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ON SOME INEQUALITIES FOR THE EXPECTATION AND VARIANCE

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Abstract. Some elementary inequalities for the expectation and variance of a continuous random variable whose probability density function is defined on a finite interval are obtained by using an identity due to P. Cerone for the Chebyshev functional and some standard results from the theory of inequalities. Thus some mistakes in the literatures are corrected.

1. INTRODUCTION

Let X be a continuous random variable having the probability density function f defined on a finite interval $[a, b]$.

By definition

$$E(X) := \int_a^b tf(t)\,dt \tag{1.1}$$

the expectation of X, and

$$\begin{aligned}\sigma^2(X) &:= \int_a^b [t - E(X)]^2 f(t)\,dt \\ &= \int_a^b t^2 f(t)\,dt - [E(X)]^2\end{aligned} \tag{1.2}$$

the variance of X.

For two integral functions $f, g : [a, b] \to \mathbf{R}$, define the Chebyshev functional

$$T(f, g) := \frac{1}{b-a}\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1.3}$$

In [1], P. Cerone has obtained the following identity that involves a Riemann-Stieltjes integral:

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Lemma 1.1. *Let $f, g : [a,b] \to \mathbf{R}$ be such that f is of bounded variation on $[a,b]$ and g is continuous on $[a,b]$. Then*

$$T(f,g) = \frac{1}{(b-a)^2} \int_a^b \Psi(t)\, df(t), \tag{1.4}$$

where

$$\Psi(t) := (t-a)A(t,b) - (b-t)A(a,t), \tag{1.5}$$

with

$$A(c,d) := \int_c^d g(x)\, dx. \tag{1.6}$$

In [1] we can also find the following useful result:

Lemma 1.2. *Let $f, g : [a,b] \to \mathbf{R}$ be such that f is of bounded variation and g is continuous on $[a,b]$. Then*

(1.7)
$$(b-a)^2|T(f,g)| \le \begin{cases} \sup_{t\in[a,b]}|\Psi(t)|\bigvee_a^b(f), & \\ L\int_a^b |\Psi(t)|\, dt, & \text{for } f \text{ } L\text{-Lipschitzian}, \\ \int_a^b |\Psi(t)|\, df(t), & \text{for } f \text{ monotonic nondecreasing,} \end{cases}$$

where $\bigvee_a^b(f)$ is the total variation of f on $[a,b]$.

The purpose of this paper is to derive some elementary inequalities for the expectation (1.1) and variance (1.2) by using Lemma 1.1 and Lemma 1.2. Thus some mistakes in [1] and [2] are corrected.

2. INEQUALITIES FOR THE EXPECTATION

We prove the following theorem by using the Lemma 1.1.

Theorem 2.1. *Let $f : [a,b] \to \mathbf{R}_+$ be an absolutely continuous probability density function associated with a random variable X, then the expectation $E(X)$ satisfies the inequalities*

$$\begin{aligned} &\left|E(X) - \tfrac{a+b}{2}\right| \\ &\le \begin{cases} \frac{(b-a)^3}{12}\|f'\|_\infty, & f' \in L_\infty[a,b]; \\ \frac{1}{2}(b-a)^{2+\frac{1}{q}}[B(q+1,q+1)]^{\frac{1}{q}}\|f'\|_p, & f' \in L_p[a,b],\ p>1, \\ & \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{8}\|f'\|_1, & f' \in L_1[a,b]. \end{cases} \end{aligned} \tag{2.1}$$

where $\|\cdot\|_p, 1 \le p \le \infty$ are the usual Lebesgue norms on $[a,b]$, i.e.,

$$\|g\|_p := \begin{cases} [\int_a^b |g(t)|^p\, dt]^{\frac{1}{p}}, & 1 \le p < \infty, \\ ess\sup_{t\in[a,b]} |g(t)|, & p = \infty. \end{cases} \tag{2.2}$$

Proof. Notice that $\int_a^b f(t)\,dt = 1$ and f is absolutely continuous on $[a,b]$, by (1.3) and (1.4)-(1.6) we get

$$E(X) - \frac{a+b}{2} = (b-a)T(t, f(t)) = \frac{1}{2}\int_a^b (t-a)(b-t)f'(t)\,dt,$$

and so

$$|E(X) - \frac{a+b}{2}| \le \frac{1}{2}\int_a^b (t-a)(b-t)|f'(t)|\,dt.$$

Using the Hölder's integral inequality, we have

$$\int_a^b (t-a)(b-t)f'(t)\,dt \le \begin{cases} \frac{1}{2}\|f'\|_\infty \int_a^b (t-a)(b-t)\,dt, & f' \in L_\infty[a,b]; \\ \frac{1}{2}\|f'\|_p [\int_a^b |(t-a)(b-t)|^q\,dt]^{\frac{1}{q}}, & f' \in L_p[a,b], \\ & p>1, \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|f'\|_1 \sup_{t\in[a,b]}(t-a)(b-t), & f' \in L_1[a,b]. \end{cases}$$

Clearly,

$$\int_a^b (t-a)(b-t)\,dt = \frac{(b-a)^3}{6},$$

$$\sup_{t\in[a,b]}(t-a)(b-t) = \frac{(b-a)^2}{4},$$

and it is easy to find by substitution $u = a + (b-a)t$ that

$$\int_a^b [(t-a)(b-t)]^q\,dt = (b-a)^{2q+1}\int_0^1 u^q(1-u)^q\,du = (b-a)^{2q+1}B(q+1, q+1).$$

Thus we have proved the inequalities (2.1).

Remark 2.1. The inequalities (2.1) provide a correction of the inequalities (3.22) in [2].

Theorem 2.2. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X. Then the expectation $E(X)$ satisfies the inequalities*

$$(2.3)\quad |E(X) - \frac{a+b}{2}| \le \begin{cases} \frac{(b-a)^2}{8}\bigvee_a^b(f), & \textit{for } f \textit{ of bounded variation,} \\ \frac{(b-a)^3}{12}L, & \textit{for } f \textit{ L-Lipschitzian,} \\ \frac{(b-a)^2}{8}[f(b)-f(a)], & \textit{for } f \textit{ monotonic nondecreasing.} \end{cases}$$

Proof. Notice that $\int_a^b f(t)\,dt = 1$, by (1.3), (1.4) and (1.6) we get

$$E(X) - \frac{a+b}{2} = (b-a)T(t, f(t)) = \frac{1}{2}\int_a^b (t-a)(b-t)\,df(t),$$

and so it follows from Lemma 1.2,

$$|E(X) - \frac{a+b}{2}| \le \begin{cases} \frac{1}{2}\sup_{t\in[a,b]}(t-a)(b-t)\bigvee_a^b(f), & \text{for } f \text{ of bounded variation,} \\ \frac{L}{2}\int_a^b (t-a)(b-t)\,dt, & \text{for } f\ L\text{-Lipschitzian,} \\ \frac{1}{2}\int_a^b (t-a)(b-t)\,df(t), & \text{for } f \text{ monotonic nondecreasing.} \end{cases}$$

We need only to calculate and estimate that

$$\begin{aligned}\int_a^b (t-a)(b-t)\,df(t) &= (t-a)(b-t)f(t)|_a^b + 2\int_a^b (t-\tfrac{a+b}{2})f(t)\,dt \\ &= 2[\int_a^{\frac{a+b}{2}} (t-\tfrac{a+b}{2})f(t)\,dt + \int_{\frac{a+b}{2}}^b (t-\tfrac{a+b}{2})f(t)\,dt] \\ &\le 2f(a)\int_a^{\frac{a+b}{2}} (t-\tfrac{a+b}{2})\,dt + 2f(b)\int_{\frac{a+b}{2}}^b (t-\tfrac{a+b}{2})\,dt \\ &= \tfrac{(b-a)^2}{4}[f(b)-f(a)].\end{aligned}$$

Consequently, the inequalities (2.2) are proved.

Remark 2.2. The inequalities (2.2) provide a correction of inequalities (3.14) in [1].

3. INEQUALITIES FOR THE VARIANCE

For convenience in further discussions, we will first to derive some technical results in what follows. Put

$$\phi(t) := (t-\gamma)^3 + \frac{1}{b-a}[(b-t)(\gamma-a)^3 - (t-a)(b-\gamma)^3] \tag{3.1}$$

for $t \in [a,b]$ and $\gamma \in \mathbf{R}$.

It is easy to find that

$$\begin{aligned}\phi(t) &= t^3 - 3\gamma t^2 - [a^2+ab+b^2-3(a+b)\gamma]t - ab[3\gamma-(a+b)] \\ &= (t-a)(t-b)(t-c),\end{aligned} \tag{3.2}$$

where $c = 3\gamma - a - b$. This implies that

$$c \begin{cases} >\gamma, & \gamma > \frac{a+b}{2}, \\ =\gamma, & \gamma = \frac{a+b}{2}, \\ <\gamma, & \gamma < \frac{a+b}{2}. \end{cases} \tag{3.3}$$

Moreover, we see that $c < a$ for $\gamma < \frac{2a+b}{3}$, $c > b$ for $\gamma > \frac{a+2b}{3}$ and $a \le c \le b$ for $\frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}$. Therefore, by (3.2) we can conclude that $\phi(t) \le 0$ for $t \in [a,b]$ if $\gamma < \frac{2a+b}{3}$, $\phi(t) \ge 0$ for $t \in [a,b]$ if $\gamma > \frac{a+2b}{3}$ and $\phi(t) > 0$ for $t \in (a,c)$ with $\phi(t) < 0$ for $t \in (c,b)$ if $\frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}$.

Thus we have

$$\int_a^b |\phi(t)|\,dt = -\int_a^b \phi(t)\,dt = \frac{1}{2}(\frac{a+b}{2}-\gamma)(b-a)^3 \tag{3.4}$$

in case $\gamma < \frac{2a+b}{3}$,

$$\int_a^b |\phi(t)|\,dt = \int_a^b \phi(t)\,dt = \frac{1}{2}(\gamma-\frac{a+b}{2})(b-a)^3 \tag{3.5}$$

in case $\frac{a+2b}{3} < \gamma$, and

(3.6)

$$\begin{aligned}\int_a^b |\phi(t)|\,dt &= \int_a^c \phi(t)\,dt - \int_c^b \phi(t)\,dt \\ &= \tfrac{1}{4}[18(\gamma-a)(b-\gamma)(b-a)^2 - 54(\gamma-a)^2(b-\gamma)^2 - (b-a)^4]\end{aligned}$$

in case $\frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}$.

Also, it is not difficult to get by elementary calculus that

$$\sup_{t\in[a,b]} |\phi(t)| = 2\{[(\gamma - \frac{a+b}{2})^2 + \frac{(b-a)^2}{12}]^{\frac{3}{2}} - (\gamma - a)(b-\gamma)|\gamma - \frac{a+b}{2}|\}, \tag{3.7}$$

for $\gamma \in \mathbf{R}$.

Now we would like to give some inequalities for the variance with different bounds.

Theorem 3.1. *Let $f : [a,b] \to \mathbf{R}_+$ be an absolutely continuous probability density function associated with a random variable X. If $f' \in L_\infty[a,b]$, then the variance $\sigma^2(X)$ satisfies the inequalities*

$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \le \|f'\|_\infty &\begin{cases} \frac{1}{6}(\frac{a+b}{2} - \gamma)(b-a)^3, & a < \gamma < \frac{2a+b}{3} \\ \frac{1}{12}[18(\gamma-a)(b-\gamma)(b-a)^2 - 54(\gamma-a)^2(b-\gamma)^2 - (b-a)^2], & \frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}, \\ \frac{1}{6}(\gamma - \frac{a+b}{2})(b-a)^3, & \frac{a+2b}{3} < \gamma < b, \end{cases} \end{aligned} \tag{3.8}$$

where $a < \gamma = E(X) < b$.

Proof. It is easy to find from (1.3)-(1.6) that

$$\sigma^2(X) - (\gamma - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12} = -\frac{1}{3}\int_a^b \phi(t) f'(t)\, dt, \tag{3.9}$$

where $\phi(t)$ is as defined in (3.1).

Thus the inequalities (3.8) follow from (3.4), (3.5) and (3.6).

Theorem 3.2. *Let $f : [a,b] \to \mathbf{R}_+$ be an absolutely continuous probability density function associated with a random variable X. If $f' \in L_1[a,b]$, then the variance $\sigma^2(X)$ satisfies the inequality*

$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \le\ &\tfrac{2}{3}\{[(\gamma - \tfrac{a+b}{2})^2 + \tfrac{(b-a)^2}{12}]^{\frac{3}{2}} - (\gamma-a)(b-\gamma)|\gamma - \tfrac{a+b}{2}|\}\|f'\|_1, \end{aligned} \tag{3.10}$$

where $a < \gamma = E(X) < b$.

Proof. The inequality (3.10) follows immediately from (3.7) and (3.9).

Remark 3.1. The inequalities (3.8) and inequality (3.10) provide a correction of inequalities (3.23) in [2].

Theorem 3.3. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X which is of bounded variation on $[a,b]$. Then the variance $\sigma^2(X)$ satisfies the inequality*

$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \le\ &\tfrac{2}{3}\{[(\gamma - \tfrac{a+b}{2})^2 + \tfrac{(b-a)^2}{12}]^{\frac{3}{2}} - (\gamma-a)(b-\gamma)|\gamma - \tfrac{a+b}{2}|\}\bigvee_a^b(f), \end{aligned} \tag{3.11}$$

where $a < \gamma = E(X) < b$ and $\bigvee_a^b(f)$ is the total variation of f on $[a,b]$.

Proof. By Lemma 1.1 and Lemma 1.2 we can conclude that

$$|\sigma^2(X) - (\gamma - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12}| \le \frac{1}{3} \sup_{t\in[a,b]} |\phi(t)| \bigvee_a^b (f),$$

where $\phi(t)$ is as defined in (3.1).

Thus the inequality (3.11) follows from (3.7).

Theorem 3.4. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X which is L-Lipschitzian on $[a,b]$. Then the variance $\sigma^2(X)$ satisfies the inequalities*

(3.12)
$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \le L &\begin{cases} \frac{1}{6}(\frac{a+b}{2} - \gamma)(b-a)^3, & a < \gamma < \frac{2a+b}{3}, \\ \frac{1}{12}[18(\gamma-a)(b-\gamma)(b-a)^2 - 54(\gamma-a)^2(b-\gamma)^2 - (b-a)^4], & \frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}, \\ \frac{1}{6}(\gamma - \frac{a+b}{2})(b-a)^3, & \frac{a+2b}{3} < \gamma < b, \end{cases} \end{aligned}$$

where $a < \gamma = E(X) < b$.

Proof. By Lemma 1.1 and Lemma 1.2 we can conclude that

$$|\sigma^2(X) - (\gamma - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12}| \le \frac{L}{3} \int_a^b |\phi(t)|\, dt,$$

where $\phi(t)$ is as defined in (3.1).

Thus the inequalities (3.12) follow from (3.4), (3.5) and (3.6).

Theorem 3.5. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X which is monotonic nondecreasing on $[a,b]$. Then the variance $\sigma^2(X)$ satisfies the inequality*

(3.13)
$$\begin{aligned} &|\sigma^2(X) - (\gamma - \tfrac{a+b}{2})^2 - \tfrac{(b-a)^2}{12}| \\ \le &\begin{cases} \frac{5b+4a-9\gamma}{18}(b-a)^2[f(b)-f(a)], & a < \gamma < \frac{2a+b}{3}, \\ \frac{3b-2a-c}{18}(c-a)^2[f(c)-f(a)] + \frac{2b+c-3a}{18}(b-c)^2[f(b)-f(c)], & \frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}, \\ \frac{9\gamma-5a-4b}{18}(b-a)^2[f(b)-f(a)], & \frac{a+2b}{3} < \gamma < b, \end{cases} \end{aligned}$$

where $a < \gamma = E(X) < b$ and $c = 3\gamma - a - b$.

Proof. By Lemma 1.1 and Lemma 1.2 we can conclude that

$$|\sigma^2(X) - (\gamma - \frac{a+b}{2})^2 - \frac{(b-a)^2}{12}| \le \frac{1}{3} \int_a^b |\phi(t)|\, df(t),$$

where $\phi(t)$ is as defined in (3.1).

Notice that

$$\phi(t) = (t-a)(t-b)(t-c)$$

for $t \in [a,b]$, where $c = 3\gamma - a - b$, it is easy to calculate that

$$\begin{aligned}\int_a^b |\phi(t)|\,df(t) &= -\int_a^b \phi(t)\,df(t) = \int_a^b \phi'(t)f(t)\,dt\\ &= \int_a^b [(t-b)(t-c)+(t-a)(t-c)+(t-a)(t-b)]f(t)\,dt\\ &\le f(a)\int_a^b (t-b)(t-c)\,dt + f(b)\int_a^b (t-a)(t-c)\,dt + f(a)\int_a^b (t-a)(t-b)\,dt\\ &= \frac{5b+4a-9\gamma}{6}(b-a)^2[f(b)-f(a)],\end{aligned}$$

in case $a < \gamma < \frac{2a+b}{3}$,

$$\begin{aligned}\int_a^b |\phi(t)|\,df(t) &= \int_a^b \phi(t)\,df(t) = -\int_a^b \phi'(t)f(t)\,dt\\ &= -\int_a^b [(t-b)(t-c)+(t-a)(t-c)+(t-a)(t-b)]f(t)\,dt\\ &\le -f(a)\int_a^b (t-b)(t-c)\,dt - f(b)\int_a^b (t-a)(t-c)\,dt - f(b)\int_a^b (t-a)(t-b)\,dt\\ &= \frac{9\gamma-5a-4b}{6}(b-a)^2[f(b)-f(a)],\end{aligned}$$

in case $\frac{a+2b}{3} < \gamma < b$, and

$$\begin{aligned}\int_a^b |\phi(t)|\,df(t) &= \int_a^c \phi(t)\,df(t) - \int_c^b \phi(t)\,df(t)\\ &= -\int_a^c \phi'(t)f(t)\,dt + \int_c^b \phi'(t)f(t)\,dt\\ &= -\int_a^c [(t-b)(t-c)+(t-a)(t-c)+(t-a)(t-b)]f(t)\,dt\\ &\quad + \int_c^b [(t-b)(t-c)+(t-a)(t-c)+(t-a)(t-b)]f(t)\,dt\\ &\le -f(a)\int_a^c (t-b)(t-c)\,dt - f(c)\int_a^c (t-a)(t-c)\,dt - f(c)\int_a^c (t-a)(t-b)\,dt\\ &\quad + f(c)\int_c^b (t-b)(t-c)\,dt + f(b)\int_c^b (t-a)(t-c)\,dt + f(c)\int_c^b (t-a)(t-b)\,dt\\ &= \frac{3b-2a-c}{6}(c-a)^2[f(c)-f(a)] + \frac{2b+c-3a}{6}(b-c)^2[f(b)-f(c)]\end{aligned}$$

in case $\frac{2a+b}{3} \le \gamma \le \frac{a+2b}{3}$.

Consequently, the inequalities (3.13) are proved.

Corollary 3.1. *Let $f : [a,b] \to \mathbf{R}_+$ be a probability density function associated with a random variable X. If $E(X) = \frac{a+b}{2}$, then the variance $\sigma^2(X)$ satisfies the inequalities*

$$|\sigma^2(X) - \frac{(b-a)^2}{12}| \le \begin{cases} \frac{(b-a)^3}{36\sqrt{3}}\bigvee_a^b(f), & f \text{ of bounded variation,}\\ \frac{(b-a)^4}{96}L, & f \text{ } L\text{-Lipschitzian,}\\ \frac{5(b-a)^3}{144}[f(b)-f(a)], & f \text{ monotonic nondecreasing.}\end{cases}$$

Proof. It is immediate from the inequalities (3.11), (3.12) and (3.13).

Remark 3.2. The inequalities (3.11), (3.12) and (3.13) provide a correction of inequalities (3.15) in [1].

Remark 3.3. The mistakes of Corollary 8 and Corollary 9 1n [2] as well as the mistakes of Corollary 3.7 and Corollary 3.8 in [1] seemed as if they are originated from having wrongly examined the behaviour of $\phi(t)$ as given by

$$\phi(t) = (t-\gamma)^{n+1} + (\frac{b-t}{b-a})(\gamma-a)^{n+1} - (\frac{t-a}{b-a})(b-\gamma)^{n+1}$$

for $t \in [a,b]$ in case n is even. (See (3.13) of Lemma 2 in [2] and also (3.6) of Lemma 3.3 in [1] and compare them with the assertions expressed at the beginning of this section as a special case of $n = 2$.)

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