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HYERS-ULAM-RASSIAS TYPE STABILITY OF POLYNOMIAL EQUATIONS

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Abstract. In this paper we introduce the concept of Hyers-Ulam-Rassias stability of polynomial equations and then we show that if x is an approximate solution of the equation $a_nx^n + a_{n-1}x^{n-1} + ...a_1x + a_0$, then there exists an exact solution of the equation near to x.

1. Introduction

The basic problem of the stability of functional equations had been first raised by Ulam [7] which Hyers in [3] gave a partial solution of Ulam's problem for the case of approximately additive mappings. And then Rassias provided a generalization of the Hyers'theorem for additive and linear mappings in [6].

Moreover the approximately mappings have been studied extensively in several papers (See for instance [4], [5]).

Li and Hua [2] investigated the Hyers-Ulam stability of the polynomial equation $x^n + \alpha x + \beta = 0$ on $[-1, 1]$. Later Bikhdam et al. in [1] proved that if $|a_1|$ is large and $|a_0|$ is small enough, then every approximate zero of the polynomial of degree n, $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a = 0$, can be approximated by a true zero within a good error bound.

In this paper, we prove the Hyers-Ulam-Rasssias stability for the following two equations

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a = 0$$

$$e^x + \alpha x + \beta = 0$$

on a Banach space X with real coefficients.

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2. **Preliminaries**

In this section, we provide a collection of definitions and related results which are essential and used in the next discussions.

Definition 2.1. One says that the equation

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a = 0$$

has the Hyrs-Ulam stability if there exists a constant $K > 0$ with the following property:

for every $\varepsilon > 0$, $y \in [-1, 1]$, if

$$|a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0| \leq \varepsilon$$

then there exists some $y \in [-1, 1]$ satisfying

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a = 0$$

such that $|x - y| \leq K\varepsilon$. One call such K a Hyers-Ulam stability constant for 2.1.

Theorem 2.1. *For a given integer $n > 1$, let the constants $a_0, a_1, ..., a_n \in \mathbb{R}$ satisfy $|a_1| > 2|a_2|+3|a_3|+...+(n-1)|a_{n-1}|+n|a_n|$ and $|a_0| < |a_1|-(|a_2|+|a_3|+...+|a_n|)$.*

If $v \in [-1, 1]$ satisfies the inequality

$$|a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0| \leq \varepsilon$$

for some $\varepsilon > 0$, then there exists a zero $y \in [-1, 1]$ of polynomial 2.1 such that

$$|y - v| \leq K\varepsilon.$$

Proof. [1] □

3. **Hyers-Ulam-Rassias Stability of power series equations**

We start our work with definition of Hyers-Ulam-Rassias stability of power series equations.

Definition 3.1. Let X be a complex Banach algebra with unit. The equation

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a = 0$$

from X into X with constant coefficient, has the Hyers-Ulam-Rassias stability if there exists a constant $K > 0$ with the following property:

for given $\varepsilon > 0$, $p \in \mathbb{R}$ and $y \in X$, if

$$|a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0| \leq \varepsilon||X||^{np}$$

then there exists some $x \in X$ satisfying

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a = 0$$

such that $||x - y|| \leq K\varepsilon$. One call such K a Hyers-Ulam-Rassias stability constant for 2.1.

Theorem 3.1. *Let X be a complex Banach algebra with unit, the constants $a_0, a_1, ..., a_n \in \mathbb{R}$ and $r \in \mathbb{R}^+$ satisfy*

$$|a_1|r > |a_0| + r^2|a_2| + ... + nr^{n-1}|a_n|,$$

and

$$|a_0| < r^2|a_2| + 3r^3|a_3| + ... + (n-1)r^n|a_n|).$$

If a $x \in \{x \in X; ||x|| \leq r\}$ *satisfies the inequality*

$$||a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0|| \leq \varepsilon ||x||^{np} \tag{3.1}$$

for some $\varepsilon > 0$ *and* $p \in \mathbb{R}$, *then there exists a zero* $x_0 \in \{x \in X; ||x|| \leq r\}$ *of polynomial 2.1 such that*

$$||x - x_0|| \leq \frac{\varepsilon r^{np}}{(1-\lambda)|a_1|}$$

where $\lambda = \lambda = \frac{2|a_2|+3|a_3|+...+(n-1)|a_{n-1}|+n|a_n|}{|a_1|}$ *is a positive constant less than* 1 *and it is independent of* ε *and* x_0.

Proof. If we set $g(x) = \frac{-1}{a_1}(a_0 + a_2x^2 + a_3x^3 + ... + a_{n-1}x^{n-1} + a_n x^n)$, for $x \in \{x \in X; ||x|| \leq r\}$, then we have

$$||g(x)|| = \frac{1}{|a_1|}||a_0 + a_2x^2 + ... + a_{n-1}x^{n-1} + a_n x^n|| \leq r.$$

Now, we show that g is a contraction map:

$$||g(x) - g(y)|| = ||\frac{1}{a_1}(a_0 + a_2x^2 + ... + a_n x^n) - \frac{1}{a_1}(a_0 + ... + a_n y^n)|| \leq \tag{3.2}$$

$$\frac{1}{|a_1|}||x-y||\{|a_2|||x+y|| + ... + |a_n|||x^{n-1} + ... + y^{n-1}||\} \leq$$
$$\frac{1}{|a_1|}||x-y||\{2r|a_2| + 3r^2|a_3| + ... + nr^{n-1}|a_n|\}.$$

Here, with $\lambda = \frac{2|a_2|+3|a_3|+...+(n-1)|a_{n-1}|+n|a_n|}{|a_1|} < 1$, g is a contraction map. By the Banach contraction mapping theorem, there exists a unique $x_0 \in \{x \in X; ||x|| \leq 1\}$ such that $g(x_0) = x_0$. It follows from (3.1) and (3.2) that

$$||x - x_0|| \leq ||x - g(x)|| + ||g(x) - g(x_0)|| \leq ||x - \frac{1}{a_1}(-a_0 - a_2x^2 - ... - a_1x + a_0|| +$$
$$\lambda||x - x_0|| = \frac{1}{|a_1|}||a_n x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0|| + \lambda||x - x_0||.$$

Thus we have

$$||x - x_0|| \leq \frac{1}{|a_1|(1-\lambda)}\varepsilon||x||^{np} \leq \frac{\varepsilon r^{np}}{(1-\lambda)|a_1|}.$$

□

Theorem 3.2. *Let* X *be a complex Banach algebra with unit,* $r \in \mathbb{R}^+$, $|\alpha| > Max\{\sum_{n=1}^{\infty} \frac{nr^n}{n!}, \frac{e^r + |\beta|}{r}\}$ *and* $x \in \{x \in X; ||x|| \leq r\}$ *satisfies*

$$||e^x + \alpha x + \beta|| \leq \varepsilon||x||^{np}.$$

Then $e^x + \alpha x + \beta = 0$ *has a unique solution* $x_0 \in \{x \in X; ||x|| \leq r\}$, *such that* $||x - x_0|| \leq \frac{\varepsilon r^{np}}{|\alpha|(1-\lambda)}$, *where* $\lambda = \frac{\sum_{n=1}^{\infty} \frac{nr^n}{n!}}{|\alpha|}$.

Proof. We define the function $g(x) = \frac{-1}{\alpha}(e^x + \beta)$. It follows that $||g(x)|| \leq \frac{1}{|alpha|}(e^r + |\beta|) \leq r$. Now, we have

$$||g(x) - g(y)|| \leq \frac{1}{|\alpha|}||e^x - e^y|| \leq \frac{1}{|\alpha|}\sum_{n=1}^{\infty} \frac{x^n - y^n}{n!} \leq$$
$$\frac{1}{|\alpha|}\sum_{n=2}^{\infty} \frac{1}{n!}||x-y||||x^{n-1} + x^{n-2}y + ... + xy^{n-2} + y^{n-1}|| \leq \frac{\sum_{n=1}^{\infty} \frac{nr^{n-1}}{n!}}{|\alpha|}||x-y||.$$

By putting $\lambda = \frac{\sum_{n=1}^{\infty} \frac{nr^{n-1}}{n!}}{|\alpha|}$ and Banach's contraction mapping theorem, g has a unique fixed point. So

$$||x - x_0|| \leq ||x - g(x)|| + ||g(x) - g(x_0)|| \leq \frac{1}{|\alpha|}\varepsilon||x||^{np} + \lambda||x - x_0||.$$

Therefore $||x - x_0|| \leq \frac{\varepsilon r^{np}}{|\alpha|(1-\lambda)}$. □

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