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ON THE INVOLUTES FOR DUAL SPLIT QUATERNIONIC CURVES

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Abstract. In this study, definition of involute-evolute curves for semi-dual quaternionic curves in semi-dual spaces $\mathbb{D}_2^4$ known as dual split quaternion and $\mathbb{D}_1^3$ are given and also some well-known theorems for involute-evolute dual split quaternionic curves are obtained.

1. Introduction

The idea of a string involute is due to C. Huygens (1658) who is also known with his work in optics. He discovered involutes while trying to build a more accurate clock [4]. Later, the relations Frenet frame of involute-evolute couple in the space E^3 were given in [10].

In recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to Lorentz manifold. For instance, in [23], the authors extended and studied the spacelike involute-evolute curves in Minkowski space-time ([2], [5], [23]).

The quaternions were first defined in 1843 by Hamilton. The dual quaternions are extension of the real quaternions by means of the dual numbers [3], [22], and they were first introduced by Clifford [6]. In $\mathbb{D}^3$ and $\mathbb{D}^4$ dual spaces, Serret Frenet Formulas had been defined by Sivridağ [21]. Inclined curves and characterization of quaternionic Lorentz manifolds were given in 1999 by Karadağ. In 2002, Serret Frenet Formulas for quaternionic curves in Semi-Euclidean space were defined by Tuna. The quaternionic inclined curves in the Semi-Euclidean space E_2^4 were given

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in 2004 by Çöken and Tuna [8]. The split quaternions were identified with Semi-Euclidean space E_2^4, while the vector part of split quaternions were identified with Minkowski 3-space [11]. In 2009, Serret Frenet Formulas for split quaternionic curves in Semi-Euclidean space E_2^4 were given in [7].

In this paper, we firstly define involute-evolute curve couples in definition of involute-evolute curves on $\mathbb{D}_1^3$ and $\mathbb{D}_2^4$. Later, we calculate Frenet frame of the evolute curve by the help of the frame of the involute curve. We use the methods expressed in [7]. (In this paper, we consider non-null curves, and a version of this adapted to null curves can be studied.)

2. Preliminaries

In this section, we will give basic definitions of the dual spaces $\mathbb{D}^3$ and $\mathbb{D}^4$ and then the semi-dual spaces $\mathbb{D}_1^3$ and $\mathbb{D}_2^4$.

A dual number has the form $a + \xi a^*$ where a and a^* are real numbers and $\xi = (0, 1)$ is the dual unit with the property that $\xi^2 = 0$. The set of all dual numbers form a comutative ring over the real number field and denoted by $\mathbb{D}$ [25].

$\mathbb{D}^3$ dual vector space ($\mathbb{D}$ - Module) can be written as

$$\mathbb{D}^3 = \{(A_1, A_2, A_3) : A_1, A_2, A_3 \in \mathbb{D}\ \}.$$

The Euclidean inner-product of two dual vectors A, $B \in \mathbb{D}^3$ is defined as

$$\begin{array}{llll} \langle , \rangle : & \mathbb{D}^3 \times \mathbb{D}^3 & \longrightarrow & \mathbb{D} \\ & (A, B) & \longrightarrow & \langle A, B\rangle = \langle a, b\rangle + \xi(\langle a^*, b\rangle + \langle a, b^*\rangle). \end{array}$$

Given a dual vector $A = a + \xi a^*$, the norm of A is

$$\|A\| = (\langle A, A\rangle)^{\frac{1}{2}} = \|a\| + \xi \frac{\langle a, a^*\rangle}{\|a\|}, \qquad a \neq 0.$$

The cross-product of two dual vectors $A, B \in \mathbb{D}^3$ is defined as,

$$A \wedge B = a \wedge b + \xi \left(a \wedge b^* + a^* \wedge b\right).$$

Similarly, $\mathbb{D}^4$ dual vector space can be written as

$$\mathbb{D}^4 = \{(A_1, A_2, A_3, A_4) : A_1, A_2, A_3, A_4 \in \mathbb{D}\ \}.$$

The same definitions of inner-product, norm and cross-product are hold for $\mathbb{D}^4$.

The Lorentzian inner-product of two dual vectors $A = a + \xi a^*$ and $B = b + \xi b^*$, a, $b \in \mathbb{R}_1^3$ is given as

$$\langle A, B\rangle = \langle a, b\rangle + \xi(\langle a^*, b\rangle + \langle a, b^*\rangle)$$

with the signature $(-, +, +)$ in $\mathbb{R}_1^3$. The $\mathbb{D}-$module $\mathbb{D}^3$ with the Lorentzian inner-product is called the semi-dual space $\mathbb{D}_1^3$ [24].

On the other hand, a semi-Euclidean inner-product of two dual vectors in $\mathbb{D}^4$, $A = a + \xi a^*$ and $B = b + \xi b^*$, a, $b \in \mathbb{R}_2^4$, can be defined as

$$\langle A, B\rangle = \langle a, b\rangle + \xi(\langle a^*, b\rangle + \langle a, b^*\rangle)$$

with the signature $(-, -, +, +)$ in $\mathbb{R}_2^4$. The dual space $\mathbb{D}^4$ with the semi-Euclidean inner-product is called the semi-dual space $\mathbb{D}_2^4$ or dual-split quaternion [12].

Let A be a dual vector in $\mathbb{D}_1^3$. If $\langle a, a\rangle < 0$, then A is called timelike, if $\langle a, a\rangle > 0$, then A is called spacelike and if $\langle a, a\rangle = 0$, then A is called lightlike (or null) vector. A smooth curve on the semi-dual space $\mathbb{D}_1^3$ is said to be timelike, spacelike or null if its tangent vectors are timelike, spacelike or null, respectively. Observe that, a timelike curve corresponds to the path of an observer moving at less than the speed of light while the spacelike curves are faster and the null curves are equal to the speed of light [17].

A real quaternion consists of a set of four ordered real numbers a, b, c, d associated with four units e_1, e_2, e_3 and 1, respectively. The three units e_1, e_2 and e_3 have the following properties:

$$\begin{array}{lll} e_1^2 = e_2^2 = e_3^2 = -1, & & \\ e_1 \times e_2 = e_3, & e_2 \times e_3 = e_1, & e_3 \times e_1 = e_2 \\ e_2 \times e_1 = -e_3, & e_3 \times e_2 = -e_1, & e_1 \times e_3 = -e_2 \end{array} \tag{1}$$

A real quaternion q may be written as $q = ae_1 + be_2 + ce_3 + d$.

Clearly, a quaternion q consists of two parts: the scalar part $S_q = d$ and the vector part $V_q = ae_1 + be_2 + ce_3$. The set of all real quaternions is denoted by $Q_{\mathbb{R}}$.

The multiplication of two real quaternions p and q is defined as

$$p \times q = V_p \wedge V_q - \langle V_p, V_q \rangle + S_p S_q + S_p V_q + S_q V_p \tag{2}$$

where $\langle , \rangle$ and $\wedge$ are the inner-product and the cross-product on $\mathbb{R}^3$, respectively. The conjugate of the quaternion q is denoted by αq and defined as $\alpha q = S_q - V_q$.

The $h-$inner-product of two quaternions is defined as

$$h(p, q) = \frac{1}{2}(p \times \alpha q + q \times \alpha p), \quad p, q \in Q_{\mathbb{R}} \tag{3}$$

The real number $[h(p, p)]^{1/2}$ is called the norm of the real quaternion p and is denoted by $\|p\|$. Hence we obtain that

$$\|p\|^2 = h(p, q) = a^2 + b^2 + c^2 + d^2. \tag{4}$$

It is easy to see that, if $p = a_1e_1 + b_1e_2 + c_1e_3 + d_1$ and $q = a_2e_1 + b_2e_2 + c_2e_3 + d_2$, then

$$h(p, q) = a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2 \text{ [1].} \tag{5}$$

Given two real quaternions p and p^*, we define the dual quaternion as P=p+ξ p^* and denote the set of dual quaternions by $Q_{\mathbb{D}}$. For given $A, B, C, D \in \mathbb{D}$, we can write $P = Ae_1 + Be_2 + Ce_3 + D$. Here $S_P = D$ is called the scalar part of P and $V_P = Ae_1 + Be_2 + Ce_3$ is called the vector part of P.

The multiplication of two dual quaternions P and Q is defined as

$$P \times Q = p \times q + \xi \,(p \times q^* + p^* \times q) \tag{6}$$

where $P = p + \xi\, p^*$ and $Q = q + \xi q^*$ and $\times$ shows the real quaternion multiplication. It is clear that

$$P \times Q = S_P S_Q + S_P V_Q + S_Q V_P - \langle V_P, V_Q \rangle + V_P \wedge V_Q \tag{7}$$

where $\langle , \rangle$ is the inner product and $\wedge$ is the cross-product on $\mathbb{D}^3$. If $P = S_P + V_P$, then the conjugate of P is defined by $\alpha P = S_P - V_P$. By using this definition, the following properties can be easily proved:

$$\begin{array}{ll} (i) & \alpha(\alpha P) = P, \\ (ii) & \alpha(P \times Q) = \alpha Q \times \alpha P. \end{array}$$

The symmetric dual-valued bilinear form H is defined as

$$H(P, Q) = \frac{1}{2}(P \times \alpha Q + Q \times \alpha P). \tag{8}$$

As a result, we obtain the followings :

1- For all elements P, Q of $Q_{\mathbb{D}}$, we have

$$H(P,Q) = h(p,q) + \xi[h(p,q^*) + h(p^*,q)]$$

where h is the symmetric real-valued bilinear form.

2- If $P = Ae_1 + Be_2 + Ce_3 + D$, then we have

$$H(P,P) = A^2 + B^2 + C^2 + D^2.$$

3- $\forall\ P \in Q_{\mathbb{D}}$, the norm of P is defined by

$$\|P\| = \|p\| + \xi \frac{h(p,p^*)}{\|p\|},$$

and so

$$\|P\|^2 = H(P,P) = P \times \alpha P. \tag{9}$$

4- $\forall\ P \in Q_{\mathbb{D}}$, the scalar part and the vector part of P is

$$S_P = \frac{1}{2}(P + \alpha P), \qquad V_P = \frac{1}{2}(P - \alpha P).$$

As a result,

$$\begin{array}{ll} (i) & \text{if } P + \alpha P = 0, \text{ then } P \in \mathbb{D} - module, \\ & \text{in this case, } P \text{ is called dual-spatial quaternion} \\ (ii) & \text{if } P - \alpha P = 0, \text{ then } P \in \mathbb{D}, \\ & \text{in this case, } P \text{ is called dual-temporal quaternion.} \end{array}$$

Let P and Q be two dual-spatial quaternion. If $H(P,Q) = 0$, we say that P and Q are $H-$orthogonal[19].

A semi-real quaternion consists of a set of four ordered real numbers a, b, c, d associated with four units e_1, e_2, e_3 and 1, respectively. The three units e_1, e_2 and e_3 have the following properties:

$$\begin{array}{llll} i) & & e_i \times e_i = -\varepsilon(e_i), & 1 \le i \le 3 \\ ii) & \text{in } \mathbb{R}^3_1, & e_i \times e_j = \varepsilon(e_i)\varepsilon(e_j)e_k & 1 \le i,j,k \le 3, \\ iii) & \text{in } \mathbb{R}^4_2, & e_i \times e_j = -\varepsilon(e_i)\varepsilon(e_j)e_k, & 1 \le i,j,k \le 3, \end{array} \tag{10}$$

where (ijk) is the even permutation of (123).

Notice here that,

$$\varepsilon(e_i) = \begin{cases} -1 & , \quad e_i \text{ timelike} \\ +1 & , \quad e_i \text{ spacelike} \end{cases}.$$

As a notation, we denote the semi-real quaternions by Q_ν with an index $\nu = 1, 2$ such that

$$Q_\nu = \left\{ \begin{array}{lll} q & | & q = ae_1 + be_2 + ce_3 + d, \quad a,b,c,d \in \mathbb{R} \\ & & e_1, e_2, e_3 \in \mathbb{R}^3_1, \quad h_\nu(e_i, e_i) = \varepsilon(e_i), \quad 1 \le i \le 3 \end{array} \right\}.$$

The multiplication of two semi-real quaternions p and q is defined as

$$p \times q = V_p \wedge V_q - \langle V_p, V_q \rangle + S_p S_q + S_p V_q + S_q V_p$$

where $\langle , \rangle$ and $\wedge$ are the inner-product and the cross-product on $\mathbb{R}^3_1$, respectively. The conjugate of the quaternion q is denoted by αq and defined as $\alpha q = S_q - V_q$.

For every $p, q \in Q_\nu$, the $h-$inner-product $h_\nu : Q_\nu \times Q_\nu \longrightarrow \mathbb{D}$ of p and q is defined as:

$$h_1(p,q) = \frac{1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) + \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)] \qquad \text{for } \mathbb{R}^3_1$$

and

$$h_2(p,q) = \frac{-1}{2}[\varepsilon(p)\varepsilon(\alpha q)(p \times \alpha q) + \varepsilon(q)\varepsilon(\alpha p)(q \times \alpha p)] \qquad \text{for } \mathbb{R}^4_2.$$

The real number $[h_\nu(p,p)]^{1/2}$ is called the norm of semi-real quaternion p and is denoted by $\|p\|$. Hence we see that

$$\|p\|^2 = |h_\nu(p,p)| = |\varepsilon(p)(p \times \alpha p)|.$$

Given $q \in Q_\nu$, if $q + \alpha q = 0$, then q is called semi-real spatial quaternion. If $q - \alpha q = 0$, q is called semi-real temporal quaternion. The set of semi-real spatial quaternions is isomorphic to $\mathbb{R}^3_1$.

In general, we can write that

$$q = \frac{1}{2}[q + \alpha q] + \frac{1}{2}[q - \alpha q].$$

For $p, q \in Q_\nu$, if $h(p,q) = 0$, p and q are called $h-$orthogonal. If the norm of q is unit, then it is called unit semi-real quaternion and denoted by q_0. So,

$$N_q = \sqrt{|q \times \alpha q|} = \sqrt{|-a^2 - b^2 + c^2 + d^2|}$$

and

$$q_0 = \frac{q}{N_q} = \frac{ae_1 + be_2 + ce_3 + d}{\sqrt{|-a^2 - b^2 + c^2 + d^2|}}$$

([8],[20]).

Let p and p^* be two semi-real quaternions. We define the semi-dual quaternion as $P = p + \xi\, p^*$ and denote the set of semi-dual quaternions by $Q_{\mathbb{D},\nu}$ with an index $\nu = 1, 2$ such that

$$Q_{\mathbb{D},\nu} = \left\{ \; P \mid \quad P = Ae_1 + Be_2 + Ce_3 + D, \quad A, B, C, D \in \mathbb{D},\, e_1, e_2, e_3 \in \mathbb{R}^3_1 \; \right\}.$$

We will use $H_1(e_i, e_i) = \varepsilon_i$, $i = 0, 1, 2$ for $\mathbb{D}^3_1$ and $H_2(e_i, e_i) = \varepsilon(e_i)$, $i = 0, 1, 2, 3$ for $\mathbb{D}^4_2$. The multiplication of two dual quaternions P and Q is defined as

$P \times Q = p \times q + \xi\,(p \times q^* + p^* \times q)$ where $P = p + \xi\, p^*$ and $Q = q + \xi q^*$ and $\times$ shows the quaternion multiplication. It is clear that

$$P \times Q = S_P S_Q + S_P V_Q + S_Q V_P - \langle V_P, V_Q \rangle + V_P \wedge V_Q \tag{12}$$

where $\langle , \rangle$ is the inner product and $\wedge$ is the cross-product on $\mathbb{D}^3_1$. If $P = S_P + V_P$, then the conjugate of P is defined by $\alpha P = S_P - V_P$. By using this, the following properties can be easily proved:

$$\begin{array}{ll} (i) & \alpha(\alpha P) = P, \\ (ii) & \alpha(P \times Q) = \alpha Q \times \alpha P. \end{array}$$

For every $P, Q \in Q_{\mathbb{D},\nu}$, we define the symmetric dual-valued bilinear form $H_\nu : Q_{\mathbb{D},\nu} \times Q_{\mathbb{D},\nu} \longrightarrow \mathbb{D}$ as

$$H_1(P,Q) = \frac{1}{2}[\varepsilon(P)\varepsilon(\alpha Q)(P \times \alpha Q) + \varepsilon(Q)\varepsilon(\alpha P)(Q \times \alpha P)] \qquad \text{for } \mathbb{D}^3_1 \tag{13}$$

and

$$H_2(P,Q) = \frac{-1}{2}[\varepsilon(P)\varepsilon(\alpha Q)(P \times \alpha Q) + \varepsilon(Q)\varepsilon(\alpha P)(Q \times \alpha P)] \qquad \text{for } \mathbb{D}^4_2. \tag{14}$$

The following results may be obtained:

1- For all elements P, Q of $Q_{\mathbb{D},\nu}$, we have

$$H_\nu(P,Q) = h_\nu(p,q) + \xi[h_\nu(p,q^*) + h_\nu(p^*,q)]$$

where h is the symmetric real-valued bilinear form.

2- If $P = Ae_1 + Be_2 + Ce_3 + D$, then we have

$$H_\nu(P,P) = -A^2 - B^2 + C^2 + D^2.$$

3- $\forall\ P \in Q_{\mathbb{D},\nu}$, the norm of P is defined by

$$\|P\| = \|p\| + \xi\frac{h_\nu(p,p^*)}{\|p\|}$$

and so

$$\|P\|^2 = |H_\nu(P,P)| = |\varepsilon(P)(P \times \alpha P)|. \tag{15}$$

4- $\forall\ P \in Q_{\mathbb{D},\nu}$, the scalar part and the vector part of P are

$$S_P = \frac{1}{2}(P + \alpha P), \qquad V_P = \frac{1}{2}(P - \alpha P).$$

As a result,

$$\begin{array}{ll} (i) & \text{if } P + \alpha P = 0, \text{ then } P \in \mathbb{D} - module, \\ & \text{in this case, } P \text{ is called semi-dual-spatial quaternion} \\ (ii) & \text{if } P - \alpha P = 0, \text{ then } P \in \mathbb{D}, \\ & \text{in this case, } P \text{ is called semi-dual-temporal quaternion.} \end{array}$$

Let P and Q be two semi-dual spatial quaternion. If $H_\nu(P,Q) = 0$, we say that P and Q are $H_\nu-$orthogonal.

Now, we give the Serret-Frenet formulas for a non-null semi-dual quaternionic curve in $\mathbb{D}^3_1$.

Consider the smooth curve $\beta \subset \mathbb{D}^3_1$, $\{\beta \in Q_\nu \mid \beta + \alpha\beta = 0\}$ given by

$$\begin{array}{rcccl} \beta & : & I \subset \mathbb{R} & \longrightarrow & Q_\nu \subset \mathbb{D}^3_1 \\ & & s & \longrightarrow & \beta(s) = \sum\limits_{i=1}^{3} \beta_i(s)e_i. \end{array}$$

Let s be the parameter along β. For any $s \in I$, if $\{t(s),\, n_1(s),\, n_2(s)\}$ is the Serret-Frenet frame and $k(s)$, $r(s)$ are the curvatures, then we have the following formulas

$$\begin{array}{rcl} t' & = & \varepsilon(n_1)kn_1 \\ n_1' & = & \varepsilon(t)[\varepsilon(t)\varepsilon(n_1)rn_2 - kt] \\ n_2' & = & -\varepsilon(n_2)rn_1 \end{array} \tag{16}$$

where $t(s) = t + \xi t^*$, $n_1(s) = n_1 + \xi n_1^*$ and $n_2(s) = n_2 + \xi n_2^*$ with the Serret-Frenet frame $\{t(s), n_1(s), n_2(s)\}$ of $\mathbb{R}^3_1$.

If a curve is a non-null semi-dual quaternionic curve, then the Serret-Frenet formulas in $\mathbb{D}^4_2$ are defined as following :

Consider the smooth curve $\gamma \subset \mathbb{D}_2^4$,

$$\begin{array}{rcccl} \gamma & : & I & \longrightarrow & Q_{\mathbb{D},\nu} \subset \mathbb{D}_2^4 \\ & & s & \longrightarrow & \gamma(s) = \sum\limits_{i=1}^{4} \beta_i(s) e_i, \quad e_4 = 1. \end{array}$$

with $\beta_4(s)e_4 = D(s)$, $D(s) = d(s) + \xi d^*(s)$. For any $s \in I$, if $\{T(s),\ N_1(s),\ N_2(s),\ N_3(s)\}$ is the Serret-Frenet frame of dual-split quaternionic curve, then

$$\begin{array}{rcl} T' & = & \varepsilon(N_1) K N_1 \\ N_1' & = & \varepsilon(n_1) k N_2 - \varepsilon(N_1)\varepsilon(t) K T \\ N_2' & = & -\varepsilon(t) k N_1 + \varepsilon(n_1)[r - \varepsilon(T)\varepsilon(t)\varepsilon(N_1) K] N_3 \\ N_3' & = & -\varepsilon(n_2)[r - \varepsilon(T)\varepsilon(t)\varepsilon(N_1) K] N_2 \end{array} \tag{17}$$

where $T(s) = T + \xi T^*$, $N_1(s) = N_1 + \xi N_1^*$, $N_2(s) = N_2 + \xi N_2^*$ and $N_3(s) = N_3 + \xi N_3^*$ with the Serret-Frenet frame $\{T(s),\ N_1(s),$
$N_2(s),\ N_3(s)\}$ of $\mathbb{R}_2^4$ and $K = \varepsilon(N_1) \|T'\|$ [7].

3. THE INVOLUTES OF THE SEMI-DUAL CURVES IN $\mathbf{D}_1^3$

Definition 3.1. Let $M_1,\ M_2 \subset \mathbb{D}_1^3$ be two curves which are given by (I, β) and (I, β^*) coordinate neighbourhoods, respectively. Let Frenet frame of M_1 and M_2 be $\{t, n_1, n_2\}$ and $\{t^*, n_1^*, n_2^*\}$, respectively. M_2 is called the involute of M_1(M_1 is called the evolute of M_2) if

$$H_1(t, t^*) = 0. \tag{18}$$

Theorem 3.1. *Let $(M_1,\ M_2)$ be the involute-evolute curve couple which are given by (I, β) and (I, β^*) coordinate neighbourhoods, respectively. The distance between the points $\beta(s) \in M_1$ and $\beta^*(s^*) \in M_2$ is given by*

$$d(\beta(s), \beta^*(s)) = \varepsilon_0 \left| c - s \right|, \quad c\text{=}\textit{dual constant.}$$

Proof. If M_2 is the involute of M_1, we have

$$\beta^*(s) = \beta(s) + \lambda(s) t(s) \tag{19}$$

Let us derivate both side with respect to s:

$$\frac{d\beta^*}{ds} = \frac{d\beta}{ds} + \frac{d\lambda}{ds} t + \lambda \frac{dt}{ds} \tag{20}$$

Because of $\dfrac{dt}{ds} = t^{'} = \varepsilon_1 k n_1$,

$$\frac{d\beta^*}{ds} = (1 + \frac{d\lambda}{ds}) t + \lambda \varepsilon_1 k n_1 \tag{21}$$

where s and $s^*$are arc parameters of M_1 and M_2, respectively.

Thus we have

$$t^* \frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds}) t + \lambda \varepsilon_1 k n_1. \tag{22}$$

By using the equation (22), we have

$$H_1(t, t^*) \frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds}) H_1(t, t) + \lambda \varepsilon_1 k H_1(t, n_1). \tag{23}$$

From the definition of the involute-evolute curve couple, $H_1(t,t^*)=0$. Thus we obtain

$$1+\frac{d\lambda}{ds}=0 \text{ and } \lambda=c-s, \quad c=\text{dual constant.} \tag{24}$$

From the definition of the distance on Lorentzian space, we easily find

$$\begin{aligned} d(\beta(s),\,\beta^*(s)) &= \|\beta^*(s)-\beta(s)\| \\ &= \varepsilon_0\,|c-s|\,. \end{aligned} \tag{25}$$

□

Theorem 3.2. *Let $(M_1,\,M_2)$ be the involute-evolute curve couple which are given by (I,β) and (I,β^*) coordinate neighbourhoods, respectively. Let Frenet frames of M_1 and M_2 in the points $\beta(s)\in M_1$ and $\beta^*(s^*)\in M_2$ be $\{t,n_1,n_2\}$ and $\{t^*,n_1^*,n_2^*\}$, respectively. For the curvature and torsion of curve M_2, we have*

$$k^*=\frac{\varepsilon_1^*}{(c-s)k}\sqrt{|\varepsilon_0k^2+\varepsilon_2r^2|}.$$

Proof. If M_2 is the involute of M_1, we have

$$\beta^*(s)=\beta(s)+\lambda(s)t(s).$$

Let us derivate both side with respect to s. From equations (22) and (24), we obtain

$$t^*\frac{ds^*}{ds}=(c-s)\,\varepsilon_1kn_1 \tag{26}$$

where s and $s^*$are arc parameters of M_1 and M_2, respectively. We can find

$$\frac{ds^*}{ds}=\varepsilon_0^*\varepsilon_1\,(c-s)\,k. \tag{27}$$

Thus we have

$$t^*=\varepsilon_0^*n_1. \tag{28}$$

Hence $\{t^*(s),\,n_1(s)\}$ is linear dependent. That's why we consider that

$$t^*(s)=n_1(s). \tag{29}$$

By derivating t^* and using equations (16), (27) and (29), then we get

$$\varepsilon_1^*k^*n_1^*=\frac{\varepsilon_0^*\varepsilon_1}{(c-s)\,k}\left[\varepsilon_0[\varepsilon_0\varepsilon_1rn_2-kt]\right]. \tag{30}$$

Then, by the norm of the both side of the equation (30), we have

$$k^*=\frac{\varepsilon_1^*}{(c-s)k}\sqrt{|\varepsilon_0k^2+\varepsilon_2r^2|}. \tag{31}$$

□

Theorem 3.3. *Let $(M_1,\,M_2)$ be the involute-evolute curve couple which are given by (I,α) and (I,β) coordinate neighbourhoods,respectively. Let Frenet frames of M_1 and M_2 in the points $\beta(s)\in M_1$ and $\beta^*(s^*)\in M_2$ be $\{t,n_1,n_2\}$ and $\{t^*,n_1^*,n_2^*\}$,* *respectively, and let the curvature and torsion of curves M_1 and M_2 be k, r and k^*, r^*, respectively. We have*

$$\begin{aligned} n_1^* &= \frac{\varepsilon_1^*\varepsilon_0^*}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}\left[\varepsilon_0(\varepsilon_0\varepsilon_1 r n_2 - kt)\right] \\ n_2^* &= \frac{1}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}(\varepsilon_2 rt+\varepsilon_1 k n_2) \\ r^* &= \frac{\varepsilon_2(k' r - k r')}{(\varepsilon_2 r^2+\varepsilon_0 k^2)(c-s)k}. \end{aligned}$$

Proof. By using equation (30) and (31), we get

$$n_1^* = \frac{\varepsilon_1^*\varepsilon_0^*}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}\left[\varepsilon_0(\varepsilon_0\varepsilon_1 r n_2 - kt)\right]. \tag{32}$$

From $n_2 = \varepsilon_0\varepsilon_1\,(t\times n_1)$, we find

$$\begin{aligned} n_2^* &= \varepsilon_0^*\varepsilon_1^*\,(t^*\times n_1^*) \\ n_2^* &= \varepsilon_0^*\varepsilon_1^*\left(n_1\times\frac{\varepsilon_1^*\varepsilon_0^*}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}\left[\varepsilon_0(\varepsilon_0\varepsilon_1 r n_2 - kt)\right]\right) \\ n_2^* &= \frac{1}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2+\varepsilon_0 k^2)|}}(\varepsilon_2 rt+\varepsilon_1 k n_2). \end{aligned} \tag{3.1}$$

By derivating n_2^* and using this result in equation (16), we obtain

$$r^* = \frac{\varepsilon_0^*\varepsilon_1^*\varepsilon_2^*\varepsilon_2(k' r - k r')}{(\varepsilon_2 r^2+\varepsilon_0 k^2)(c-s)k}. \tag{34}$$

□

4. THE INVOLUTES OF THE SEMI-DUAL CURVES IN $\mathbb{D}_2^4$

Definition 4.1. Let M_1, $M_2 \subset \mathbb{D}_2^4$ be two curves which are given by (I,β) and (I,β^*) coordinate neighbourhoods, respectively. Let Frenet frame of M_1 and M_2 be $\{T, N_1, N_2, N_3\}$ and $\{T^*, N_1^*, N_2^*, N_3^*\}$, respectively. M_2 is called the involute of M_1(M_1 is called the evolute of M_2) if

$$H_2(T,T^*) = 0. \tag{35}$$

Theorem 4.1. *Let (M_1, M_2) be the involute-evolute curve couple which are given by (I,γ) and (I,γ^*) coordinate neighbourhoods, respectively. The distance between the points $\gamma(s)\in M_1$ and $\gamma^*(s^*)\in M_2$ is given by*

$$d(\gamma(s),\, \gamma^*(s)) = |c-s|\,, \quad c = \text{dual constant}.$$

Proof. If M_2 is the involute of M_1, we have

$$\gamma^*(s) = \gamma(s) + \lambda(s)T(s). \tag{36}$$

Let us derivate both side with respect to s:

$$\frac{d\gamma^*}{ds} = \frac{d\gamma}{ds} + \frac{d\lambda}{ds}T + \lambda\frac{dT}{ds}. \tag{37}$$

Because of $\dfrac{dT}{ds} = T' = \varepsilon(N_1)KN_1$,

$$\frac{d\gamma^*}{ds} = (1 + \frac{d\lambda}{ds})T + \lambda\varepsilon(N_1)KN_1 \tag{38}$$

where s and $s^*$are arc parameters of M_1 and M_2, respectively. Thus we have

$$T^*\frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds})T + \lambda\varepsilon(N_1)KN_1. \tag{39}$$

Taking inner product with t this equation's both side, we have

$$H_2(T, T^*)\frac{ds^*}{ds} = (1 + \frac{d\lambda}{ds})H_2(T, T^*) + \lambda\varepsilon_1 KH(T, N_1). \tag{40}$$

From the definition of the involute-evolute curve couple, $H_2(T, T^*) = 0$. Thus we obtain

$$1 + \frac{d\lambda}{ds} = 0 \text{ and } \lambda = c - s, \quad \text{c=dual constant.} \tag{41}$$

From the definition of the distance on Lorentzian space, we easily find

$$\begin{aligned} d(\gamma(s),\, \gamma^*(s)) &= \|\gamma^*(s) - \gamma(s)\| \\ &= |c - s|\,. \end{aligned} \tag{42}$$

□

Theorem 4.2. *Let $(M_1,\, M_2)$ be the involute-evolute curve couple which are given by (I, γ) and (I, γ^*) coordinate neighbourhoods, respectively. Let Frenet frames of M_1 and M_2 in the points $\gamma(s) \in M_1$ and $\gamma^*(s^*) \in M_2$ be $\{T, N_1, N_2, N_3\}$ and $\{T^*, N_1^*, N_2^*, N_3^*\}$, respectively. For the curvature and torsion of curve M_2, we have*

$$K^*(s^*) = \frac{|\varepsilon(N_1^*)|\sqrt{|\varepsilon(N_2)k^2 + \varepsilon(T)K^2|}}{(c - s)K}.$$

Proof. If M_2 is the involute of M_1, we have

$$\gamma^*(s) = \gamma(s) + \lambda(s)T(s).$$

Let us derivate both side with respect to s. From equations (39) and (41), we obtain

$$T^*\frac{ds^*}{ds} = \varepsilon(N_1)\,(c - s)\,KN_1 \tag{43}$$

where s and $s^*$are arc parameter of M_1 and M_2, respectively. We can find

$$\frac{ds^*}{ds} = |\varepsilon(T^*)|\,|(c - s)|\,K. \tag{44}$$

Thus we have

$$T^* = |\varepsilon(T^*)|\,\varepsilon(N_1)N_1. \tag{45}$$

Hence $\{T^*(s),\, N_1(s)\}$ is linear dependent. We consider that

$$T^*(s) = N_1(s). \tag{46}$$

By derivating T^* and using equations (17), (44) and (46), then we get

$$\varepsilon(N_1^*)K^*N_1^* = \frac{|\varepsilon(T^*)|}{|(c - s)|\,k}\left(\varepsilon_1 kN_2 - \varepsilon(N_1)\varepsilon_1 KT\right). \tag{47}$$

Then, by the norm of the both side of equation (47), we have

$$K^*(s^*) = \frac{|\varepsilon(N_1^*)| \sqrt{|\varepsilon(N_2)k^2 + \varepsilon(T)K^2|}}{(c-s)K}. \tag{48}$$

□

Theorem 4.3. *Let $(M_1,\, M_2)$ be the involute-evolute curve couple which are given by (I,γ) and (I,γ^*) coordinate neighbourhoods, respectively. Let Frenet frames of M_1 and M_2 in the points $\gamma(s) \in M_1$ and $\gamma^*(s^*) \in M_2$ be $\{T, N_1, N_2, N_3\}$ and $\{T^*, N_1^*, N_2^*, N_3^*\}$, respectively, and let the curvature and torsion of curves M_1 and M_2 be K, r, k and K^*, r^*, k ,respectively. we have*

$$\begin{aligned}
N_1^* &= \frac{\varepsilon(N_1^*)\,|\varepsilon(N_1)|}{\sqrt{|\varepsilon(N_2)k^2 + \varepsilon(T)K^2|}}\,(\varepsilon_1 k N_2 - \varepsilon(N_1)\varepsilon_1 K T)\\
N_2^* &= \frac{\varepsilon(T^*)}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2 + \varepsilon_0 k^2)|}}(\varepsilon_2\varepsilon_0 r N_2 + \varepsilon(T)kT)\\
N_3^* &= \frac{\varepsilon(T^*)\varepsilon_2\varepsilon_0}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2 + \varepsilon_0 k^2)|}}(-\varepsilon(T)T + \varepsilon_1 k N_2).
\end{aligned}$$

Proof. By using equations (47) and (48), we get

$$N_1^* = \frac{\varepsilon(N_1^*)\,|\varepsilon(N_1)|}{\sqrt{||\varepsilon(N_2)|\,k^2 + |\varepsilon(T)|\,K^2|}}\,(\varepsilon_1 k N_2 - \varepsilon(N_1)\varepsilon_1 K T)\,. \tag{49}$$

From equalities $N_2 = \varepsilon(T)\,(n_1 \times T)$, $n_2 \times N_1 = \varepsilon_1\varepsilon_2 N_2$ and $t \times N_1 = -\varepsilon(T)T$, we find

$$\begin{aligned}
N_2^* &= \varepsilon(T^*)\,(n_1^* \times T^*)\\
N_2^* &= \frac{\varepsilon(T^*)\varepsilon_1^*\varepsilon_0^*}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2 + \varepsilon_0 k^2)|}}(\varepsilon_2\varepsilon_0 r N_2 + \varepsilon(T)kT).50
\end{aligned} \tag{4.1}$$

If we use similar step as equation (50) and equality $N_3 = \varepsilon(T)\,(n_2 \times T)$, then

$$\begin{aligned}
N_3^* &= \varepsilon(T^*)\,(n_2^* \times T^*)\\
N_3^* &= \frac{\varepsilon(T^*)\varepsilon_2\varepsilon_0}{\sqrt{|\varepsilon_1^*(\varepsilon_2 r^2 + \varepsilon_0 k^2)|}}(-\varepsilon(T)T + \varepsilon_1 k N_2).
\end{aligned}$$

□

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