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SYMMETRY REDUCTIONS AND EXACT SOLUTIONS TO THE SEVENTH-ORDER KDV TYPES OF EQUATION

YOUWEI ZHANG

Abstract. In present paper, the seventh-order KdV types of equation is considered by the Lie symmetry analysis. All of the geometric vector fields of the KdV equation are obtained, then the symmetry reductions and exact solutions to the KdV equation are investigated by the dynamical system and the power series method.

1. Introduction

Recently, mathematics and physics field have devoted considerable effort to the study of solutions to ordinary and partial differential equations (ODEs and PDEs). Among many powerful methods for solving the equation, Lie symmetry analysis provides an effective procedure for integrability, conservation laws, reducing equations and exact solutions of a wide and general class of differential systems representing real physical problems [12, 15]. Sinkala et al [14] have performed the group classification of a bond-pricing PDE of mathematical finance to discover the combinations of arbitrary parameters that allow the PDE to admit a nontrivial symmetry Lie algebra, and computed the admitted Lie point symmetries, identify the corresponding symmetry Lie algebra and solve the PDE. Under the condition of the symmetry group of the PDE is nontrivial, it contains a standard integral transform of the fundamental solution for PDEs, and fundamental solution can be reduced to inverting a Laplace transform or some other classical transform in [1]. In [7], by the direct construction method, all of the first-order multipliers of the the generalized nonlinear second-order equation are obtained, and the corresponding complete conservation laws of such equations are provided. Furthermore, Lie symmetry analysis helps to study their group theoretical properties, and effectively assists to derive several mathematical characteristics related with their complete integrability [10]. Also, Lie symmetry analysis and dynamical system method is a feasible approach to dealing with exact explicit solutions to nonlinear PDEs and systems, (see, e.g.,

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[2, 3, 8, 11]). Liu et al have derived the symmetries, bifurcations and exact explicit solutions to the KdV equation by using Lie symmetry analysis and the dynamical system method [5, 6]. The KdV equation models the dust-ion-acoustic waves in such cosmic environments as those in the supernova shells and Saturn's F-ring [4], etc., In present paper, we will investigate the vector fields, symmetry reductions and exact solutions to the KdV equation with power law nonlinearity and linear damping with dispersion

$$u_t + u^2 u_x + u u_{4x} + 2u_x u_{3x} + u_{xx}^2 + u_{7x} = 0, \tag{1.1}$$

where $u = u(x,t)$ is the unknown functions, x is the spatial coordinate in the propagation direction and t is the temporal coordinates, which occur in different contexts in mathematical physics.

The rest of this paper is organized as follows: in Section 2, the vector fields of Eqs. (1.1) are presented by using Lie symmetry analysis method. Based on the optimal system method, all the similarity reductions to the Eqs. (1.1) are obtained. In Section 3, the exact analytic solutions to the equations are investigated by means of the power series method. Finally, the conclusions will be given in Section 4.

2. Lie symmetry analysis and similarity reductions

Recall that the geometric vector field of a PDE equation is as follows:

$$V = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u, \tag{2.1}$$

where the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$, $\eta(x,t,u)$ of the vector field are to be determined later.

If the vector field (2.1) generates a symmetry of the equation (1.1), then V must satisfy the Lie symmetry condition

$$\mathrm{Pr}V(\Delta)|_{\Delta=0} = 0,$$

where $\mathrm{Pr}V$ denotes the 7-th prolongation of V , and $\Delta = u_t + u^2 u_x + u u_{4x} + 2u_x u_{3x} + u_{xx}^2 + u_{7x}$. Moreover, the prolongation $\mathrm{Pr}V$ depends on the equation

$$\mathrm{Pr}V = \eta\partial_u + \eta^x \partial_{u_x} + \eta^{xx}\partial_{u_{xx}} + \eta^{3x}\partial_{u_{3x}} + \eta^{4x}\partial_{u_{4x}} + \eta^{7x}\partial_{u_{7x}},$$

where the coefficient functions $\eta^{kx}(k = 1,2,3,4,7)$ are given as

$$\eta^{kx} = D_x^k(\eta - \tau u_t - \xi u_x) + \tau u_{kxt} + \xi u_{(k+1)x}, \quad k = 1,2,3,4,7,$$

here symbol D_x denotes the total differentiation operator and is defined as

$$D_x = \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \dots.$$

Then, in terms of the Lie symmetry analysis method, we obtain that all of the geometric vector fields of Eq. (1.1) are as follows:

$$V_1 = x\partial_x + 7t\partial_t - 3u\partial_u, \quad V_2 = \partial_x, \quad V_3 = \partial_t.$$

Moreover, it is necessary to show that the vector fields of Eq. (1.1) are closed under the Lie bracket, we have

$$[V_i, V_i] = 0, \quad i = 1,2,3,$$
$$[V_1,V_2] = -[V_2,V_1] = V_2, \quad [V_1,V_3] = -[V_3,V_1] = 7V_3, \quad [V_2,V_3] = -[V_3,V_2] = 0.$$

In the preceding section, we obtained the vector fields and the optimal systems of Eq. (1.1). Now, we deal with the symmetry reductions and exact solutions to the equations. We will consider the following similarity reductions and group-invariant solutions based on the optimal system method. From an optimal system of group-invariant solutions to an equation, every other such solution to the equation can be derived.

For the generator V_1, we have

$$u = t^{-\frac{3}{7}} f(z), \tag{2.2}$$

where $z = xt^{-\frac{1}{7}}$. Substituting (2.2) into Eq. (1.1), we reduce it into the following ODE

$$f^{(7)} + f^2 f'^{(4)} + 2f'^{(3)} + f''^2 - \frac{1}{7} z f' - \frac{3}{7} f = 0, \tag{2.3}$$

where $f' = \frac{df}{dz}$.

For the generator V_2, we get the trivial solution to Eq. (1.1) is $u(x,t) = c$, where c is an arbitrary constant.

For the generator V_3, we have

$$u = f(z), \tag{2.4}$$

where $z = x$. Substituting (2.4) into Eq. (1.1), we reduce it into the following ODE

$$f^{(7)} + f^2 f'^{(4)} + 2f'^{(3)} + f''^2 = 0, \tag{2.5}$$

where $f' = \frac{df}{dz}$.

For the generator $V_3 + \upsilon V_2$, υ is an arbitrary constant, we have

$$u = f(z), \tag{2.6}$$

where $z = x - \upsilon t$. Substituting (2.6) into Eq. (1.1), we reduce it into the following ODE

$$f^{(7)} + f^2 f'^{(4)} + 2f'^{(3)} + f''^2 - \upsilon f' = 0, \tag{2.7}$$

where $f' = \frac{df}{dz}$.

3. The exact power series solutions

By seeking for exact solutions of the PDEs, we mean that those can be obtained from some ODEs or, in general, from PDEs of lower order than the original PDE. In terms of this definition, the exact solutions to Eq. (1.1) are obtained actually in both of the preceding Sections 2. In spite of this, we still want to detect the explicit solutions expressed in terms of elementary or, at least, known functions of mathematical physics, in terms of quadratures, and so on. But this is not always the case, even for simple semilinear PDEs. However, we know that the power series can be used to solve differential equations, including many complicated differential equations [9, 13]. In this section, we will consider the exact analytic solutions to the reduced equations by using the power series method. Once we get the exact analytic solutions of the reduced ODEs, the exact power series solutions to the original PDEs are obtained, now we consider the solutions of ODEs (2.3), (2.5) and (2.7).

3.1 Exact analytic solutions to Eq. (2.3)

In view of (2.3), we seek a solution in a power series of the form

$$f(z)=\sum_{n=0}^{\infty}c_n z^n. \tag{3.1}$$

Substituting (3.1) into (2.3), and comparing coefficients, then we obtain the following recursion formula:

$$\begin{aligned}c_{n+7}={}&-\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)}\\ &\times\Bigg(\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1)c_ic_{k-i}c_{n-k+1}\\ &+\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)(n-k+4)c_kc_{n-k+4}\\ &+2\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)(k+1)c_{k+1}c_{n-k+3}\\ &+\sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)(k+2)c_{k+2}c_{n-k+2}\\ &-\frac{1}{7}nc_n-\frac{3}{7}c_n\Bigg),\end{aligned} \tag{3.2}$$

for all $n=0,1,2,\ldots$.

Thus, for arbitrarily chosen constants c_i $(i=0,1,\ldots,6)$, we obtain

$$c_7=-\frac{1}{5040}\big(c_0^2c_1+24c_0c_4+12c_1c_3+4c_2^2-\frac{3}{7}c_1\big). \tag{3.3}$$

Furthermore, from (3.2), it yield

$$\begin{aligned}c_8&=-\frac{1}{20160}\big(c_0c_1^2+c_0^2c_2+60c_0c_5+36c_1c_4+24c_2c_3-\frac{2}{7}c_1\big),\\ c_9&=-\frac{1}{181440}\big(3c_0^2c_3+c_1^3+6c_0c_1c_2+360c_0c_6+240c_1c_5+144c_2c_4\\ &\quad+72c_3^2+24c_0c_4-\frac{5}{7}c_2\big),\end{aligned} \tag{3.4}$$

and so on.

Thus, for arbitrary chosen constant numbers c_i $(i=0,1,\ldots)$, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined successively from (3.3) and (3.4) in a unique manner. This implies that for Eq. (2.3), there exists a power series solution (3.1) with the coefficients given by (3.3) and (3.4). Furthermore, it is easy to prove the convergence of the power series (3.1) with the coefficients given by (3.3) and (3.4). Therefore, this power series solution (3.1) to Eq. (2.3) is an exact analytic solution.

Hence, the power series solution of Eq. (2.3) can be written as

$$\begin{aligned}
f(z) &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + c_7 z^7 + \sum_{n=1}^{\infty} c_{n+7} z^{n+7} \\
&= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 \\
&\quad - \frac{1}{5040}\big(c_0^2 c_1 + 24 c_0 c_4 + 12 c_1 c_3 + 4 c_2^2 - \frac{3}{7} c_1\big) z^7 \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \\
&\quad \times \Bigg(\sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1) c_i c_{k-i} c_{n-k+1} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(n-k+4) c_k c_{n-k+4} \\
&\quad + 2 \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(k+1) c_{k+1} c_{n-k+3} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)(k+2) c_{k+2} c_{n-k+2} - \frac{1}{7} n c_n - \frac{3}{7} c_n \Bigg) z^{n+7}.
\end{aligned}$$

Thus, the exact power series solution of Eq. (1.1) is

$$\begin{aligned}
u(x,t) &= c_0 t^{-\frac{3}{7}} + c_1 x t^{-\frac{4}{7}} + c_2 x^2 t^{-\frac{5}{7}} + c_3 x^3 t^{-\frac{6}{7}} + c_4 x^4 t^{-1} + c_5 x^5 t^{-\frac{8}{7}} \\
&\quad + c_6 x^6 t^{-\frac{9}{7}} - \frac{1}{5040}\big(c_0^2 c_1 + 24 c_0 c_4 + 12 c_1 c_3 + 4 c_2^2 - \frac{3}{7} c_1\big) x^7 t^{-\frac{10}{7}} \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)(n+7)} \\
&\quad \times \Bigg(\sum_{k=0}^{n} \sum_{i=0}^{k} (n-k+1) c_i c_{k-i} c_{n-k+1} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(n-k+4) c_k c_{n-k+4} \\
&\quad + 2 \sum_{k=0}^{n} (n-k+1)(n-k+2)(n-k+3)(k+1) c_{k+1} c_{n-k+3} \\
&\quad + \sum_{k=0}^{n} (n-k+1)(n-k+2)(k+1)(k+2) c_{k+2} c_{n-k+2} - \frac{1}{7} n c_n \\
&\quad - \frac{3}{7} c_n \Bigg) \times x^{n+7} t^{-\frac{n+10}{7}}.
\end{aligned} \tag{3.5}$$

3.2 Exact analytic solutions to Eq. (2.5)

In view of (2.5), we have

$$\frac{1}{3} f^3 + f f''' + f' f''^{(6)} + c = 0, \tag{3.6}$$

where c is an integration constant.

We seek a solution of Eq. (3.6) in a power series of the form (3.1). Substituting (3.1) into (3.6), and comparing coefficients, we obtain

$$\begin{aligned} c_{n+6} = &-\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}\bigg(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\ &+\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\ &+\sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2}\bigg), \end{aligned} \tag{3.7}$$

for all $n = 1, 2, \ldots$.

Thus, for arbitrarily chosen constants c_i $(i = 0, 1, \ldots, 5)$, we have

$$c_6 = -\frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 + c\big).$$

Furthermore, from (3.7), we have

$$\begin{aligned} c_7 &= -\frac{1}{5040}\Big(c_0^2c_1 + 24c_0c_4 + 12c_1c_3 + 4c_1c_2\Big), \\ c_8 &= -\frac{1}{20160}\Big(c_0^2c_2 + c_0c_1^2 + 60c_0c_5 + 36c_1c_4 + 24c_2c_3\Big), \end{aligned}$$

and so on.

Hence, the power series solution of Eq. (2.5) can be written as

$$\begin{aligned} f(z) &= c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 + c_6z^6 + \sum_{n=1}^{\infty} c_{n+6}z^{n+6} \\ &= c_0 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + c_5z^5 - \frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 + c\big)z^6 \\ &\quad -\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}\bigg(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\ &\quad +\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\ &\quad +\sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2}\bigg)z^{n+6}. \end{aligned}$$

Thus, the exact power series solution of Eq. (1.1) is

$$
\begin{aligned}
u(x,t) = {} & c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 \\
& - \frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 + c\big)x^6 \\
& - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \\
& \times \Big(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
& + \sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\
& + \sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2}\Big)x^{n+6}.
\end{aligned} \tag{3.8}
$$

3.3 Exact analytic solutions to Eq. (2.7)

In view of (2.7), we have

$$
\frac{1}{3}f^3 + ff''' + f'f''^{(6)} - \upsilon f + c = 0, \tag{3.9}
$$

where c is an integration constant.

Similarly, we seek a solution of Eq. (3.9) in a power series of the form (3.1). Substituting (3.1) into (3.9), and comparing coefficients, we obtain

$$
\begin{aligned}
c_{n+6} = & -\frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \\
& \times \Big(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
& + \sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\
& + \sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2} - \upsilon c_n\Big),
\end{aligned} \tag{3.10}
$$

for all $n = 1, 2, \ldots$.

Thus, for arbitrarily chosen constants c_i $(i = 0, 1, \ldots, 5)$, we can get

$$
c_6 = -\frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 - \upsilon c_0 + c\big).
$$

Furthermore, from (3.10), we have

$$
\begin{aligned}
c_7 &= -\frac{1}{5040}\Big(c_0^2c_1 + 24c_0c_4 + 12c_1c_3 + 4c_1c_2 - \upsilon c_1\Big), \\
c_8 &= -\frac{1}{20160}\Big(c_0^2c_2 + c_0c_1^2 + 60c_0c_5 + 36c_1c_4 + 24c_2c_3 - \upsilon c_2\Big),
\end{aligned}
$$

and so on.

Hence, the power series solution of Eq. (2.7) can be written as

$$\begin{aligned}
f(z) &= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + \sum_{n=1}^{\infty} c_{n+6} z^{n+6} \\
&= c_0 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 - \frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2 - \upsilon c_0\big)z^6 \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \Bigg(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
&\quad + \sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\
&\quad + \sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2} - \upsilon c_n\Bigg) z^{n+6}.
\end{aligned}$$

Thus, we obtain the traveling wave solution to Eq. (1.1) as follows

$$\begin{aligned}
u(x,t) &= c_0 + c_1(x-\upsilon t) + c_2(x-\upsilon t)^2 + c_3(x-\upsilon t)^3 + c_4(x-\upsilon t)^4 \\
&\quad + c_5(x-\upsilon t)^5 - \frac{1}{720}\big(\frac{1}{3}c_0^3 + 6c_0c_3 + 2c_1c_2\big)(x-\upsilon t)^6 \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)} \\
&\quad \times \Bigg(\frac{1}{3}\sum_{k=0}^{n}\sum_{i=0}^{k} c_i c_{k-i} c_{n-k} \\
&\quad + \sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_k c_{n-k+3} \\
&\quad + \sum_{k=0}^{n}(n-k+1)(n-k+2)(k+1)c_{k+1}c_{n-k+2} - \upsilon c_n\Bigg) \\
&\quad \times (x-\upsilon t)^{n+6}.
\end{aligned} \tag{3.11}$$

Remark 3.1. We would like to reiterate that the power series solutions which have been obtained in this section are exact analytic solutions to the equations. Moreover, we can see that these power series solutions converge for the given chosen constants c_i $(i = 0, 1, \ldots, 6)$ of (3.5), c_i $(i = 0, 1, \ldots, 5)$ of (3.8) and (3.11), respectively, it is actual value for mathematical and physical applications.

4. Summary and discussion

In this paper, we have obtained the symmetries and similarity reductions of the seventh-order KdV types of equations by using Lie symmetry analysis method. All the group-invariant solutions to the equations are considered based on the optimal system method. Then the exact analytic solutions are investigated by using the power series method, and we can see that these power series solutions converge.

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School of Mathematics and Statistics, Hexi University, Gansu 734000, China
E-mail address: ywzhang0288@163.com