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AN INEQUALITY OF GRÜSS LIKE VIA VARIANT OF POMPEIU'S MEAN VALUE THEOREM

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Abstract. The main of this paper is to establish an integral inequality of Grüss type by using a mean value theorem.

1. Introduction

In 1935, G. Grüss [4] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\le\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma),$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi\le f(x)\le\Phi \text{ and } \gamma\le g(x)\le\Gamma \text{ for all } x\in[a,b].$$

The constant $\frac{1}{4}$ is best possible.

In 1882, P. L. Čebyšev [2] gave the following inequality:

$$|T(f,g)|\le\frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty,$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty=ess\sup_{t\in[a,b]}|p(t)|$.

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For a differentiable function $f:[a,b]\to\mathbb{R},\ a{\cdot}b>0$, Pachpatte has in [6] proved, using Pompeiu's mean value theorem [9], the following Grüss type inequality:

$$\left|\int_a^b f(t)g(t)dt-\frac{1}{b^2-a^2}\left(\int_a^b f(t)dt.\int_a^b tg(t)dt+\int_a^b g(t)dt.\int_a^b tf(t)dt\right)\right|$$
$$\leq\ \left\|f-\ell f'\right\|_\infty\int_a^b|g(t)|\left|\frac{1}{2}-\frac{t}{a+b}\right|dt+\left\|g-\ell g'\right\|_\infty\int_a^b|f(t)|\left|\frac{1}{2}-\frac{t}{a+b}\right|dt$$

where $\ell(t)=t,\ t\in[a,b]$.

In [7], Pecaric and Ungar proved a general estimate with the p-norm, $1<p<\infty$, which will for $p=\infty$ give the Pachpatte [6] result.

The interested reader is also referred to ([1], [3], [5]-[11]) for integral inequalities by using Pompeiu's mean value theorem. In this paper, we establish a new integral inequality of Grüss like via Pompeiu's mean value theorem.

2. Main Results

Before starting the main results, we will give the following lemma proved by Pecaric and Ungar in [7]:

Lemma 2.1. *For* $\frac{1}{p}+\frac{1}{q}=1,\ 1\leq p,q\leq\infty,$ *and* $0<a\leq x\leq b,$ *denote*

$$A(x,q):=\left(\int_a^x\left(\int_t^x\frac{t^q du}{u^{2q}}\right)dt\right)^{\frac{1}{q}}+\left(\int_x^b\left(\int_x^t\frac{t^q du}{u^{2q}}\right)dt\right)^{\frac{1}{q}}\tag{2.1}$$

where for $p=1$, *i.e.* $q=\infty$, *the integrals are to be interpreted as the* ∞*-norms, i.e. as maxima of the function* $(u,t)\mapsto\frac{1}{u^2}$ *on the corresponding domains of integration. Then,*

$$\begin{aligned}A(x,q) &= \left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}\\ &\quad+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}},\end{aligned}$$

for $1<p,q<\infty,\ p,q\neq 2$;

$$A(x,2)=\frac{1}{3}\left[\left(\ln\left(\frac{x}{a}\right)^3+\frac{a^3}{x^3}-1\right)^{\frac{1}{2}}+\left(\ln\left(\frac{x}{b}\right)^3+\frac{b^3}{x^3}-1\right)^{\frac{1}{2}}\right]=\lim_{q\to 2}A(x,q);$$

$$A(x,\infty)=\frac{a^2+b^2}{2x}+x-a-b=\lim_{q\to\infty}A(x,q);$$

$$A(x,1)=\frac{1}{a}+\frac{b}{x^2}=\lim_{q\to 1}A(x,1).$$

To prove our theorems, we need the following lemma proved by Sarikaya in [12]:

Lemma 2.2. *$f:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$. Then for any $t,x\in[a,b]$, we have*

$$tf(x)-xf(t)+xt\frac{f'(t)-f'(x)}{2}=\frac{xt}{2}\int\limits_x^t\left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du. \tag{2.2}$$

Theorem 2.1. *$f,g:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$. Then for $\frac{1}{p}+\frac{1}{q}=1$, with $1<p,q<\infty$ any $t,x\in[a,b]$, we have*

$$\begin{aligned} &\left|\frac{1}{b-a}\int\limits_a^b f(x)g(x)dx\right.\\ &-\left[\frac{3}{b-a}\int f(t)dt-\frac{bf(b)-af(a)}{b-a}\right]\left(\frac{2}{5\left(b^2-a^2\right)}\int\limits_a^b xg(x)dx\right)\\ &-\left[\frac{3}{b-a}\int g(t)dt-\frac{bg(b)-ag(a)}{b-a}\right]\left(\frac{2}{5\left(b^2-a^2\right)}\int\limits_a^b xf(x)dx\right)\\ &\left.-\frac{bf(b)g(b)-af(a)g(a)}{5(b-a)}\right|\\ \leq\ &\frac{2(b-a)^{\frac{1}{p}-2}}{5\left(b+a\right)}\left\{\left\|2f-2lf'+l^2f''\right\|_p\int\limits_a^b xg(x)A(x,q)dx\right.\\ &\left.+\left\|2g-2lg'+l^2g''\right\|_p\int\limits_a^b xf(x)A(x,q)dx\right\} \end{aligned} \tag{2.3}$$

where $l(t)=t$ for $t\in[a,b]$.

Proof. Applying (2.2) to the function g, we have

$$tg(x)-xg(t)+xt\frac{g'(t)-g'(x)}{2}=\frac{xt}{2}\int\limits_x^t\left[2g(u)-2ug'(u)+u^2g''(u)\right]\frac{1}{u^2}du. \tag{2.4}$$

Multiplying (2.2) by $g(x)$, (2.4) by $f(x)$,summing the resultant equalities, then integrating with respect to t on $[a,b]$, we have

(2.5)
$$\begin{aligned}
&\left(b^2-a^2\right) f(x)g(x)-\frac{3xg(x)}{2}\int_a^b f(t)dt-\frac{3xf(x)}{2}\int_a^b g(t)dt+\frac{xg(x)}{2}\left[bf(b)-af(a)\right]\\
&-\frac{b^2-a^2}{4}xg(x)f'(x)+\frac{xf(x)}{2}\left[bg(b)-ag(a)\right]-\frac{b^2-a^2}{4}xf(x)g'(x)\\
=\;&\frac{xg(x)}{2}\int_a^b t\left[\int_x^t \left[2f(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du\right]dt\\
&+\frac{xf(x)}{2}\int_a^b t\left[\int_x^t \left[2g(u)-2ug'(u)+u^2g''(u)\right]\frac{1}{u^2}du\right]dt.
\end{aligned}$$

Integrating with respect to x on $[a,b]$ and adding notations $F(u)=2f(u)-2uf'(u)+u^2f''(u)$ and $G(u)=2g(u)-2ug'(u)+u^2g''(u)$, we obtain

(2.6)
$$\begin{aligned}
&\left(b^2-a^2\right)\int_a^b f(x)g(x)dx\\
&-\frac{3}{2}\left(\int_a^b xg(x)dx\right)\left(\int_a^b f(t)dt\right)-\frac{3}{2}\left(\int_a^b xf(x)dx\right)\left(\int_a^b g(t)dt\right)\\
&+\frac{bf(b)-af(a)}{2}\left(\int_a^b xg(x)dx\right)+\frac{bg(b)-ag(a)}{2}\left(\int_a^b xf(x)dx\right)\\
&-\frac{b^2-a^2}{4}\int_a^b xg(x)f'(x)dx-\frac{b^2-a^2}{4}\int_a^b xf(x)g'(x)dx\\
=\;&\frac{1}{2}\int_a^b xg(x)\left(\int_a^b t\left[\int_x^t F(u)\frac{du}{u^2}\right]dt\right)dx+\frac{1}{2}\int_a^b xf(x)\left(\int_a^b t\left[\int_x^t G(u)\frac{du}{u^2}\right]dt\right)dx.
\end{aligned}$$

$$\int_a^b xg(x)f'(x)dx=bf(b)g(b)-af(a)g(a)-\int_a^b f(x)g(x)dx-\int_a^b xf(x)g'(x)dx. \tag{2.7}$$

Adding (2.7) in (2.6), we have

$$\begin{aligned}
&\frac{5\left(b^2-a^2\right)}{4}\int_a^b f(x)g(x)dx \\
&-\left[\frac{3}{2}\int_a^b f(t)dt-\frac{bf(b)-af(a)}{2}\right]\left(\int_a^b xg(x)dx\right) \\
&-\left[\frac{3}{2}\int_a^b g(t)dt-\frac{bg(b)-ag(a)}{2}\right]\left(\int_a^b xf(x)dx\right) \\
&-\frac{\left(b^2-a^2\right)}{4}\left[bf(b)g(b)-af(a)g(a)\right] \\
=\;&\frac{1}{2}\int_a^b xg(x)\left(\int_a^b t\left[\int_x^t F(u)\frac{du}{u^2}\right]dt\right)dx+\frac{1}{2}\int_a^b xf(x)\left(\int_a^b t\left[\int_x^t G(u)\frac{du}{u^2}\right]dt\right)dx.
\end{aligned}$$

Taking modulus, we have

$$\begin{aligned}
&(2.8) \\
&\left|\frac{5\left(b^2-a^2\right)}{4}\int_a^b f(x)g(x)dx\right. \\
&-\left[\frac{3}{2}\int_a^b f(t)dt-\frac{bf(b)-af(a)}{2}\right]\left(\int_a^b xg(x)dx\right) \\
&-\left[\frac{3}{2}\int_a^b g(t)dt-\frac{bg(b)-ag(a)}{2}\right]\left(\int_a^b xf(x)dx\right) \\
&\left.-\frac{\left(b^2-a^2\right)}{4}\left[bf(b)g(b)-af(a)g(a)\right]\right| \\
\le\;&\frac{1}{2}\left|\int_a^b xg(x)\left(\int_a^b t\left[\int_x^t F(u)\frac{du}{u^2}\right]dt\right)dx\right|+\frac{1}{2}\left|\int_a^b xf(x)\left(\int_a^b t\left[\int_x^t G(u)\frac{du}{u^2}\right]dt\right)\right|dx \\
\le\;&\frac{1}{2}\int_a^b |xg(x)|\left|\int_a^b t\left[\int_x^t F(u)\frac{du}{u^2}\right]dt\right|dx+\frac{1}{2}\int_a^b |xf(x)|\left|\int_a^b t\left[\int_x^t G(u)\frac{du}{u^2}\right]dt\right|dx \\
\le\;&\frac{1}{2}\int_a^b |xg(x)|\left(\int_a^b\left|\int_x^t |F(u)|\,\frac{t}{u^2}du\right|dt\right)dx+\frac{1}{2}\int_a^b |xf(x)|\left(\int_a^b\left|\int_x^t |G(u)|\,\frac{t}{u^2}du\right|dt\right)dx.
\end{aligned}$$

In the last line (2.8), we have

$$\int_a^b\left|\int_x^t |F(u)|\,\frac{t}{u^2}du\right|dt=\int_a^x\int_t^x |F(u)|\,\frac{t}{u^2}dudt+\int_x^b\int_x^t |F(u)|\,\frac{t}{u^2}dudt. \tag{2.9}$$

Using Hölder's inequality in (2.9), we obtain

$$
\begin{aligned}
(2.10)\qquad & \int_a^b \left| \int_x^t |F(u)| \frac{t}{u^2} du \right| dt \\
\leq\ & \left(\int_a^x \int_t^x |F(u)|^p \, dudt \right)^{\frac{1}{p}} \left(\int_a^x \int_t^x \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \\
& + \left(\int_x^b \int_x^t |F(u)|^p \, dudt \right)^{\frac{1}{p}} \left(\int_x^b \int_x^t \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \\
\leq\ & \left(\int_a^b \int_a^b |F(u)|^p \, dudt \right)^{\frac{1}{p}} \left\{ \left(\int_a^x \int_t^x \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} + \left(\int_x^b \int_x^t \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \right\} \\
=\ & (b-a)^{\frac{1}{p}} \left\| 2f - 2lf' + l^2 f'' \right\|_p A(x,q).
\end{aligned}
$$

Similarly, we get

$$
(2.11)\qquad \int_a^b \left| \int_x^t |G(u)| \frac{t}{u^2} du \right| dt \leq (b-a)^{\frac{1}{p}} \left\| 2g - 2lg' + l^2 g'' \right\|_p A(x,q).
$$

Adding (2.10) and (2.11) in (2.8), we obtain

$$
\begin{aligned}
(2.12)\qquad & \left| \frac{5\left(b^2-a^2\right)}{4} \int_a^b f(x)g(x)dx \right. \\
& - \left[\frac{3}{2} \int_a^b f(t)dt - \frac{bf(b)-af(a)}{2} \right] \left(\int_a^b xg(x)dx \right) \\
& - \left[\frac{3}{2} \int_a^b g(t)dt - \frac{bg(b)-ag(a)}{2} \right] \left(\int_a^b xf(x)dx \right) \\
& \left. - \frac{\left(b^2-a^2\right)}{4} \left[bf(b)g(b) - af(a)g(a) \right] \right| \\
\leq\ & \frac{1}{2} (b-a)^{\frac{1}{p}} \left\{ \left\| 2f - 2lf' + l^2 f'' \right\|_p \int_a^b |xg(x)| \, A(x,q)dx \right. \\
& \left. + \left\| 2g - 2lg' + l^2 g'' \right\|_p \int_a^b |xf(x)| \, A(x,q)dx \right\}.
\end{aligned}
$$

Dividing (2.12) by $\frac{5\left(b^2-a^2\right)(b-a)}{4}$, we obtain the required inequality (2.3). □

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