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TRIVARIATE FIBONACCI AND LUCAS POLYNOMIALS

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Abstract. In this article, we study the Trivariate Fibonacci and Lucas polynomials. The classical Tribonacci numbers and Tribonacci polynomials are the special cases of the trivariate Fibonacci polynomials. Also, we obtain some properties of the trivariate Fibonacci and Lucas polynomials. Using these properties, we give some results for the Tribonacci numbers and Tribonacci polynomials.

1. Introduction

In [4], the Tribonacci sequence originally was studied in 1963 by M. Feinberg. For any integer $n > 2$, the Tribonacci numbers T_n were defined by the recurrence relation

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}; \quad T_0 = 0,\ T_1 = 1,\ T_2 = 1.$$

In [2], the author derived the different recurrence relations on the Tribonacci numbers and their sums and got some identities of the Tribonacci numbers and their sums by using the companion matrices and generating matrices. In [5], the authors defined the generalized Tribonacci numbers and derived an explicit formula for the generalized Tribonacci numbers with negative subscripts. In [6], Lin obtained the Binet's formula and De Moivre types identities for the Tribonacci Numbers. In [7], the author got a formula for Tribonacci numbers by using an analytic method. In [8], the author obtained some identities for the Tribonacci numbers. Also, Pethe defined the complex Tribonacci numbers at Gaussian integers. In [10], Spickerman got the Binet's formula and generating function for the Tribonacci sequence and obtained an application for the Tribonacci numbers.

In [1], the authors got the Tribonacci Numbers from Tribonacci triangles and discussed the properties of functions related to Tribonacci Numbers. Also, Alladi

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and Hoggatt defined the Tribonacci triangle as follows

$n \diagdown i$	0	1	2	3	4	5	...
0	1						
1	1	1					
2	1	3	1				
3	1	5	5	1			
4	1	7	13	7	1		
5	1	9	25	25	9	1	
$\vdots$							

Table1: Tribonacci Triangle

It is interesting to note that, the sum of the elements on the rising diagonal lines in the Tribonacci triangle is $1, 1, 2, 4, 7, 13, 24, \ldots$which are the Tribonacci numbers.

In 1973, the Tribonacci polynomials was defined by Hoggatt and Bicknell [3]. For any integer $n > 2$, the recurrence relation of the Tribonacci polynomials is as follows

$$t_n(x) = x^2 t_{n-1}(x) + x t_{n-2}(x) + t_{n-3}(x)$$

where $t_0(x) = 0,\ t_1(x) = 1,\ t_2(x) = x^2$.

Some of Tribonacci polynomials are $0,\ 1,\ x^2,\ x^4 + x,\ x^6 + 2x^3 + 1,\ x^8 + 3x^5 + 3x^2,\ x^{10} + 4x^7 + 6x^4 + 2x,\ \ldots$. It's clear that $t_n(1) = T_n$, where T_n is $n - th$ Tribonacci number.

In [3], the authors gave the generating matrices for the Tribonacci, quadranacci and $r-$ bonacci polynomials. Also, they obtained the interesting determinantal properties for these polynomials. In [11], the authors defined the bivariate and trivariate Fibonacci polynomials and obtained the some properties of these polynomials.

There are different studies associated with the Tribonacci numbers and polynomials. One of them is incomplete Tribonacci numbers and polynomials in [9]. Ramirez and Sirvent defined the Tribonacci polynomial triangle as follows

$n \diagdown i$	0	1	2	3	4	5	...
0	1						
1	x^2	x					
2	x^4	$2x^3 + 1$	x^2				
3	x^6	$3x^5 + 2x^2$	$3x^4 + 2x$	x^3			
4	x^8	$4x^7 + 3x^4$	$6x^6 + 6x^3 + 1$	$4x^5 + 3x^2$	x^4		
5	x^{10}	$5x^9 + 4x^6$	$10x^8 + 12x^5 + 3x^2$	$10x^7 + 12x^4 + 3x$	$5x^6 + 4x^3$	x^5	
$\vdots$							

Table 2: Tribonacci Polynomial Triangle

In this study, based on the definition of Tan and Zhang [11], we make a new genaralization of the Tribonacci polynomials.

2. Trivariate Fibonacci and Lucas Polynomials

Definition 2.1. Let $n > 2$ be integer. The recurrence relation of the trivariate Fibonacci and Lucas polynomials are as follows

$$H_n(x, y, z) = xH_{n-1}(x, y, z) + yH_{n-2}(x, y, z) + zH_{n-3}(x, y, z) \tag{2.1}$$

with the initial conditions

$$H_0(x,y,z)=0,\quad H_1(x,y,z)=1,\quad H_2(x,y,z)=x$$

and

$$K_n(x,y,z)=xK_{n-1}(x,y,z)+yK_{n-2}(x,y,z)+zK_{n-3}(x,y,z) \tag{2.2}$$

with the initial conditions

$$K_0(x,y,z)=3,\quad K_1(x,y,z)=x,\quad K_2(x,y,z)=x^2+2y$$

respectively.

It is not difficult to see that $H_n(1,1,1)=T_n$, where T_n is $n-th$ Tribonacci number and $H_n\left(x^2,x,1\right)=t_n(x)$, where $t_n(x)$ is $n-th$ Tribonacci polynomial, are special cases of the trivariate Fibonacci polynomials.

The characteristic equation of the recurrences in (2.1) and (2.2) is as

$$\lambda^3-x\lambda^2-y\lambda-z=0. \tag{2.3}$$

The Binet's formula for the trivariate Fibonacci and Lucas polynomials are as follows

$$H_n(x,y,z)=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

and

$$K_n(x,y,z)=\alpha^n+\beta^n+\gamma^n$$

where α, β and γ are roots of the characteristic equation (2.3), respectively.

Now, we show that some of trivariate Fibonacci and Lucas polynomials in Table 3.

n	$H_n(x,y,z)$	$K_n(x,y,z)$
0	0	3
1	1	x
2	x	x^2+2y
3	x^2+y	$x^3+3xy+3z$
4	$x^3+2xy+z$	$x^4+4x^2y+4xz+2y^2$
5	$x^4+3x^2y+2xz+y^2$	$x^5+5x^3y+5xy^2+5x^2z+5yz$
6	$x^5+4x^3y+3xy^2+3x^2z+2yz$	$x^6+6x^4y+9x^2y^2+6x^3z+12xyz+2y^3+3z^2$
$\vdots$	$\vdots$	$\vdots$

Table 3:Trivariate Fibonacci and Lucas Polynomials

The generating functions of the trivariate Fibonacci and Lucas poynomials are as follows

$$h(t)=\sum_{n=0}^{\infty}H_n(x,y,z)\,t^n=\frac{t}{1-xt-yt^2-zt^3} \tag{2.4}$$

and

$$k(t)=\sum_{n=0}^{\infty}K_n(x,y,z)\,t^n=\frac{3-2xt-yt^2}{1-xt-yt^2-zt^3}. \tag{2.5}$$

Taking $x=y=z=1$ in (2.4), we obtain the generating function of the Tribonacci numbers. Writing x^2 instead of x, x instead of y and taking $z=1$ in (2.4), we have the generating function of the Tribonacci polynomials.

Theorem 2.1. *Let $H_n(x,y,z)$ and $K_n(x,y,z)$ be $n-th$ trivariate Fibonacci and Lucas polynomials, respectively. Then, we get*

$$K_n(x,y,z) = xH_n(x,y,z) + 2yH_{n-1}(x,y,z) + 3zH_{n-2}(x,y,z). \tag{2.6}$$

Proof. Using the generating function of the trivariate Lucas polynomials, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} K_n(x,y,z)\,t^n &= \frac{3-2xt-yt^2}{1-xt-yt^2-zt^3} \\
&= 3\frac{1}{1-xt-yt^2-zt^3} - 2x\frac{t}{1-xt-yt^2-zt^3} - y\frac{t^2}{1-xt-yt^2-zt^3} \\
&= 3\sum_{n=0}^{\infty} H_{n+1}(x,y,z)\,t^n - 2x\sum_{n=0}^{\infty} H_n(x,y,z)\,t^n - y\sum_{n=0}^{\infty} H_{n-1}(x,y,z)\,t^n \\
&= \sum_{n=0}^{\infty} \left(3H_{n+1}(x,y,z) - 2xH_n(x,y,z) - yH_{n-1}(x,y,z)\right) t^n.
\end{aligned}$$

From the recurrence relation in (2.1), we can write

$$\sum_{n=0}^{\infty} K_n(x,y,z)\,t^n = \sum_{n=0}^{\infty} \left(xH_n(x,y,z) + 2yH_{n-1}(x,y,z) + 3zH_{n-2}(x,y,z)\right) t^n.$$

Comparing of the coefficients of t^n, we have the desired result. □

Theorem 2.2. *The sum of the trivariate Fibonacci and Lucas polynomials are as follows*

$$\sum_{s=0}^{n} H_s(x,y,z) = \frac{H_{n+2}(x,y,z) + (1-x)H_{n+1}(x,y,z) + zH_n(x,y,z) - 1}{x+y+z-1} \tag{2.7}$$

and

$$\sum_{s=0}^{n} K_s(x,y,z) = \frac{K_{n+2}(x,y,z) + (x-1)K_{n+1}(x,y,z) + zK_n(x,y,z) - (3-2x-y)}{x+y+z-1} \tag{2.8}$$

for $x+y+z \neq 1$, respectively.

Proof. Using the Binet's formulas, it can be proved. □

Taking $x=y=z=1$ in (2.7), we have the sum of the Tribonacci numbers as

$$\sum_{s=0}^{n} T_s = \frac{T_{n+2} + T_n - 1}{2}.$$

Similarly, we obtain the sum of the Tribonacci polynomials as

$$\sum_{s=0}^{n} t_s(x) = \frac{t_{n+2}(x) + \left(1-x^2\right)t_{n+1}(x) + t_n(x) - 1}{x^2+x}.$$

Similar to Table 1 and Table 2, we can give the trivariate Fibonacci polynomial triangle as follows

$n \diagdown i$	0	1	2	3	4	$\ldots$
0	1					
1	x	y				
2	x^2	$2xy+z$	y^2			
3	x^3	$3x^2y+2xz$	$3xy^2+2yz$	y^3		
4	x^4	$4x^3y+3x^2z$	$6x^2y^2+6xyz+z^2$	$4xy^3+3y^2z$	y^4	
$\vdots$						

Table 4: Trivariate Fibonacci Polynomial Triangle

$G(n,i,x,y,z)$ is the element in the $n-th$ row and $i-th$ column of the trivariate Fibonacci polynomial triangle. Then, we get

$$G(n,i,x,y,z)=\sum_{j=0}^{i}\binom{i}{j}\binom{n-j}{i}x^{n-i-j}y^{i-j}z^{j} \tag{2.9}$$

and

$$G(n+1,i,x,y,z)=xG(n,i,x,y,z)+yG(n,i-1,x,y,z)+zG(n-1,i-1,x,y,z)$$

where

$$G(n,0,x,y,z)=x^{n},\quad G(n,n,x,y,z)=y^{n}.$$

The sum of elements on the rising diagonal lines in the trivariate Fibonacci polynomial triangle is the trivariate Fibonacci polynomial $H_n(x,y,z)$. Thus, we have

$$H_n(x,y,z)=\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}G(n-i-1,i,x,y,z).$$

Consequently, we obtain an explicit formula for the trivariate Fibonacci polynomial $H_n(x,y,z)$ as

$$H_n(x,y,z)=\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}x^{n-2i-j-1}y^{i-j}z^{j}. \tag{2.10}$$

Taking $x=y=z=1$ in (2.10), we obtain the explicit formula for the Tribonacci numbers as

$$H_n(1,1,1)=\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}$$

Also, we have

$$H_n\left(x^2,x,1\right)=\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}x^{2n-3(i+j)-2}$$

which is the explicit formula for the Tribonacci polynomials in [9].

Similarly, we have an explicit formula for the trivariate Lucas polynomials as follows

$$K_n(x,y,z)=\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\sum_{j=0}^{i}\frac{n}{n-i-j}\binom{i}{j}\binom{n-i-j}{i}x^{n-2i-j}y^{i-j}z^{j}. \tag{2.11}$$

Theorem 2.3. *Let $H_n(x,y,z)$ and $K_n(x,y,z)$ be $n-th$ trivariate Fibonacci and Lucas polynomials, respectively. Then, we get*

$$x\frac{\partial K_n(x,y,z)}{\partial x}+y\frac{\partial K_n(x,y,z)}{\partial y}+z\frac{\partial K_n(x,y,z)}{\partial z}=nH_{n+1}(x,y,z).$$

Proof. Using partial derivations of the explicit formula of the trivariate Lucas polynomial $K_n(x,y,z)$, we have

$$\begin{aligned}\frac{\partial K_n(x,y,z)}{\partial x} &= \frac{\partial}{\partial x}\left(\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{i}\frac{n}{n-i-j}\binom{i}{j}\binom{n-i-j}{i}x^{n-2i-j}y^{i-j}z^{j}\right)\\ &= \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{j=0}^{i}\frac{n}{n-i-j}(n-2i-j)\binom{i}{j}\binom{n-i-j}{i}x^{n-2i-j-1}y^{i-j}z^{j}\\ &= n\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j-1}{i}x^{n-2i-j-1}y^{i-j}z^{j}\\ &= nH_n(x,y,z).\end{aligned}$$

Similarly, we obtain

$$\frac{\partial K_n(x,y,z)}{\partial y}=nH_{n-1}(x,y,z)$$

and

$$\frac{\partial K_n(x,y,z)}{\partial z}=nH_{n-2}(x,y,z).$$

Using the recurrence relation (2.1), we have

$$x\frac{\partial K_n(x,y,z)}{\partial x}+y\frac{\partial K_n(x,y,z)}{\partial y}+z\frac{\partial K_n(x,y,z)}{\partial z}=nH_{n+1}(x,y,z).$$

□

The generating matrix of the Tribonacci polynomials was introduced in [3, 4]. Similarly, the trivariate Fibonacci polynomials are generated by the matrix Q,where

$$Q=\begin{pmatrix} x & 1 & 0\\ y & 0 & 1\\ z & 0 & 0\end{pmatrix}$$

with the help of mathematical induction on n, we get

$$Q^n=\begin{pmatrix} H_{n+1} & H_n & H_{n-1}\\ yH_n+zH_{n-1} & yH_{n-1}+zH_{n-2} & yH_{n-2}+zH_{n-3}\\ zH_n & zH_{n-1} & zH_{n-2}\end{pmatrix},$$

where H_n is $n-th$ trivariate Fibonacci polynomial, namely $H_n(x,y,z)=H_n$.

Theorem 2.4. *Let m and n be positive integers. Then, we get*

$$\begin{aligned}H_{m+n}(x,y,z) &= H_{m+1}(x,y,z)H_n(x,y,z)+H_m(x,y,z)H_{n+1}(x,y,z)\\ &\quad +zH_{m-1}(x,y,z)H_{n-1}(x,y,z)\\ &\quad -xH_m(x,y,z)H_n(x,y,z). \qquad (2.12)\end{aligned}$$

Proof. It can be proved by using the identity $Q^{n+m}=Q^nQ^m$ and matrix equality.

□

The identity in (2.12) is similar to Honsberger formula for the Fibonacci like sequences. From the special cases of (2.12), we obtain some identities for the trivariate Fibonacci polynomials. Therefore, taking $m = n$ in (2.12), we have

$$H_{2n}(x,y,z) = zH_{n-1}^{2}(x,y,z) - xH_{n}^{2}(x,y,z) + 2H_{n+1}(x,y,z)\,H_{n}(x,y,z)$$

Writing $n+1$ instead of m in (2.12), and using the recurennce relation in (2.1), we obtain

$$H_{2n+1}(x,y,z) = H_{n+1}^{2}(x,y,z) + yH_{n}^{2}(x,y,z) + 2zH_{n}(x,y,z)\,H_{n-1}(x,y,z)\,.$$

Theorem 2.5. *Let $H_n(x,y,z)$ be $n-th$ trivariate Fibonacci polynomial. Then, we get*

$$\left|\begin{array}{ccc} H_{n+2}(x,y,z) & H_{n+1}(x,y,z) & H_{n}(x,y,z) \\ H_{n+1}(x,y,z) & H_{n}(x,y,z) & H_{n-1}(x,y,z) \\ H_{n}(x,y,z) & H_{n-1}(x,y,z) & H_{n-2}(x,y,z) \end{array}\right| = -z^{n-1}. \tag{2.13}$$

Proof. It's note that $\det(Q) = z$, $\det(Q^n) = z^n$. Using the determinants of the matrices Q and Q^n , we obtain

$$\left|\begin{array}{ccc} H_{n+1} & H_{n} & H_{n-1} \\ yH_{n} + zH_{n-1} & yH_{n-1} + zH_{n-2} & yH_{n-2} + zH_{n-3} \\ zH_{n} & zH_{n-1} & zH_{n-2} \end{array}\right| = z^{n}.$$

Multiplying the first row of Q^n by x and then adding to second row, then, exchanging rows 1 and 2, we have

$$\left|\begin{array}{ccc} H_{n+2}(x,y,z) & H_{n+1}(x,y,z) & H_{n}(x,y,z) \\ H_{n+1}(x,y,z) & H_{n}(x,y,z) & H_{n-1}(x,y,z) \\ zH_{n}(x,y,z) & zH_{n-1}(x,y,z) & zH_{n-2}(x,y,z) \end{array}\right| = -z^{n}$$

From the properties of determinant, we obtain

$$\left|\begin{array}{ccc} H_{n+2}(x,y,z) & H_{n+1}(x,y,z) & H_{n}(x,y,z) \\ H_{n+1}(x,y,z) & H_{n}(x,y,z) & H_{n-1}(x,y,z) \\ H_{n}(x,y,z) & H_{n-1}(x,y,z) & H_{n-2}(x,y,z) \end{array}\right| = -z^{n-1}.$$

□

In this way, we obtain the interesting determinantal property for the trivariate Fibonacci polynomials. The result of the determinant in (2.13) is similar to the Cassini like identity for the trivariate Fibonacci polynomials. Taking $x = y = z = 1$ in (2.13), we obtain the determinantal property for the Tribonacci numbers. Writing x^2 instead of x, x instead of y and taking $z = 1$, we have the determinantal property for the Tribonacci polynomials in [3].

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