The $\Delta-$convergence of proximal point iteration based on the variational inequality in CAT(0) Spaces

Yue Zhang$^a$, Dingping Wu$^a$.

$^a$ Department of Applied Mathematics, Chengdu University of Information Technology, Chengdu, 610225, China

Abstract

In this paper, we build the inexact proximal point algorithm of Mann-type and Ishikawa-type iteration based on the variational inequality in Hadamard spaces and prove $\Delta-$convergence to a fixed point of the nonexpansive mapping.

Keywords: Mann-type, Ishikawa-type, variational inequality, nonexpansive mapping, Hadamard spaces, inexact, $\Delta-$convergence.

2010 MSC: 54H25, 47H10.

1. Introduction

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $f$ from a closed interval $[0, l] \subset \mathbb{R}$ to $X$ such that $f(0) = x$, $f(l) = y$ and $d(f(t), f(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $f$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $f$ is called a geodesic (or metric) segment joining $x$ and $y$. When it is unique this geodesic is denoted $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points.

A geodesic space $(X, d)$ is a CAT(0) space if it satisfies the following $CN$-inequality for $x, z_0, z_1, z_2 \in X$ such that $d(z_0, z_1) = d(z_0, z_2) = \frac{1}{2}d(z_1, z_2)$:

$$d^2(x, z_0) \leq \frac{1}{2}d^2(x, z_1) + \frac{1}{2}d^2(x, z_2) - \frac{1}{4}d^2(z_1, z_2).$$

A complete CAT(0) space is called a Hadamard space.
Berg and Nikolaev\cite{3} introduced the concept of quasi-linearization in CAT(0) space $X$. They denoted a vector by $\overrightarrow{ab}$ for $(a, b) \in X \times X$ and defined the quasi-linearization map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ as follow:
\[\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}[d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)],\]
for $a, b, c, d \in X$. We can verify $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{a\epsilon}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, e \in X$. For a space $X$, it satisfies the Cauchy-Schwarz inequality if
\[\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)\]
for all $a, b, c, d \in X$. It is known\cite{3} that a geodesically connected metric space $X$ is a CAT(0) space if and only if it satisfies the Cauchy-Schwarz inequality.

Let $(X, d)$ be a complete CAT(0) space, $K$ is convex and closed in $X$ and $T : K \to X$ is a nonexpansive mapping. We formulate the variational inequality problem in a Hadamard space $X$ and defined the quasi-linearization map
\[\langle -\rightarrow, \rightarrow \rangle, T_x = \langle Ty, y \rangle - \langle Tx, y \rangle - \langle x, y \rangle \geq 0, \quad \forall z \in X,\]
(1.1)

Hadi Khatibzadeh and Sajad Ranjbar\cite{2} have introduced the existence of solutions for the variational inequality problem (1.1), which is also a fixed point of the nonexpansive mapping $T$. Meanwhile, they also proved the strong convergence of the inexact proximal point algorithm
\[\left\{\begin{array}{c}
x_n = \alpha_n u + (1 - \alpha_n)y_n, \\
x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n,
\end{array}\right.\]
with suitable assumptions on the parameter sequences $\alpha_n$ and $\lambda_n$, to a fixed point of the nonexpansive mapping $T$. In this paper, we build the Mann-type and Ishikawa-type regularization as follow, Mann-type:
\[\left\{\begin{array}{c}
x_0 \in K, \\
\langle Ty_n, y_n \rangle - \frac{1}{\lambda_n} y_n x_n, y_n z \rangle \geq -\varepsilon_n, \quad \forall z \in X, \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n,
\end{array}\right.\]
(1.4)
with $(\lambda_n) \in (0, \infty)$, $(\alpha_n) \subset [0, 1]$, $(\varepsilon_n) \subset [0, \infty)$ and Ishikawa-type:
\[\left\{\begin{array}{c}
x_0 \in K, \\
\langle Ty_n, y_n \rangle - \frac{1}{\lambda_n} y_n x_n, y_n z \rangle \geq -\varepsilon_n, \quad \forall z \in X, \\
\langle Ty_n, y_n \rangle - \frac{1}{\lambda_n} y_n x_n, y_n z \rangle \geq -\varepsilon_n, \quad \forall z \in X, \\
u_n = \alpha_n x_n + (1 - \alpha_n)y_n, \\
u_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n, \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n,
\end{array}\right.\]
(1.5)
with $(\lambda_n) \in (0, \infty)$, $(\eta_n) \in (0, \infty)$, $(\alpha_n) \subset [0, 1]$, $(\beta_n) \subset [0, 1]$, $(\varepsilon_n) \subset [0, \infty)$ and $(\delta_n) \subset [0, \infty)$. Therefore, we prove the two sequences generated by the algorithm (1.4) and (1.5) are $\Delta$-convergent to a fixed point of the nonexpansive mapping $T$. Then, we improve and extend their results.
2. Preliminaries

Definition 2.1 A sequence \((x_n)\) in a complete CAT(0) space \((X, d)\) is said to be \(\Delta\)-convergent to \(x \in X\) if \(A((x_n)) = \{x\}\) for every subsequence \((x_{n_k})\) of \((x_n)\).

Lemma 2.2 Every bounded sequence in a complete CAT(0) space has a \(\Delta\)-convergent subsequence.

Lemma 2.3 Let \((X, d)\) be a CAT(0) space. Then

\[
d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)
\]

for all \(x, y, z \in X\) and \(t \in [0, 1]\).

Lemma 2.4 Let \((X, d)\) be a CAT(0) space. Then

\[
d^2((1 - t)x \oplus ty, z) \leq (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y)
\]

for all \(t \in [0, 1]\) and \(x, y, z \in X\).

Lemma 2.5 Let \((X, d)\) be a CAT(0) space and and let \((x_n)\) be a sequence in \(X\). If there exists a nonempty subset \(F\) of \(X\) verifying the following conditions:

(i) for each \(z \in F\), \(\lim_{n \to \infty} d(x_n, z)\) exists,

(ii) if a subsequence \((x_{n_k})\) of \((x_n)\) is \(\Delta\)-convergent to \(x \in X\), then \(x \in F\), then \((x_n)\) \(\Delta\)-converges to an element of \(F\).

Lemma 2.6 Let \(K\) be bounded. Then there exists a solution of the variational inequality problem (1.1).

Lemma 2.7 Let \(x \in \text{int}(K)\) (interior of \(K\)) be a solution of problem (1.1), then \(x \in F(T)\).

Lemma 2.8 For each \(x \in K\) and each \(\lambda > 0\), there exists a unique \(y \in K\) such that

\[
\langle Ty - \frac{1}{\lambda} \overrightarrow{y, y_n}, x_n \rangle \geq 0, \quad \forall z \in X.
\]

Lemma 2.9 Suppose that \(\{a_n\}\) and \(\{b_n\}\) are two sequences of nonnegative numbers such that \(a_{n+1} \leq a_n + b_n\) for all \(n \geq 1\). If \(\sum b_n\) converges, then \(\lim_{n} a_n\) exists.

3. Main results

Theorem 3.1 Let \(X\) be a Hadamard space. \(K \subset X\) is convex and closed and \(T : K \to X\) is a nonexpansive mapping and \(F(T) \neq \emptyset\). The sequence \(\{x_n\}\) is generated by (1.4) and let the conditions hold: \(\limsup_n \alpha_n < 1\), \(\liminf_n \lambda_n > 0\) and \(\sum \epsilon_n \lambda_n < \infty\). Then the sequence \(\{x_n\}\) is \(\Delta\)-convergent to a fixed point of \(T\).

Proof. Let \(z \in F(T)\), then by (1.4), we have

\[
\langle Ty_n - \frac{1}{\lambda_n} \overrightarrow{y_n x_n}, y_n \rangle \geq -\frac{\epsilon_n}{2}.
\]

Hence, we can obtain

\[
-\epsilon_n \leq 2 \langle Ty_n, y_n \rangle - \frac{2}{\lambda_n} \langle y_n x_n, y_n \rangle
\]

\[
= d^2(z, Ty_n) - d^2(y_n, Ty_n) - d^2(y_n, z)
\]

\[
- \frac{1}{\lambda_n} (d^2(y_n, z) + d^2(y_n, x_n) - d^2(x_n, z)).
\]

By the nonexpansiveness of \(T\), we get

\[
\lambda_n d^2(y_n, Ty_n) + d^2(y_n, x_n) \leq d^2(x_n, z) - d^2(y_n, z) + \epsilon_n \lambda_n,
\]

(3.1)
which implies
\[ d^2(y_n, z) \leq d^2(x_n, z) + \varepsilon_n \lambda_n. \] (3.2)

By the lemma 2.4, we have
\[
d^2(x_{n+1}, z) = d^2(\alpha_n x_n + (1 - \alpha_n) y_n, z)
\leq \alpha_n d^2(x_n, z) + (1 - \alpha_n) d^2(y_n, z) - \alpha_n (1 - \alpha_n) d^2(x_n, y_n)
\leq \alpha_n d^2(x_n, z) + (1 - \alpha_n) (d^2(x_n, z) + \varepsilon_n \lambda_n)
= d^2(x_n, z) + (1 - \alpha_n) \varepsilon_n \lambda_n
\leq \cdots \leq d^2(x_1, z) + \sum_{i=1}^{n} (1 - \alpha_i) \varepsilon_i \lambda_i.
\]

Therefore, according to the lemma 2.9, we know that \( \{x_n\} \) and \( \{y_n\} \) are bounded and \( \lim_n d(x_n, z) \) exists. Then, let \( \lim_n d(x_n, z) = a \). If \( a = 0 \), it denotes that the sequence \( \{x_n\} \) is strongly convergent to a fixed point of \( T \); if \( a > 0 \), consequently, by (3.2), we have
\[
(d(y_n, z) - d(x_n, z))(d(y_n, z) + d(x_n, z)) \leq \varepsilon_n \lambda_n,
\]
which means
\[
d(y_n, z) \leq d(x_n, z) + \frac{\varepsilon_n \lambda_n}{d(y_n, z) + d(x_n, z)} \leq d(x_n, z) + M \varepsilon_n \lambda_n,
\]
where
\[
M = \sup_n \{\frac{1}{d(y_n, z) + d(x_n, z)}\}.
\]

Since \( \lim_n d(x_n, z) \) exists and by the lemma 2.3, we obtain
\[
0 = \lim_n (d(x_{n+1}, z) - d(x_n, z)) \leq \lim_n \inf_n (\alpha_n d(x_n, z) + (1 - \alpha_n) d(y_n, z) - d(x_n, z))
= \lim_n \inf_n (1 - \alpha_n) (d(y_n, z) - d(x_n, z))
\leq \lim_n \sup_n (1 - \alpha_n) (d(y_n, z) - d(x_n, z))
\leq \lim_n \sup_n (1 - \alpha_n) M \varepsilon_n \lambda_n = 0.
\]

By the assumptions \( \limsup_n \alpha_n < 1 \), we get
\[
\lim_n (d(y_n, z) - d(x_n, z)) = 0.
\]

Also, by the assumptions and (3.1), we have
\[
\lim_n d(y_n, Ty_n) = 0, \quad \lim_n d(y_n, x_n) = 0.
\]

Therefore, by the nonexpansiveness of \( T \), we may obtain
\[
\lim_n d(x_n, Tx_n) \leq \lim_n d(x_n, y_n) + d(y_n, Ty_n) + d(Ty_n, Tx_n)
\leq \lim_n d(x_n, y_n) + d(y_n, Ty_n) + d(y_n, x_n) = 0,
\]
which implies
\[
\lim_n d(x_n, Tx_n) = 0.
\]

Thus, if a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) is \( \Delta \)-convergent to \( q \in X \), then
\[
d(x_{n_j}, Tx_{n_j}) \to 0.
\]

Hence, by \( \Delta \)-demiclosedness of nonexpansive mappings, we have \( q \in F(T) \). Finally, by the lemma 2.5, the sequence \( \{x_n\} \) is \( \Delta \)-convergent to \( z \in F(T) \). This completes the proof. \( \square \)
Theorem 3.2 Let $X$ be a Hadamard space. $K \subset X$ is convex and closed and $T : K \to X$ is a nonexpansive mapping and $F(T) \neq \emptyset$. The sequence $\{x_n\}$ is generated by (1.5) and let the conditions hold: $\lim \sup_n \beta_n < 1$, $\lim \inf_n \lambda_n > 0$, $\lim \inf \eta_n > 0$, $\sum_n \varepsilon_n \lambda_n < \infty$ and $\sum_n \delta_n \eta_n < \infty$. Then the sequence $\{x_n\}$ is $\Delta-$convergent to a fixed point of $T$.

Proof. Let $z \in F(T)$, then by (1.5), we have

$$
\left\langle Ty_ny_n - \frac{1}{\lambda_n}y_nx_n, y_nz \right\rangle \geq -\frac{\varepsilon_n}{2}.
$$

Hence, we can obtain

$$
-\varepsilon_n \leq 2 \left\langle Ty_ny_n, y_nz \right\rangle - \frac{2}{\lambda_n} \left\langle y_nx_n, y_nz \right\rangle = d^2(z, Ty_n) - d^2(y_n, Ty_n) - d^2(y_n, z) - \frac{1}{\lambda_n}(d^2(y_n, z) + d^2(y_n, x_n) - d^2(x_n, z)).
$$

By the nonexpansiveness of $T$, we get

$$
\lambda_n d^2(y_n, Ty_n) + d^2(y_n, x_n) \leq d^2(x_n, z) - d^2(y_n, z) + \varepsilon_n \lambda_n,
$$

which implies

$$
d^2(y_n, z) \leq d^2(x_n, z) + \varepsilon_n \lambda_n.
$$

Similarly, we obtain

$$
\eta_n d^2(w_n, Tw_n) + d^2(u_n, w_n) \leq d^2(u_n, z) - d^2(w_n, z) + \delta_n \eta_n,
$$

which also implies

$$
d^2(w_n, z) \leq d^2(u_n, z) + \delta_n \eta_n.
$$

By the lemma 2.4, we get

$$
d^2(x_{n+1}, z) = d^2(\beta_n x_n + (1 - \beta_n)w_n, z)
\leq \beta_n d^2(x_n, z) + (1 - \beta_n)d^2(w_n, z) - \beta_n(1 - \beta_n)d^2(x_n, w_n)
\leq \beta_n d^2(x_n, z) + (1 - \beta_n)d^2(u_n, z) + (1 - \beta_n)\delta_n \eta_n,
$$

and

$$
d^2(u_n, z) = d^2(\alpha_n x_n + (1 - \alpha_n)y_n, z)
\leq \alpha_n d^2(x_n, z) + (1 - \alpha_n)d^2(y_n, z) - \alpha_n(1 - \alpha_n)d^2(x_n, y_n)
\leq \alpha_n d^2(x_n, z) + (1 - \alpha_n)(d^2(x_n, z) + \varepsilon_n \lambda_n)
= d^2(x_n, z) + (1 - \alpha_n)\varepsilon_n \lambda_n,
$$

then, we obtain

$$
d^2(x_{n+1}, z) \leq \beta_n d^2(x_n, z) + (1 - \beta_n)(d^2(x_n, z) + (1 - \alpha_n)\varepsilon_n \lambda_n) + (1 - \beta_n)\delta_n \eta_n
= d^2(x_n, z) + (1 - \beta_n)(1 - \alpha_n)\varepsilon_n \lambda_n + (1 - \beta_n)\delta_n \eta_n
\leq \cdots \leq d^2(x_1, z) + \sum_{i=1}^{n} (1 - \beta_i)(1 - \alpha_i)\varepsilon_i \lambda_i + \sum_{i=1}^{n} (1 - \beta_i)\delta_i \eta_i.
$$
Therefore, according to the lemma 2.9, we know that \( \{x_n\}, \{y_n\}, \{u_n\} \) and \( \{w_n\} \) are bounded and \( \lim_n d(x_n, z) \) exists. Then, let \( \lim_n d(x_n, z) = a \). If \( a = 0 \), it denotes that the sequence \( \{x_n\} \) is strongly convergent to a fixed point of \( T \); if \( a > 0 \), consequently, by (3.4) and (3.6), we have

\[
(d(y_n, z) - d(x_n, z))(d(y_n, z) + d(x_n, z)) \leq \varepsilon_n \lambda_n
\]

and

\[
(d(w_n, z) - d(u_n, z))(d(w_n, z) + d(u_n, z)) \leq \delta_n \eta_n,
\]

which mean

\[
d(y_n, z) \leq d(x_n, z) + \frac{\varepsilon_n \lambda_n}{d(y_n, z) + d(x_n, z)} \leq d(x_n, z) + M_1 \varepsilon_n \lambda_n
\]

and

\[
d(w_n, z) \leq d(u_n, z) + \frac{\delta_n \eta_n}{d(w_n, z) + d(u_n, z)} \leq d(u_n, z) + M_2 \delta_n \eta_n,
\]

where

\[
M_1 = \sup_n \left\{ \frac{1}{d(y_n, z) + d(x_n, z)} \right\}, \quad M_2 = \sup_n \left\{ \frac{1}{d(w_n, z) + d(u_n, z)} \right\}.
\]

In the following, we consider three possible cases for the sequence \( \{\alpha_n\} \).

(1) \( \limsup_n \alpha_n < 1 \).

Since \( \lim_n d(x_n, z) \) exists and by the lemma 2.3, we obtain

\[
0 = \lim_n (d(x_{n+1}, z) - d(x_n, z)) \leq \liminf_n (\beta_n d(x_n, z) + (1 - \beta_n) d(w_n, z) - d(x_n, z)) = \liminf_n (1 - \beta_n) (d(w_n, z) - d(x_n, z)) \leq \liminf_n (1 - \beta_n) (d(w_n, z) + M_2 \delta_n \eta_n - d(x_n, z)) \leq \liminf_n (1 - \beta_n) (\alpha_n d(x_n, z) + (1 - \alpha_n) d(y_n, z) + M_2 \delta_n \eta_n - d(x_n, z)) = \liminf_n (1 - \beta_n) (1 - \alpha_n) (d(y_n, z) - d(x_n, z) + \frac{M_2 \delta_n \eta_n}{1 - \alpha_n}) \leq \limsup_n (1 - \beta_n) (1 - \alpha_n) (d(y_n, z) - d(x_n, z) + \frac{M_2 \delta_n \eta_n}{1 - \alpha_n}) \leq \limsup_n (1 - \beta_n) (1 - \alpha_n) (M_1 \varepsilon_n \lambda_n + \frac{M_2 \delta_n \eta_n}{1 - \alpha_n}) = 0.
\]

By the condition \( \limsup_n \alpha_n < 1 \) and assumption \( \limsup_n \beta_n < 1 \), we get

\[
\lim_n (d(y_n, z) - d(x_n, z)) = 0.
\]

Also, by the assumptions and (3.3), we have

\[
\lim_n d(y_n, Ty_n) = 0, \quad \lim_n d(y_n, x_n) = 0.
\]

Therefore, by the nonexpansiveness of \( T \), we may obtain

\[
\lim_n d(x_n, Tx_n) \leq \lim_n (d(x_n, y_n) + d(y_n, Ty_n) + d(Ty_n, Tx_n)) \leq \lim_n (d(x_n, y_n) + d(y_n, Ty_n) + d(y_n, x_n)) = 0,
\]

which implies

\[
\lim_n d(x_n, Tx_n) = 0.
\]
Thus, if a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) is \( \Delta \)-convergent to \( q \in X \), then
\[
d(x_{n_j}, Tx_{n_j}) \to 0.
\]

Hence, by \( \Delta \)-demiclosedness of nonexpansive mappings, we have \( q \in F(T) \). Finally, by the lemma 2.5, the sequence \( \{x_n\} \) is \( \Delta \)-convergent to \( z \in F(T) \).

(2) \( \limsup_n \alpha_n = 1 \).

Hence, there exists a \( \{\alpha_{n_j}\} \) of \( \{\alpha_n\} \) satisfied \( \limsup_j \alpha_{n_j} < 1 \). Similar to (1), there exist a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) and a subsequence \( \{y_{n_j}\} \) of \( \{y_n\} \) such that
\[
\lim_j (d(y_{n_j}, z) - d(x_{n_j}, z)) = 0.
\]

Also, by the assumptions and (3.3), we still have
\[
\lim_j d(y_{n_j}, Ty_{n_j}) = 0, \lim_j d(y_{n_j}, x_{n_j}) = 0.
\]

Therefore, we also obtain
\[
\lim_j d(x_{n_j}, Tx_{n_j}) = 0.
\]

Thus, if the subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) is \( \Delta \)-convergent to \( q \in X \), then
\[
d(x_{n_j}, Tx_{n_j}) \to 0.
\]

Hence, by \( \Delta \)-demiclosedness of nonexpansive mappings, we have \( q \in F(T) \). Finally, by the lemma 2.5, the sequence \( \{x_n\} \) is \( \Delta \)-convergent to \( z \in F(T) \).

(3) \( \lim_n \alpha_n = 1 \).

Since \( \lim_n d(x_n, z) \) exists and by the lemma 2.3, we obtain
\[
0 = \lim_n (d(x_{n+1}, z) - d(x_n, z)) \\
\leq \liminf_n (\beta_n d(x_n, z) + (1 - \beta_n) d(w_n, z) - d(x_n, z)) \\
= \liminf_n (1 - \beta_n) (d(w_n, z) - d(x_n, z)) \\
\leq \liminf_n (1 - \beta_n) (d(u_n, z) + M_2 \delta_n \eta_n - d(x_n, z)) \\
\leq \limsup_n (1 - \beta_n) (d(u_n, z) + M_2 \delta_n \eta_n - d(x_n, z)) \\
\leq \limsup_n (1 - \beta_n) (\alpha_n d(x_n, z) + (1 - \alpha_n) d(y_n, z) + M_2 \delta_n \eta_n - d(x_n, z)) \\
\leq \limsup_n (1 - \beta_n) (1 - \alpha_n) (d(y_n, z) - d(x_n, z)) + \limsup_n (1 - \beta_n) M_2 \delta_n \eta_n \\
= \limsup_n (1 - \beta_n) M_2 \delta_n \eta_n = 0.
\]

By the assumption \( \limsup_n \beta_n < 1 \), we get
\[
\lim_n (d(u_n, z) - d(x_n, z)) = 0.
\]

In fact, it is easy to prove
\[
\lim_n (d(w_n, z) - d(x_n, z)) = 0,
\]

hence, we may obtain
\[
\lim_n (d(w_n, z) - d(u_n, z)) = 0.
\]
Then, by the assumptions and (3.5), we have

$$\lim_n d(w_n, Tw_n) = 0, \lim_n d(u_n, w_n) = 0.$$ 

Therefore, by the nonexpansiveness of $T$, we may obtain

$$\lim_n d(x_n, Tx_n) \leq \lim_n (d(x_n, u_n) + d(u_n, w_n) + d(w_n, Tw_n) + d(Tw_n, Tx_n))$$

$$\leq \lim_n ((1 - \alpha_n)d(x_n, y_n) + d(u_n, w_n) + d(w_n, Tw_n) + d(w_n, x_n))$$

$$\leq \lim_n (1 - \alpha_n)d(x_n, y_n) + \lim_n d(u_n, w_n) + \lim_n d(w_n, Tw_n)$$

$$+ \limsup_n d(w_n, x_n)$$

$$= \limsup_n d(w_n, x_n).$$

By the lemma 2.4 again, we have

$$d^2(x_{n+1}, z) = d^2(\beta_n x_n + (1 - \beta_n)w_n, z)$$

$$\leq \beta_n d^2(x_n, z) + (1 - \beta_n)d^2(w_n, z) - \beta_n(1 - \beta_n)d^2(x_n, w_n),$$

which implies

$$\beta_n(1 - \beta_n)d^2(x_n, w_n) \leq \beta_n d^2(x_n, z) + (1 - \beta_n)d^2(w_n, z) - d^2(x_{n+1}, z)$$

$$= d^2(x_n, z) - d^2(x_{n+1}, z)$$

$$+ (1 - \beta_n)(d^2(w_n, z) - d^2(x_n, z)).$$

Then, we have

$$\lim_n \beta_n(1 - \beta_n)d^2(x_n, w_n) \leq \lim_n (d^2(x_n, z) - d^2(x_{n+1}, z))$$

$$+ \lim_n (1 - \beta_n)(d^2(w_n, z) - d^2(x_n, z)),$$

which implies

$$0 \leq \lim_n \beta_n(1 - \beta_n)d^2(x_n, w_n) \leq 0,$$

therefore, we obtain

$$\lim_n \beta_n(1 - \beta_n)d^2(x_n, w_n) = 0.$$ 

By the assumption, we know $\limsup_n \beta_n < 1$. Hence, as long as $\limsup_n \beta_n \neq 0$, then there exists a subsequence $\{\beta_{n_j}\}$ of $\{\beta_n\}$ such that $\liminf_j \beta_{n_j} > 0$, we have

$$\lim_j \beta_{n_j}(1 - \beta_{n_j})d^2(x_{n_j}, w_{n_j}) = 0,$$

which implies

$$\lim_j d(x_{n_j}, w_{n_j}) = 0.$$

Thus, there also exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfied

$$\lim_j d(x_{n_j}, Tx_{n_j}) \leq \limsup_j d(w_{n_j}, x_{n_j})$$

$$= \lim_j d(w_{n_j}, x_{n_j}) = 0,$$
which implies
\[ \lim_{j} d(x_{n_j}, Tx_{n_j}) = 0. \]

Thus, if the subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) is \( \Delta - \)convergent to \( q \in X \), then
\[ d(x_{n_j}, Tx_{n_j}) \to 0. \]

Hence, by \( \Delta - \)demiclosedness of nonexpansive mappings, we have \( q \in F(T) \). Finally, by the lemma 2.5, the sequence \( \{x_n\} \) is \( \Delta - \)convergent to \( z \in F(T) \). This completes the proof. \( \square \)

This work is supported by Applied Basic Research Foundation of Sichuan Province of China (Grant No. 2018JY0169).

References


