(x_l,u))<\varphi(d(u,Tu))<d(u,Tu)=\tau$$

Thus, $d(u,Tu)=\tau=0$, therefore $Tu=u$. Suppose now, that u and v are two fixed points of T, $u\neq v$. Then, $R(u,v)=d(u,v)>0$. We have

$$0<d(u,v)=d(Tu,Tv)\le\varphi(R(u,v))<R(u,v)=d(u,v),$$

which is a contradiction. So, we obtain that $d(u,v)=0$. □

Theorem 2.5. *Let (X,d) be a complete metric space and $T:X\to X$. Assume that there exists a function $\varphi\in\Phi$ such that,*

$$d(Tx,Ty)<\varphi\left(S(x,y)\right), \tag{2.34}$$

for all $x,y\in X$, where $S(x,y)=\max\left\{d(x,y),\dfrac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}\right\}$. Then, T has a unique fixed point u. Moreover for all $x\in X$ the sequence $\{T^nx\}$ converges to u.

Proof. Since

$$S(x,Tx)=\max\left\{d(x,Tx),\frac{d(x,Tx)d(x,T^2x)+d(Tx,T^2x)d(Tx,Tx)}{d(x,T^2x)+d(Tx,Tx)}\right\}=d(x,Tx),$$

following the lines of the proof of Theorem 2.1, we shall construct a sequence $\{x_n\}\subset X$ where $x_{n+1}=Tx_n$ for which $x_n\neq x_{n+1}$ and

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0. \tag{2.35}$$

In what follows, we shall prove that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon>0$ fixed. From $(\varphi 2')$, there exists $\delta>0$ satisfying the following

$$t<\varepsilon+\delta \text{ implies } \varphi(t)\le\varepsilon$$

(We may assume that $\delta<\varepsilon$.) By the convergence of the sequence $d(x_n,x_{n+1})$ to 0, there exists $k_0\in\mathbb{N}$ such that $d(x_m,x_{m+1})<\delta$, for all $m\ge k_0$. We will show by induction that

$$d(x_m,x_{m+k})<\varepsilon+\delta, \tag{2.36}$$

for $k\in\mathbb{N}$. Obviously, when $k=1$, inequality (2.36) holds. We assume that (2.36) holds for some $k\in\mathbb{N}$. Then we have

$$\begin{aligned}S(x_m,x_{m+k})&=\max\left\{d(x_m,x_{m+k}),\frac{d(x_m,Tx_m)d(x_m,Tx_{m+k})+d(x_{m+k},Tx_{m+k})d(x_{m+k},Tx_m)}{d(x_m,Tx_{m+k})+d(x_{m+k},Tx_m)}\right\}\\&=\max\left\{d(x_m,x_{m+k}),\frac{d(x_m,x_{m+1})d(x_m,x_{m+k+1})+d(x_{m+k},x_{m+k+1})d(x_{m+k},x_{m+1})}{d(x_m,x_{m+k+1})+d(x_{m+k},x_{m+1})}\right\}\\&\le\max\left\{\varepsilon+\delta,\delta\frac{d(x_m,x_{m+k+1})+d(x_{m+k},x_{m+1})}{d(x_m,x_{m+k+1})+d(x_{m+k},x_{m+1})}\right\}=\varepsilon+\delta\end{aligned}$$

and hence

$$d(x_m, x_{m+k+1}) \le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+k+1}) \le d(x_m, x_{m+1}) + \varphi\left(S(x_m, x_{m+k}\right) < \varepsilon + \delta.$$

Thus, by induction (2.36) holds for every $k \in \mathbb{N}$. Since $\varepsilon > 0$ is arbitrary, we obtain that $\{x_n\}$ is Cauchy sequence. By completeness of (X, d), there exists $u \in X$ such that $\lim_{n\to\infty}(d(x_n, u) = 0$. To prove that u is a fixed point for T, we suppose, on contrary, that $d(u, Tu) = \tau > 0$. Since $\lim_{n\to\infty} d(x_n, u) = 0$ and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ we can choose $l \in \mathbb{N}$ satisfying

$$d(x_l, x_{l+1}) < \frac{\tau}{2}, d(x_l, u) < \frac{\tau}{2}.$$

We have

$$\begin{aligned} S(x_l, u) &= \max\left\{d(x_l, u), \frac{d(x_l, x_{l+1})d(x_l, Tu) + d(u, Tu)d(u, x_{l+1})}{d(x_l, Tu) + d(u, x_{l+1})}\right\} \\ &< \max\left\{\tfrac{\tau}{2}, \frac{\frac{\tau}{2}d(x_l, Tu) + \tau d(u, x_{l+1})}{d(x_l, Tu) + d(u, x_{l+1})}\right\} \\ &\le \max\left\{\tfrac{\tau}{2}, \tau\right\} = \tfrac{\tau}{2} + \tfrac{\tau}{2}. \end{aligned}$$

Taking into account $(\varphi 2')$, (where $\delta = \frac{\tau}{2}$ and $\varepsilon = \frac{\tau}{2}$) and (2.34), we get

$$\tau = d(u, Tu) \le d(u, x_{l+1}) + d(Tx_l, Tu) \le d(u, x_{l+1}) + \varphi(S(x_l, u)) < \frac{\tau}{2} + \frac{\tau}{2} = \tau.$$

Thus, $d(u, Tu) = 0$. Therefore $Tu = u$.

Now, suppose that there exists another point $v \in X$, $u \ne v$ such that $Tv = v$. Then,

$$S(u, v) = \max\left\{d(u, v), \frac{d(u, Tu)d(u, Tv) + d(v, Tv)d(v, Tu)}{d(u, Tv) + d(v, Tu)}\right\} = d(u, v)$$

and from (2.34) together with the hypothesis $\varphi \in \Phi$ we get

$$0 < d(u, v) = d(Tu, Tv) \le \varphi(S(u, v)) = \varphi(d(u, v)) < d(u, v),$$

which is a contradiction. □

Example 2.6. Let X be a finite set defined by $X = \left\{\frac{1}{4}, \frac{1}{2}, 1\right\}$. We endow X with usual metric. Define $T : X \to X$, $\varphi : [0, \infty) \to [0, \infty)$ by

$$T\frac{1}{2} = T\frac{1}{4} = \frac{1}{4},\ T1 = \frac{1}{2}$$

and

$$\varphi(t) = \begin{cases} t^2, & \text{if } t \in [0, \frac{1}{2}] \\ 1 - t, & \text{if } t \in (\frac{1}{2}, 1] \\ \frac{t}{2}, & \text{if } t \in (1, \infty) \end{cases}$$

Since $d(T\frac{1}{2}, T\frac{1}{4}) = 0$ obviously (2.34) holds. We consider the following two cases:
Case (1). Let $x = \frac{1}{2}$ and $y = 1$. In this case, we have:

$$d(T\frac{1}{2}, T1) = \frac{1}{4},\ d(\frac{1}{2}, 1) = \frac{1}{2},\ d(1, T1) = \frac{1}{2},$$

$$d(\frac{1}{2}, T\frac{1}{2}) = \frac{1}{4},\ d(1, T\frac{1}{2}) = \frac{3}{4},\ d(\frac{1}{2}, T1) = 0$$

and

$$\begin{aligned} S\left(\tfrac{1}{2}, 1\right) &= \max\left\{d(\tfrac{1}{2}, 1), \frac{d(\frac{1}{2}, T\frac{1}{2})d(\frac{1}{2}, T1) + d(1, T1)d(1, T\frac{1}{2})}{d(\frac{1}{2}, T1) + d(1, T\frac{1}{2})}\right\} \\ &= \max\left\{\tfrac{1}{2}, \frac{\frac{1}{4}\cdot 0 + \frac{1}{2}\cdot\frac{3}{4}}{0 + \frac{3}{4}}\right\} = \tfrac{1}{2}. \end{aligned}$$

In this case
$$d(T\frac{1}{2},T1)=\frac{1}{4}\leq\frac{1}{4}=\varphi(\frac{1}{2})=\varphi\left(S\left(\frac{1}{2},1\right)\right).$$
Case (2). For $x=\frac{1}{4}$ and $y=1$. In this case, we have:
$$d(T\frac{1}{4},T1)=\frac{1}{4},\ d(\frac{1}{4},1)=\frac{3}{4},\ d(1,T1)=\frac{1}{2},$$
$$d(\frac{1}{4},T\frac{1}{4})=0,\ d(1,T\frac{1}{4})=\frac{3}{4},\ d(\frac{1}{4},T1)=\frac{1}{4}$$
and
$$\begin{aligned}S\left(\tfrac{1}{4},1\right)&=\max\left\{d(\tfrac{1}{4},1),\frac{d(\frac{1}{4},T\frac{1}{4})d(\frac{1}{4},T1)+d(1,T1)d(1,T\frac{1}{4})}{d(\frac{1}{4},T1)+d(1,T\frac{1}{4})}\right\}\\&=\max\left\{\tfrac{3}{4},\frac{0\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}}{\frac{1}{4}+\frac{3}{4}}\right\}=\tfrac{3}{4}.\end{aligned}$$
Therefore,
$$d(T\frac{1}{4},T1)=\frac{1}{4}\leq\frac{1}{4}=\varphi(\frac{3}{4})=\varphi\left(S\left(\frac{1}{4},1\right)\right).$$
Hence all the conditions of Theorem 2.5are satisfied and T has a unique fixed point, $x=\frac{1}{4}$.

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