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SOME PARANORMED SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION OVER N-NORMED SPACES

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Abstract. In this paper we present new classes of sequence spaces using lacunary sequences and a Musielak-Orlicz function over n-normed spaces. We examine some topological properties and prove some interesting inclusion relations between them.

1. Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [5] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [14]. Since then, many others have studied this concept and obtained various results, see Gunawan ([6], [7]) and Gunawan and Mashadi [8]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
(2) $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
(3) $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
(4) $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly

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independent set in X. Then the following function $||\cdot,\cdots,\cdot||_\infty$ on X^{n-1} defined by

$$||x_1,x_2,\cdots,x_{n-1}||_\infty = \max\{||x_1,x_2,\cdots,x_{n-1},a_i|| : i=1,2,\cdots,n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1,a_2,\cdots,a_n\}$.

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to converge to some $L\in X$ if

$$\lim_{k\to\infty} ||x_k-L,z_1,\cdots,z_{n-1}|| = 0 \text{ for every } z_1,\cdots,z_{n-1}\in X.$$

A sequence (x_k) in a n-normed space $(X,||\cdot,\cdots,\cdot||)$ is said to be Cauchy if

$$\lim_{\substack{k\to\infty\\ p\to\infty}} ||x_k-x_p,z_1,\cdots,z_{n-1}|| = 0 \text{ for every } z_1,\cdots,z_{n-1}\in X.$$

If every cauchy sequence in X converges to some $L\in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let X be a linear metric space. A function $p: X\to\mathbb{R}$ is called paranorm, if

(1) $p(x)\geq 0$ for all $x\in X$,
(2) $p(-x)=p(x)$ for all $x\in X$,
(3) $p(x+y)\leq p(x)+p(y)$ for all $x,y\in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n\to\lambda$ as $n\to\infty$ and (x_n) is a sequence of vectors with $p(x_n-x)\to 0$ as $n\to\infty$, then $p(\lambda_n x_n-\lambda x)\to 0$ as $n\to\infty$.

A paranorm p for which $p(x)=0$ implies $x=0$ is called total paranorm and the pair (X,p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19, Theorem 10.4.2, pp. 183]).

For more details about sequence spaces (see [1], [2], [3], [17], [18]) and references therein.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\longrightarrow\infty$ as $x\longrightarrow\infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x=(x_k)$, then

$$\ell_M = \Big\{x\in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho>0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big)\leq 1\Big\}.$$

It is shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(p\geq 1)$. The Δ_2-condition is equivalent to $M(Lx)\leq kLM(x)$ for all values of $x\geq 0$, and for $L>1$. A sequence $\mathcal{M}=(M_k)$ of Orlicz function is called a Musielak-Orlicz function (see [13], [16]). A sequence $\mathcal{N}=(N_k)$ is defined by

$$N_k(v) = \sup\{|v|u-(M_k) : u\geq 0\},\ k=1,2,\ldots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace h_M are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty}(M_k)(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let ℓ_∞, c and c_0 denotes the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [9], it was shown that

$$\hat{c} = \Big\{x = (x_k) : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_{k+s} \text{ exists, uniformly in } s\Big\}.$$

In ([11], [12]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number L such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |x_{k+s} - L| = 0, \text{ uniformly in } s.$$

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, \cdots$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $g_r = (i_r - i_{r-1}) \to \infty$ $(r \to \infty)$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman [4] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{g_r}\sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L\Big\}.$$

Mursaleen and Noman [15] introduced the notion of λ-convergent and λ-bounded sequences as follows :

Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [15] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m} \left(\sum_{k=1}^{m} (\lambda_k - \lambda_{k-1}) |x_k - a| \right) = 0.\right.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m} \sum_{k=1}^{m} (\lambda_k - \lambda_{k-1})(x_k - a)| = 0$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.

Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $w(n - X)$ denotes the space of X-valued sequences. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then we define the following sequence spaces in the present paper:

$[\ c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]^\theta =$

$$\Big\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} = 0,$$

$$\text{for some } \ \rho > 0, L \in X \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\},$$

$[\ c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]_0^\theta =$

$$\Big\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} = 0,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\}$$

and

$[\ c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]_\infty^\theta =$

$$\Big\{ x = (x_k) \in w(n - X) : \sup_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[M_k \Big(|| \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1} || \Big) \Big]^{p_k} < \infty,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\}.$$

When, $\mathcal{M}(x) = x$, we get

$[\ c, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]^\theta =$

$$\Big\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big(|| \frac{\Lambda_k(x) - L}{\rho}, z_1, \cdots, z_{n-1} || \Big)^{p_k} = 0,$$

$$\text{for some } \ \rho > 0, L \in X \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\},$$

$[\ c, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]_0^\theta =$

$$\Big\{ x = (x_k) \in w(n - X) : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big(|| \frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1} || \Big)^{p_k} = 0,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X \Big\}$$

and

$[\ c, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]^\theta_\infty =$

$$\Big\{x = (x_k) \in w(n-X) : \sup_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)^{p_k} < \infty,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X\Big\}.$$

If we take $p = (p_k) = 1$ for all k, then we get
$[\ c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\ ]^\theta =$

$$\Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{for some } \ \rho > 0, L \in X \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$[\ c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\ ]^\theta_0 =$

$$\Big\{x = (x_k) \in w(n-X) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X\Big\}$$

and
$[\ c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\ ]^\theta_\infty =$

$$\Big\{x = (x_k) \in w(n-X) : \sup_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \infty,$$

$$\text{for some } \ \rho > 0 \ \text{ and for every } \ z_1, \cdots, z_{n-1} \in X\Big\}.$$

The following inequality will be used throughout the paper. If $0 \le \inf_k p_k = H_0 \le p_k \le \sup_k = H < \infty$, $K = \max(1, 2^{H-1})$ and $H = \sup_k p_k < \infty$, then

$$|x_k + y_k|^{p_k} \le K(|x_k|^{p_k} + |y_k|^{p_k}), \tag{1.1}$$

for all $k \in \mathbb{N}$ and $x_k, y_k \in \mathbb{C}$. Also $|x_k|^{p_k} \le \max(1, |x_k|^H)$ for all $x_k \in \mathbb{C}$.

2. Some properties of difference sequence spaces

Theorem 2.1. *Let $M = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $[\ c, M, p, \Lambda, ||.,\cdots,\cdot||\ ]^\theta$, $[\ c, M, p, \Lambda, ||\cdot,\cdots,\cdot||\ ]^\theta_0$ and $[\ c, M, p, \Lambda, ||\cdot,\cdots,\cdot||\ ]^\theta_\infty$ are linear spaces over the field of complex numbers C.*

Proof. Let $x = (x_k), y = (y_k) \in [\ c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\ ]^\theta_0$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0,$$

and

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0,.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_k)$ is non-decreasing convex function, by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(\alpha x+\beta y)}{\rho_3}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad = \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\alpha\Lambda_k(x)}{\rho_3}, z_1,\cdots,z_{n-1}|| + \frac{\beta\Lambda_k(y)}{\rho_3}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad \le K\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\quad + K\frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad \le K\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad\quad + K\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_1}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\qquad \longrightarrow 0 \text{ as } r\longrightarrow\infty.
\end{aligned}$$

Thus, we have $\alpha x+\beta y \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot,\cdots,\cdot||\,]_0^\theta$. Hence $[\, c, \mathcal{M}, p, \Lambda, ||\cdot,\cdots,\cdot||\,]_0^\theta$ is a linear space. Similarly, we can prove that $[\, c, \mathcal{M}, p, \Lambda, ||\cdot,\cdots,\cdot||\,]^\theta$ and $[\, c, \mathcal{M}, p, \Lambda, ||\cdot,\cdots,\cdot||\,]_\infty^\theta$ are linear spaces. □

Theorem 2.2. *For any Musielak-Orlicz function $M = (M_k)$ and a bounded sequence $p = (p_k)$ of positive real numbers, $[\, c, M, p, \Lambda, ||\cdot,\cdots,\cdot||\,]_0^\theta$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k(||\frac{\Lambda_k(x)}{\rho}, z_1,\cdots,z_{n-1}||)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r\in\mathbb{N}\Big\},$$

where $H = \max(1, \sup_k p_k < \infty)$.

Proof. Clearly $g(x)\ge 0$ for $x = (x_k)\in [\, c, \mathcal{M}, p, \Lambda, ||\cdot,\cdots,\cdot||\,]_0^\theta$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$\inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1, r\in\mathbb{N}\Big\} = 0.$$

This implies that for a given $\epsilon > 0$, there exists some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_\epsilon}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1.$$

Thus

$$\begin{aligned}
\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\epsilon}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} &\le \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_\epsilon}, z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\\
&\le 1,
\end{aligned}$$

for each r. Suppose that $x \neq 0$ for each $k \in N$. This implies that $\Lambda_k(x) \neq 0$, for each $k \in N$. Let $\epsilon \longrightarrow 0$, then $||\frac{\Lambda_k(x)}{\epsilon}, z_1, \cdots, z_{n-1}|| \longrightarrow \infty$. It follows that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\epsilon}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \longrightarrow \infty,$$

which is a contradiction. Therefore, $\Lambda_k(x) = 0$ for each k and thus $x = 0$ for each $k \in N$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

and

$$\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1$$

for each r. Let $\rho = \rho_1 + \rho_2$. Then, by Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x+y)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\\
&\leq \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)+\Lambda_k(y)}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\\
&\leq \Big(\sum_{k\in I_r}\Big[\frac{\rho_1}{\rho_1+\rho_2}M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\\
&\quad + \frac{\rho_2}{\rho_1+\rho_2}M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p^k}\Big)^{\frac{1}{H}}\\
&\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\\
&\quad + \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}}\\
&\leq 1
\end{aligned}$$

Since $\rho's$ are non-negative, so we have

$$\begin{aligned}
g(x+y) &= \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x+y)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r\in\mathbb{N}\Big\},\\
&\leq \inf\Big\{\rho_1^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\ \Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r\in\mathbb{N}\Big\}\\
&\quad + \inf\Big\{\rho_2^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(y)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r\in\mathbb{N}\Big\}.
\end{aligned}$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number. By definition,

$$g(\mu x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(\mu x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r \in \mathbb{N}\Big\}.$$

Then

$$g(\mu x) = \inf\Big\{(|\mu|t)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r \in \mathbb{N}\Big\},$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_r} \leq \max(1, |\mu|^{\sup p_r})$, we have

$$g(\mu x) \leq \max(1, |\mu|^{\sup p_r}) \inf\Big\{t^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{t}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1, r \in \mathbb{N}\Big\}.$$

So, the fact that scalar multiplication is continuous follows from the above inequality.

This completes the proof of the theorem. □

Theorem 2.3. *Let $M = (M_k)$ be a Musielak-Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then $[\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta \subset [\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta$.*

Proof. Let $x = (x_k) \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta$. There exists some positive ρ_1 such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Define $\rho = 2\rho_1$. Since $\mathcal{M} = (M_k)$ is non-decreasing and convex, by using inequality (1.1), we have

$$\begin{aligned}
&\sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&= \sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x) - L + L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\leq K \sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[\frac{1}{2^{p_k}}M_k\Big(||\frac{\Lambda_k(x) - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&+ K \sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[\frac{1}{2^{p_k}}M_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&\leq K \sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x) - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&+ K \sup_r \frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\\
&< \infty.
\end{aligned}$$

Hence $x = (x_k) \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta$. □

Theorem 2.4. *Let $0 < \inf p_k = g \leq p_k \leq \sup p_k = H < \infty$ and $M = (M_k)$, $M' = (M'_k)$ are Musielak-Orlicz functions satisfying Δ_2-condition, then we have*

(i) $[\, c, M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta \subset [\, c, M \circ M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta$,
(ii) $[\, c, M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta \subset [\, c, M \circ M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]_0^\theta$,
(iii) $[\, c, M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta \subset [\, c, M \circ M', p, \Lambda, ||\cdot, \cdots, \cdot||\,]_\infty^\theta$.

Proof. Let $x=(x_k)\in[c,\mathcal{M}',p,\Lambda,||\cdot,\cdots,\cdot||]^{\theta}$. Then we have

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M'_k\Big(||\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}=0,\ \text{ for some }\ L.$$

Let $\epsilon>0$ and choose δ with $0<\delta<1$ such that $M_k(t)<\epsilon$ for $0\le t\le\delta$. Let

$$y_k=M'_k\Big(||\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\ \text{ for all }\ k\in\mathbb{N}.$$

We can write

$$\frac{1}{h_r}\sum_{k\in I_r}[M_k(y_k)]^{p_k}=\frac{1}{h_r}\sum_{k\in I_r,y_{k\le\delta}}[M_k(y_k)]^{p_k}+\frac{1}{h_r}\sum_{k\in I_r,y_{k>\delta}}[M_k(y_k)]^{p_k}.$$

Since $\mathcal{M}=(M_k)$ satisfies Δ_2-condition, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r,y_k\le\delta}[M_k(y_k)]^{p_k}&\le[M_k(1)]^H\frac{1}{h_r}\sum_{k\in I_r,y_k\le\delta}[M_k(y_k)]^{p_k}\\ &\le[M_k(2)]^H\frac{1}{h_r}\sum_{k\in I_r,y_k\le\delta}[M_k(y_k)]^{p_k}\qquad(2.1)\end{aligned}$$

For $y_k>\delta$

$$y_k<\frac{y_k}{\delta}<1+\frac{y_k}{\delta}.$$

Since $\mathcal{M}=(M_k)$ is non-decreasing and convex, it follows that

$$M_k(y_k)<M_k\Big(1+\frac{y_k}{\delta}\Big)<\frac{1}{2}M_k(2)+\frac{1}{2}M_k\Big(\frac{2y_k}{\delta}\Big).$$

Since (M_k) satisfies Δ_2-condition, we can write

$$\begin{aligned}M_k(y_k)&<\frac{1}{2}T\frac{y_k}{\delta}M_k(2)+\frac{1}{2}T\frac{y_k}{\delta}M_k(2)\\ &=T\frac{y_k}{\delta}M_k(2).\end{aligned}$$

Hence,

$$\frac{1}{h_r}\sum_{k\in I_r,y_k>\delta}[M_k(y_k)]^{p_k}\le\max\Big(1,\Big(\frac{TM_k(2)}{\delta}\Big)^H\Big)\frac{1}{h_r}\sum_{k\in I_r,y_k>\delta}[(y_k)]^{p_k}\qquad(2.2)$$

from equations (2.1) and (2.2), we have

$$x=(x_k)\in[\,c,\mathcal{M}\circ\mathcal{M}',p,\Lambda,||\cdot,\cdots,\cdot||]^{\theta}.$$

This completes the proof of (i). Similarly, we can prove that

$$[\,c,\mathcal{M}'^{\theta}_0\subset[\,c,\mathcal{M}\circ\mathcal{M}'^{\theta}_0$$

and

$$[\,c,\mathcal{M}'^{\theta}_\infty\subset[\,c,\mathcal{M}\circ\mathcal{M}',p,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}_\infty.$$

□

Corollary 2.1. *Let $0<\inf p_k=h\le p_k\le\sup p_k=H<\infty$ and $M=(M_k)$ be a Musielak-Orlicz function satisfying Δ_2-condition, then we have*

$$[\,c,\mathcal{M}',p,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}_0\subset[\,c,\mathcal{M},p,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}_0$$

and

$$[\,c,\mathcal{M}',p,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}_\infty\subset[\,c,\mathcal{M},p,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}_\infty.$$

Proof. Taking $\mathcal{M}'(x) = x$ in Theorem 2.4, we get the required result. □

Theorem 2.5. *Let $M = (M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*
(i) $[\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty \subset [c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$,
(ii) $[c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$,
(iii) $\sup_r \frac{1}{h_r} \sum_{k\in I_r} [M_k(\frac{t}{\rho})]^{p_k} < \infty \;\; (t, \rho > 0)$.

Proof. (i) ⇒ (ii) The proof is obvious in view of the fact that $[\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$.
(ii) ⇒(iii) Let $[\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$. Suppose that (iii) does not hold. Then for some $t,\ \rho > 0$

$$\sup_r \frac{1}{h_r} \sum_{k\in I_r} [M_k(\frac{t}{\rho})]^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(j)}} \sum_{k\in I_{r(j)}} \Big[M_k\Big(\frac{j^{-1}}{\rho}\Big)\Big]^{p_k} > j, j = 1, 2, \tag{2.3}$$

Define the sequence $x = (x_k)$ by

$$\Lambda_k(x) = \begin{cases} j^{-1}, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases} \quad \text{for all } \; s \in \mathbb{N}.$$

Then $x = (x_k) \in [c, p, \Lambda, ||\cdot, \cdots, \cdot||]^\theta_0$ but by equation (2.3), $x = (x_k) \notin [c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$, which contradicts (ii). Hence (iii) must hold.
(iii) ⇒ (i) Let (iii) hold and $x = (x_k) \in [c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$. Suppose that $x = (x_k) \notin [c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$. Then

$$\sup_r \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = \infty \tag{2.4}$$

Let $t = ||\Lambda_k(x), z_1, \cdots, z_{n-1}||$ for each k, then by equations (2.4)

$$\sup_r \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big] = \infty,$$

which contradicts (iii). Hence (i) must hold. □

Theorem 2.6. *Let $1 \le p_k \le \sup p_k < \infty$ and $M = (M_k)$ be a Musielak Orlicz function. Then the following statements are equivalent:*
(i) $[\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [c, p, \Lambda, ||\cdot, \cdots, \cdot||]^\theta_0$,
(ii) $[\, c, M, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_0 \subset [\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^\theta_\infty$,
(iii) $\inf_r \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big]^{p_k} > 0 \;\; (t, \rho > 0)$.

Proof. (i) ⇒ (ii) It is trivial.
(ii) ⇒ (iii) Let (ii) hold. Suppose that (iii) does not hold. Then

$$\inf_r \frac{1}{h_r} \sum_{k\in I_r} \Big[M_k\Big(\frac{t}{\rho}\Big)\Big]^{p_k} = 0 \;\; (t, \rho > 0),$$

so we can find a subinterval $I_{r(j)}$ of the set of interval I_r such that

$$\frac{1}{h_{r(j)}}\sum_{k\in I_{r(j)}}\Big[M_k\Big(\frac{j}{\rho}\Big)\Big]^{p_k} < j^{-1},\ \ j=1,2,. \tag{2.5}$$

Define the sequence $x=(x_k)$ by

$$\Lambda_k(x)=\begin{cases} j, & k\in I_{r(j)}\\ 0, & k\notin I_{r(j)}\end{cases} \text{ for all } s\in\mathbb{N}.$$

Thus by equation (2.5), $x=(x_k)\in[\,c,\mathcal{M},p,\Lambda,||\cdot,\cdots,\cdot||]_0^{\theta}$, but by equation (2.3), $x=(x_k)\notin[c,p,\Lambda,||\cdot,\cdots,\cdot||\,]_{\infty}^{\theta}$, which contradicts (ii). Hence (iii) must hold.

(iii) $\Rightarrow$ (i) Let (iii) hold and suppose that $x=(x_k)\in[\,c,\mathcal{M},p,\Lambda,||\cdot,\cdots,\cdot||\,]_0^{\theta}$, i.e,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}=0,\ \text{ for some } \rho>0. \tag{2.6}$$

Again, suppose that $x=(x_k)\notin[\,c,p,\Lambda,||\cdot,\cdots,\cdot||\,]_0^{\theta}$. Then, for some number $\epsilon>0$ and a subinterval $I_{r(j)}$ of the set of interval I_r, we have $||\Lambda_k(x),z_1,\cdots,z_{n-1}||\geq\epsilon$ for all $k\in\mathbb{N}$ and some $s\geq s_0$. Then, from the properties of the Orlicz function, we can write

$$M_k\Big(||\frac{\Lambda_k(x)}{\rho},z_1,\cdots,z_{n-1}||\Big)_k^{p}\geq M_k\Big(\frac{\epsilon}{\rho}\Big)^{p_k}$$

and consequently by (2.6)

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(\frac{\epsilon}{\rho}\Big)\Big]^{p_k}=0,$$

which contradicts (iii). Hence (i) must hold. □

Theorem 2.7. *Let $0<p_k\leq q_k$ for all $k\in N$ and $\left(\frac{q_k}{p_k}\right)$ be bounded. Then,* $[\,c,M,q,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}\subset[\,c,M,p,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}$.

Proof. Let $x\in[\,c,\mathcal{M},q,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}$. Write

$$t_k=\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{q_k}$$

and $\mu_k=\frac{p_k}{q_k}$ for all $k\in\mathbb{N}$. Then $0<\mu_k\leq 1\ \ for\ \ k\in\mathbb{N}$. Take $0<\mu<\mu_k$ for $k\in\mathbb{N}$. Define the sequences (u_k) and (v_k) as follows: For $t_k\geq 1$, let $u_k=t_k$ and $v_k=0$ and for $t_k<1$, let $u_k=0$ and $v_k=t_k$. Then clearly for all $k\in\mathbb{N}$, we have

$$t_k=u_k+v_k,\qquad t_k^{\mu_k}=u_k^{\mu_k}+v_k^{\mu_k}$$

Now it follows that $u_k^{\mu_k}\leq u_k\leq t_k$ and $v_k^{\mu_k}\leq v_k^{\mu}$. Therefore,

$$\frac{1}{h_r}\sum_{k\in I_r}t_k^{\mu_k}=\frac{1}{g_h}\sum_{k\in I_r}(u_k^{\mu_k}+v_k^{\mu_k})\leq\frac{1}{h_r}\sum_{k\in I_r}t_k+\frac{1}{h_r}\sum_{k\in I_r}v_k^{\mu}.$$

Now for each k,

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu} &= \sum_{k\in I_r}\Big(\frac{1}{h_r}v_k\Big)^{\mu}\Big(\frac{1}{h_r}\Big)^{1-\mu}\\ &\le \Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}v_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu}\\ &= \Big(\frac{1}{h_r}\sum_{k\in I_r} v_k\Big)^{\mu}\end{aligned}$$

and so

$$\frac{1}{h_r}\sum_{k\in I_r} t_k^{\mu_k} \le \frac{1}{h_r}\sum_{k\in I_r} t_k + \Big(\frac{1}{h_r}\sum_{k\in I_r} v_k\Big)^{\mu}.$$

Hence $x \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}$. □

Theorem 2.8. *(a) If* $0 < \inf p_k \le p_k \le 1$ *for all* $k \in N$, *then*

$$[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta} \subset [\, c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}.$$

(b) If $1 \le p_k \le \sup p_k < \infty$ *for all* $k \in N$. *Then*

$$[\, c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta} \subset [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}.$$

Proof. (a) Let $x \in [\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}$, then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0.$$

Since $0 < \inf p_k \le p_k \le 1$. This implies that

$$\begin{aligned}&\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]\\ &\qquad\le \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k},\end{aligned}$$

therefore, $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] = 0.$

This shows that $x \in [\, c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}\cdot$ Therefore,

$$[\, c, \mathcal{M}, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta} \subset [\, c, \mathcal{M}, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}.$$

This completes the proof.

(b) Let $p_k \ge 1$ for each k and $\sup p_k < \infty$. Let $x \in [\, c, p, \Lambda, ||\cdot, \cdots, \cdot||\,]^{\theta}$. Then for each $\epsilon > 0$ there exists a positive integer N such that

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} = 0 < 1.$$

Since $1 \le p_k \le \sup p_k < \infty$, we have

$$\begin{aligned}\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\ &\le \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\Big[M_k\Big(||\frac{\Lambda_k(x)-L}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]\\ &=0\\ &<1.\end{aligned}$$

Therefore $x\in[\,c,\mathcal{M},p,\Lambda,||\cdot,\cdots,\cdot||\,]^{\theta}$. □

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