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# SCALING ANALYSIS AND SELF-SIMILARITY OF ONE-DIMENSIONAL TRANSPORT PROCESS

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**Abstract:** Convection-diffusion equation has been widely used to model a variety of flow and transport processes in earth sciences, including spread of pollutants in rivers, dispersion of dissolved material in estuaries and coastal waters, flow and transport in porous media, and transport of pollutants in the atmosphere. In this study, the conditions under which one-dimensional convection-diffusion equation becomes self-similar are investigated by utilizing one-parameter Lie group of point scaling transformations. By the numerical simulations, it is shown that the one-dimensional point source transport process in an original domain can be self-similar with that of a scaled domain. In fact, by changing the scaling parameter or the scaling exponents of the length dimension, one can obtain several different down-scaled or up-scaled self-similar domains. The derived scaling relations obtained by the Lie group scaling approach may provide additional understanding of transport phenomena at different space and time scales and may provide additional flexibility in setting up physical models in which one dimensional transport is significant.

**Keywords:** Lie group transformations, scaling, self-similarity, convection-diffusion equation

**Bir Boyutlu Taşınım Süreçlerinde Ölçekleme Analizi ve Kendine Benzeşim**

**Öz:** Konveksiyon-difüzyon denklemi nehirlerdeki kirleticilerin yayılması, çözülmüş maddenin haliç ve sahil sularına dağılımı, gözenekli ortamda akış ve taşınım, ve atmosferdeki kirleticilerin taşınması gibi yer bilimlerindeki çeşitli akım ve taşınım süreçlerini modellemek için yaygın bir şekilde kullanılmaktadır. Bu çalışmada tek boyutlu konveksiyon-difüzyon denkleminin kendine benzeşim koşulları tek parametreli Lie grubu nokta ölçeklendirme dönüşümleri kullanılarak araştırılmıştır. Sayısal simülasyonlarla, tek boyutlu noktasal kaynaklı taşıma sürecinin ölçeklendirilmiş bir mekanla özdeşleşebileceği gösterilmiştir. Ölçeklendirme parametresi veya uzunluk boyutunun ölçekleme katsayısı değiştirilerek daha büyük veya daha küçük mekansal boyutlarda taşınım sürecinin gerçekleştiği simetrik problemler elde edebilir. Lie grubu ölçeklendirme yaklaşımı ile elde edilen ölçeklendirme ilişkileri, farklı mekan ve zaman ölçeklerindeki taşınım süreçlerini anlamamızı kolaylaştırabilir ve bir boyutlu taşınımın önemli olduğu süreçlerinde fiziksel modellerin oluşturulmasında ilave esneklik sağlayabilir.

**Anahtar Kelimeler:** Lie grup değişim yöntemi, ölçekleme analizi, kendine benzerlik, konveksiyon-difizyon denklemi

## 1. INTRODUCTION

Convection-diffusion equation has been widely used to describe a variety of physical processes in nature, such as water transport in soils (e.g., Parlarge, 1980), dispersion of tracers in porous media (e.g., Fattah and Hoopes, 1985), the sea water intrusion into coastal aquifers (Bolster et al., 2007), the dispersion of pollutants and dissolved material in rivers, estuaries,

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shallow lakes and coastal waters (e.g., Salmon et al., 1980; Chatwin and Allen, 1985; James, 2002), transport of pollutants in the atmosphere (e.g., Ermak, 1977; Zlatev, 2012), flow in porous media (e.g., Kumar, 1983), and solute transport in groundwater (e.g., Guvanasen and Volker, 1983). Commonly, the diffusion coefficient is constant, there are no source/sink terms, and the velocity field describes an incompressible flow (Bird et al., 2007). Then, one-dimensional transport can be represented by the convection-diffusion equation in the form of

$$\frac{\partial C(x,t)}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial C(x,t)}{\partial x} - V \frac{\partial C(x,t)}{\partial x} \tag{1}$$

with initial condition

$$C(x,0) = f(x), \quad 0 \le x \le L \tag{2}$$

and boundary conditions

$$C(0,t) = g_0(t), \text{ and } C(L,t) = g_L(t), \; 0 \le t \le T \tag{3}$$

Here, C(x.t) is the concentration at location x at time t, D is the diffusion coefficient, V is the convection velocity, and f, $g_0$, $g_L$ are known functions.
In this manuscript, the conditions under which one-dimensional convection-diffusion process, as an initial and boundary value problem (Equations 1-3), becomes self-similar will be investigated by utilizing one-parameter Lie group of point scaling transformations. This approach is complementary to the dimensional analysis, which was widely utilized in physical model application in engineering. As suggested by Rayleigh (1892) and Buckingham (1914), dimensional analysis is performed by means of dimensionless products, which reduce the number of variables to be considered in a physical phenomenon. The method of dimensional analysis is widely used in experimental fluid mechanics. Refer also Sedov (1959) and Yalin (1971) for further engineering and hydraulic applications of dimensional analysis. By means of dimensional analysis, the ratio of the governing or dominant forces (such as Froude Number, Reynolds Number, or Weber number) is preserved between the original and the scaled domains. Such scaling based on the preserving one or a set of properties of the governing process is named with respect to the preserved dimensionless quantity, such as Reynolds scaling, or Froude scaling, which usually lead to the deviations of the other properties of the process. Guidelines on minimizing such scale effects in engineering applications have been offered by Yalin (1971), Martins (1989), and Heller (2011), etc.

One-parameter Lie group of point scaling transformations is the formal way of obtaining the self-similarity conditions of a physical process, as long as the governing process can be written in terms of governing equations with the corresponding initial and boundary conditions. The traditional approaches lack the process based formal self-similarity analysis. Readers can refer to Polsinelli and Kavvas (2016) for comparison of the modern Lie scaling method with classical scaling techniques. Recently, one-parameter Lie group of point scaling transformations were applied to investigate scale invariance and self-similarity of various hydrologic processes (Haltas and Kavvas 2011), one-dimensional open channel flow process (Ercan et. al, 2014), and one-dimensional suspended sediment transport process (Carr et al. 2015), two dimensional hydrodynamics (Ercan and Kavvas, 2015a), and 3-dimensional Navier-Stokes equations (Ercan and Kavvas, 2015b). The Main advantage of Lie group scaling transformations is that the self-similarity conditions due to the initial and boundary conditions could also be investigated in addition to those due to the governing equations. Many researchers including Hansen (1964), Bluman and Cole (1974), Ibragimov (1994, 1995), and Bluman and Anco (2002) discussed the Lie groups and symmetries and presented their applications.

In the next section, the one-parameter Lie group of point scaling transformations is applied to the governing partial differential equation of one dimensional transport (i.e., Equation 1), as well as its general initial and boundary conditions (i.e., Equations 2 and 3). From the transformed initial-boundary value problem, the self-similarity conditions are obtained. In the following section, numerical simulations are performed to show the validity of the obtained self-similarity criteria. Finally, a summary and conclusions are provided.

## 2. MATERIALS AND METHODS

As outlined in Bluman and Anco (2002) and Polyanin and Manzhirov (2006), the one-parameter Lie group of point scaling transformations in (x, t, C, D, V) space of one-dimensional convection-diffusion process that is governed by Equation (1) can be written as

$$x = \beta^{\alpha_x}\bar{x},\ t = \beta^{\alpha_t}\bar{t},\ C = \beta^{\alpha_C}\bar{C},\ D = \beta^{\alpha_D}\bar{D},\ V = \beta^{\alpha_V}\bar{V} \tag{4}$$

Here $\beta$ is the scaling parameter and $\alpha_t, \alpha_x, \alpha_C,\ \alpha_D, \alpha_V$ are scaling exponents of time, space, concentration, diffusion coefficient, and convective velocity. The equation (4) maps the original (x, t, C, D, V) space to scaled ($\bar{x}$, $\bar{t}$, $\bar{C}$, $\bar{D}$, $\bar{V}$) space. Then the scaling ratios of the variables x, t, C, D, V can be defined as

$$x_r = \frac{x}{\bar{x}} = \beta^{\alpha_x},\ t_r = \frac{t}{\bar{t}} = \beta^{\alpha_t},\ C_r = \frac{C}{\bar{C}} = \beta^{\alpha_C},\ D_r = \frac{D}{\bar{D}} = \beta^{\alpha_D},\ V_r = \frac{V}{\bar{V}} = \beta^{\alpha_V} \tag{5}$$

The self-similarity conditions for the one-dimensional convection-diffusion process can be found when the initial and boundary value problem of the governing process in the original domain, subjected to the Lie group of point scaling transformations, remains invariant in the new transformed variables.

### 2.1. Transformation of the Governing Equation

Applying the one-parameter Lie scaling transformations on the one-dimensional convection-diffusion (Equation 1) yields:

$$\beta^{\alpha_C-\alpha_x}\frac{\partial C(x,t)}{\partial t} = \beta^{\alpha_D+\alpha_C-2\alpha_x}\frac{\partial}{\partial x}D\frac{\partial C(x,t)}{\partial x} - \beta^{\alpha_V+\alpha_C-\alpha_x}V\frac{\partial C(x,t)}{\partial x} \tag{6}$$

In order for above equation to be invariant, it has to be in the same form as equation (1). Hence, the following conditions must hold

$$\alpha_C - \alpha_t = \alpha_D + \alpha_C - 2\alpha_x = \alpha_V + \alpha_C - \alpha_x \tag{7}$$

Then one can obtain $\alpha_V = \alpha_x - \alpha_t$ and $\alpha_D = 2\alpha_x - \alpha_t$. The scaling conditions of the one-dimensional convection-diffusion process are provided in Table 1.

### 2.2. Transformation of the initial and boundary conditions

Applying the one-parameter Lie scaling transformations on the initial and boundary conditions of the one-dimensional convection-diffusion process (Equations 2-3) yields:

$$C(x,0) = \beta^{\alpha_C}\bar{C}(\bar{x},0) = f(\beta^{\alpha_x}\bar{x}),\quad 0 \le \beta^{\alpha_x}\bar{x} \le L \tag{8}$$

$$C(0,t)=\beta^{\alpha_C}\bar{C}(0,\bar{t})=g_0(\beta^{\alpha_t}\bar{t}),\ \text{and}\ C(L,t)=\beta^{\alpha_C}\bar{C}(\beta^{-\alpha_x}L,\bar{t})=g_L(\beta^{\alpha_t}\bar{t}),\quad 0\le\beta^{\alpha_t}\bar{t}\le T \tag{9}$$

which may then be expressed as

$$\bar{C}(\bar{x},0)=\beta^{-\alpha_C}f(\beta^{\alpha_x}\bar{x}),\ \ 0\le\bar{x}\le\beta^{-\alpha_x}L \tag{10}$$

$$\bar{C}(0,\bar{t})=\beta^{-\alpha_C}g_0(\beta^{\alpha_t}\bar{t})\ ,\ \text{and}\ \bar{C}(\beta^{-\alpha_x}L,\bar{t})=\beta^{-\alpha_C}g_L(\beta^{\alpha_t}\bar{t})\ ,\quad 0\le\bar{t}\le\beta^{-\alpha_t}T \tag{11}$$

**Table 1. The scaling exponents obtained by the one-parameter Lie group of point scaling transformations for the variables of the 1-dimensional convection-diffusion equation.**

| **Variable** | **Scaling Conditions in terms of $\alpha_x$, $\alpha_t$, and $\alpha_M$** |
|---|---|
| Length x | $\alpha_x$ |
| Time, t | $\alpha_t$ |
| Mass per area, M | $\alpha_M$ |
| Diffusivity D | $\alpha_D=2\alpha_x-\alpha_t$ |
| Convective Velocity, V | $\alpha_V=\alpha_x-\alpha_t$ |
| Concentration, C | $\alpha_C=\alpha_M-\alpha_x$ |

## 3. RESULTS AND DISCUSSION

The analytical solution of the convection-diffusion equation can be derived from the Gaussian distribution

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-0.5\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{12}$$

where $\mu$ is the mean and $\sigma$ is the standard deviation, and such models are named Gaussian models (Fischer, 1966; Holzbecher, 2007). When the initial condition of the concentration for the total mass per area M can be described by $\delta-\text{function}$ in the form of

$$\delta(x)=\begin{bmatrix}\infty & \text{for}\ \ x=0\\ 0 & \text{for}\ \ x\neq 0\end{bmatrix}\ \ \text{and}\ \ \int_{-\infty}^{\infty}\delta(x)dx=M \tag{13}$$

the analytical solution of the convection-diffusion equation can be obtained from (Fischer, 1966; Holzbecher, 2007)

$$C(x,t)=\frac{M}{\sqrt{4\pi tD}}\exp\left[-\frac{(x-Vt)^2}{4tD}\right] \tag{14}$$

In order for the initial condition given by Equation 13 to be invariant, the condition $\alpha_C=\alpha_M-\alpha_x$ must hold.

Let us now explore the self-similarity of one-dimensional convection-diffusion equation by solving the above analytical solution. The above analytical solution (Equation 14) has been applied in many transport problems, such as tracer experiments in karstic aquifers (Maloszewski et al., 1994), dispersion in lowland rivers (Sukhodolov et al., 1997), miscible displacement of initial solute distributions in soil columns (Wang and Persaud, 2004), and pollution problems in aquifers (Bear, 1976).

First, the one-dimensional transport problem is simulated over an original domain (i.e. Domain 1, or D1) of 0,2 m length for a duration of 1 second, diffusivity coefficient of 6,25E-04 $m^2/s$, and convective velocity of 0,1 m/s when a total of 1 $kg/m^2$ mass is released at x=0. Utilizing the scaling exponents and ratios given in Table 2, which follow the scaling conditions provided in Table 1, transport characteristics of five self-similar domains (D2, D3, D4, D5, D6) are obtained as presented in Table 3.

It is possible to obtain both larger (e.g. D6) and smaller (e.g. D2, D3, D4, and D5) self-similar domains by selecting the scaling parameter β and scaling exponent of length $\alpha_x$, which result in shorter (e.g. D2, D3, D4, and D5) and longer (e.g. D6) simulation times. A self-similar domain which is larger than the original domain (e.g. D6) can be obtained when the scaling parameter β is less than 1 and scaling exponent of length to be positive, which is equivalent to the case when the scaling parameter β is greater than 1 but scaling exponent of length $\alpha_x$ to be negative. For example, β=0,5 and $\alpha_x$ =1 is equivalent to β=2 and $\alpha_x$=-1 since both cases result in the same length scaling ratio of 0,5.

Concentration contours as a function of the whole simulation time and length for Domains 1-6 are depicted in Figure 1. As depicted in subfigures of Figure 1, self-similar results of concentration are obtained following the derived scaling conditions for Domain 1 (D1) in which the simulation length L is 0,2 m and simulation time T is 1 second, D2 in which the simulation length L is 0,1 m and simulation time T is 0,5 second, D3 in which the simulation length L is 0,1 m and simulation time T is 0,25 second, D4 in which the simulation length L is 0,025 m and simulation time T is 0,25 second, D5 in which the simulation length L is 0,1 m and simulation time T is 0,25 second, and D6 in which the simulation length L is 0,4 m and simulation time T is 4 second.

Concentration versus length at various times (t=T/5, t=2T/5, t=3T/5, t=4T/5, and t=T) for Domains 1-6 are depicted in Figure 2. Following the derived scaling conditions, self-similar results of concentration at various times (t=T/5, t=2T/5, t=3T/5, t=4T/5, and t=T) are obtained for D1 (in which L is 0,2 m and T is 1 second), D2 (in which L is 0,1 m and T is 0,5 second), D3 (in which L is 0,1 m and T is 0,25 second), D4 (in which L is 0,025 m and T is 0,25 second), D5 (in which L is 0,1 m and T is 0,25 second), and D6 (in which L is 0,4 m and T is 4 second). Furthermore, concentration in the original domain (D1) against the corresponding concentration in the scaled domains (Domain 2-6) at times t=T/5, t=2T/5, t=3T/5, t=4T/5, and t=T are presented in Figure 3. In the case of perfect self-similarity, the plotted concentrations in Figures 3 should follow perfect lines with slopes being the concentration scaling ratios $\beta^{\alpha_C}$ (1 for D1-D2, 1 for D1-D3, 0,25 for D1-D4, 4 for D1-D5, and 0,25 for D1-D6), and with intercept being 0. The simulated concentrations in Figures 3 follow almost perfect lines with slopes 1,00000000000000 for D1-D2, 1,00000000000000 for D1-D3, 2,50000000000000e-01 for D1-

D4, 4,00000000000000e-00 for D1-D5, and 2,50000000000000e-01 for D1-D6), and with absolute value of the intercepts being less than 3,97e-16.

**Table 2. Scaling exponents and ratios to obtain Domains 2, 3, 4, and 5 from Domain 1.**

| | Domain 2 | Domain 3 | Domain 4 | Domain 5 | Domain 6 |
|---|---|---|---|---|---|
| Scaling parameter, $\beta$ | 2 | 2 | 2 | 2 | 0,5 |
| scaling exponents | | | | | |
| Length, $\alpha_x$ | 1 | 1 | 3 | 1 | 1 |
| Time, $\alpha_t$ | 1 | 2 | 2 | 2 | 2 |
| Mass, $\alpha_M$ | 1 | 1 | 1 | 3 | 3 |
| Diffusivity, $\alpha_D$ | 1 | 0 | 4 | 0 | 0 |
| Convective Velocity, $\alpha_V$ | 0 | -1 | 1 | -1 | -1 |
| Concentration, $\alpha_C$ | 0 | 0 | -2 | 2 | 2 |
| scaling ratios | | | | | |
| Length, $\beta^{\alpha_x}$ | 2 | 2 | 8 | 2 | 0,5 |
| Time, $\beta^{\alpha_t}$ | 2 | 4 | 4 | 4 | 0,25 |
| Mass, $\beta^{\alpha_M}$ | 2 | 2 | 2 | 8 | 0,125 |
| Diffusivity, $\beta^{\alpha_D}$ | 2 | 1 | 16 | 1 | 1 |
| Convective Velocity, $\beta^{\alpha_V}$ | 1 | 0,5 | 2 | 0,5 | 2 |
| Concentration, $\beta^{\alpha_C}$ | 1 | 1 | 0,25 | 4 | 0,25 |

**Table 3. Summary of the simulation characteristics for the original domain (Domain 1) and its self-similar domains (Domains 2, 3, 4, and 5).**

| | Domain 1 | Domain 2 | Domain 3 | Domain 4 | Domain 5 | Domain 6 |
|---|---|---|---|---|---|---|
| Diffusivity, D ($m^2$/s) | 6,25E-04 | 3,13E-04 | 6,25E-04 | 3,91E-05 | 6,25E-04 | 6,25E-04 |
| Convective Velocity, V (m/s) | 0,1 | 0,1 | 0,2 | 0,05 | 0,2 | 0,05 |
| Mass per area, M (kg/$m^2$) | 1 | 0,5 | 0,5 | 0,5 | 0,125 | 8 |
| Length, L (m) | 0,2 | 0,1 | 0,1 | 0,025 | 0,1 | 0,4 |
| Simulation Time, T (s) | 1 | 0,5 | 0,25 | 0,25 | 0,25 | 4 |

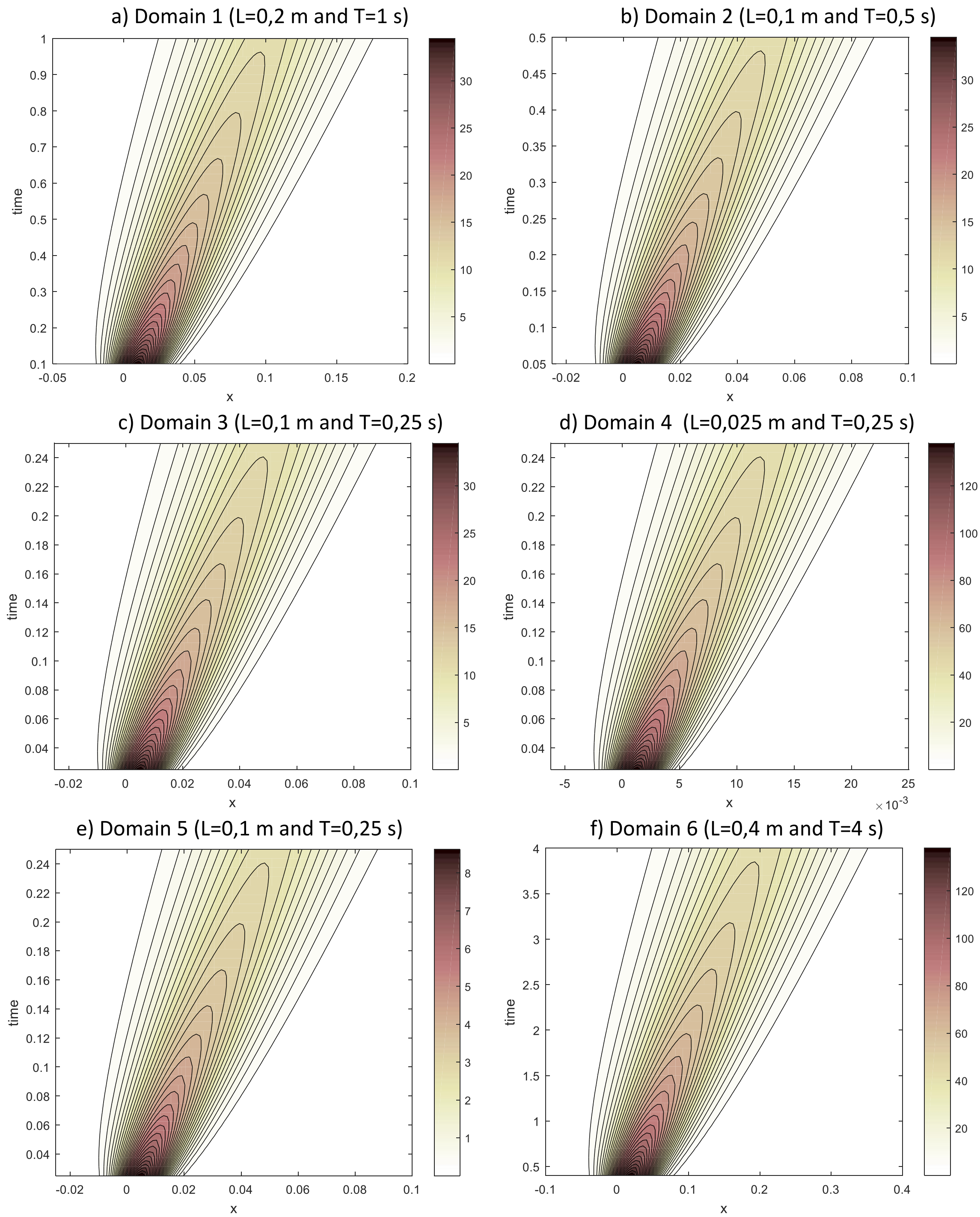


**Figure 1:**

*Concentration contours as a function of time (seconds) and location x (m) for a) Domain 1, b) Domain 2, c) Domain 3, d) Domain 4, e) Domain 5, and f) Domain 6.*

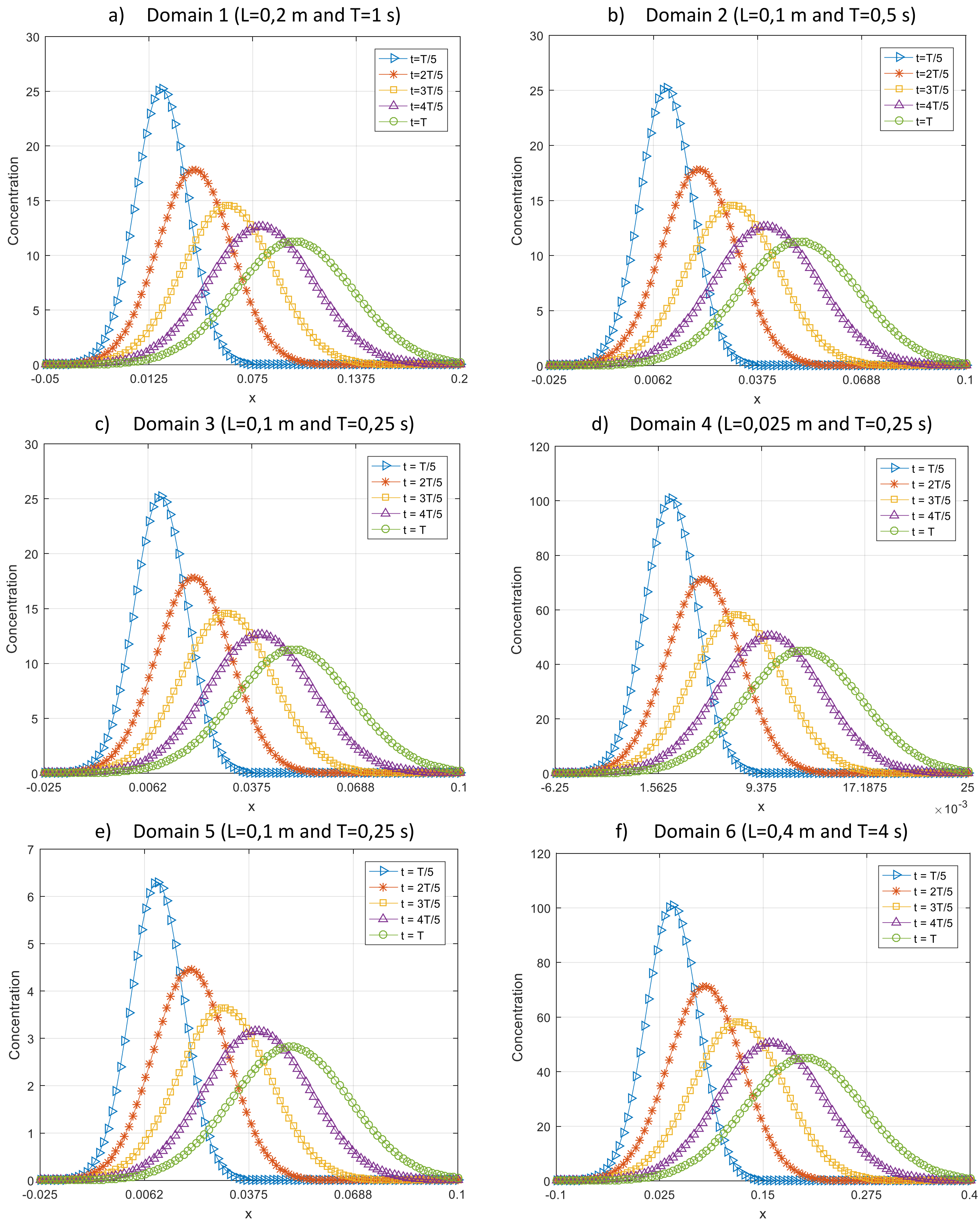


***Figure 2:***
*Concentration versus length x (m) at various times (t=T/5, t=2T/5, t=3T/5, t=4T/5, and t=T) for a) Domain 1, b) Domain 2, c) Domain 3, d) Domain 4, e) Domain 5, and f) Domain 6.*

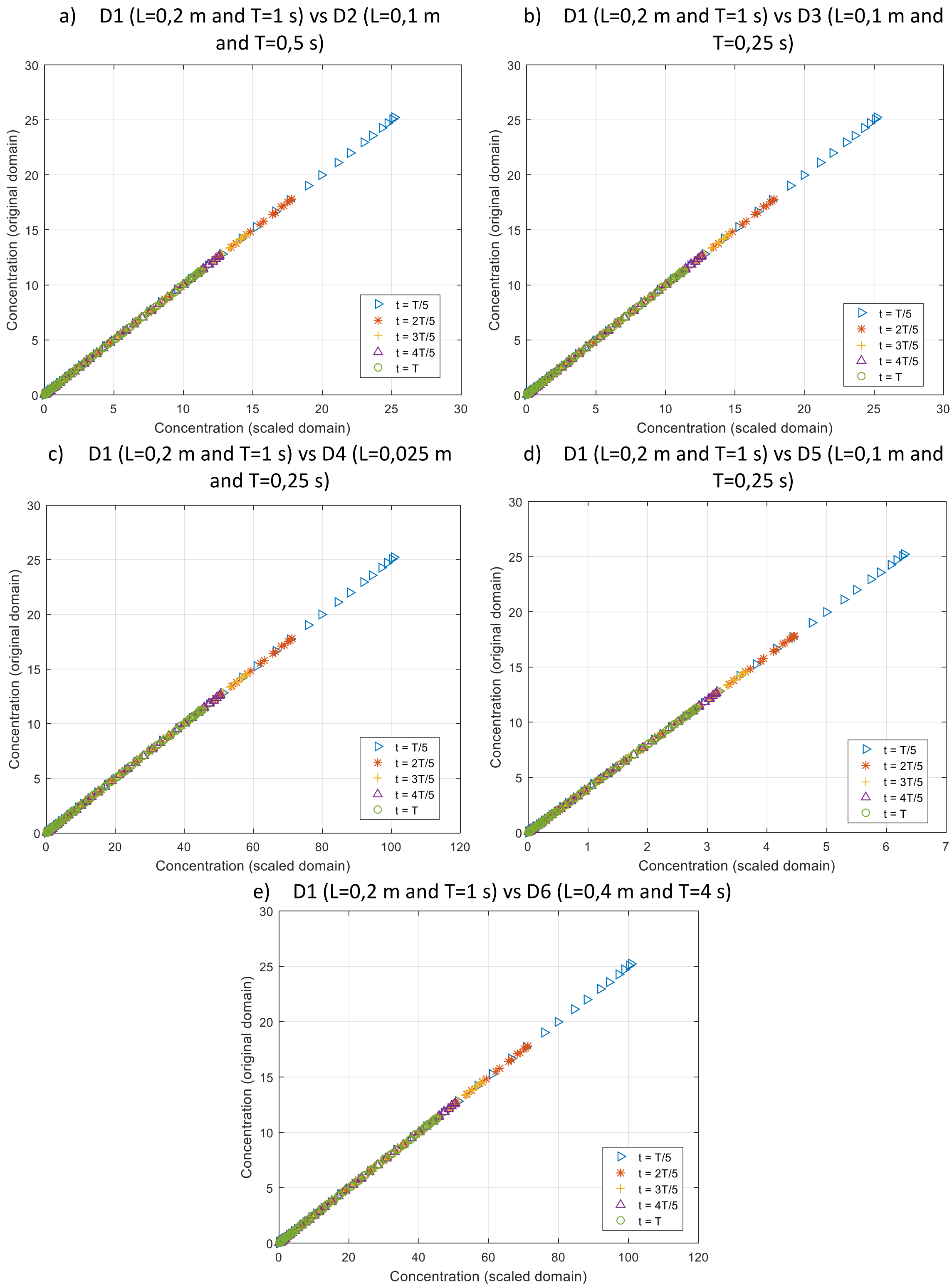


**Figure 3:**

*Concentration in the original domain against the corresponding concentration in the scaled domain at times t=T/5, t=2T/5, t=3T/5, t=4T/5, and t=T: a) Domain 1 vs Domain 2, b) Domain 1 vs Domain 3, c) Domain 1 vs Domain 4, d) Domain 1 vs Domain 5, and e) Domain 1 vs Domain 6.*

## 4. CONCLUSIONS

The conditions under which the one-dimensional convection-diffusion equation system, as an initial-boundary value problem, becomes self-similar were investigated by utilizing the one-parameter Lie group of point scaling transformations. The self-similarity conditions due to the initial and boundary conditions were also investigated in addition to the conditions due to the governing equation. The derived self-similarity conditions for the one-dimensional convection-diffusion transport process were then evaluated by a hypothetical numerical problem simulating the transport for six different domains. By the numerical simulations, it was shown that the one-dimensional convection-diffusion transport process in a original domain can be self-similar with that of a scaled (up-scaled or down-scaled) domain. In fact, it was demonstrated that one can obtain several different scaled domains by changing the scaling parameter or the scaling exponents of the length dimension.

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