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Univalence of certain integral operators involving generalized Struve functions

Muhey U Din*, Hari Mohan Srivastava† and Mohsan Raza‡§

Abstract

In this paper, we are mainly interested to find sufficient conditions for some integral operators defined by generalized Struve functions. These operators are normalized and as well as univalent in the open unit disc $\mathcal{U}$. Some special cases of Struve functions and modified Struve functions are also a part of our investigations.

Keywords: Univalence conditions, Integral operators, Generalized Struve functions, Modified Struve functions, Ahlfors-Becker and Becker univalence criteria, Schwarz lemma.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ denotes the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

*Department of Mathematics, Government College University Faisalabad, Pakistan
Email : muheyudin@yahoo.com

†Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada; Department of Medical Research, China Medical University Hospital, China Medical University
Email : harimsri@math.uvic.ca

‡Department of Mathematics, Government College University Faisalabad, Pakistan
Email : mohsan976@yahoo.com

§Corresponding Author.

The Struve functions H_v and L_v appeared as special solutions of the second order inhomogeneous differential equations of the form

$$z^2w''(z)+zw(z)+(z^2-v^2)w(z)=\frac{4\left(\frac{1}{2}z\right)^{v+1}}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}, \tag{1.2}$$

$$z^2w''(z)+zw(z)-(z^2-v^2)w(z)=\frac{4\left(\frac{1}{2}z\right)^{v+1}}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}, \tag{1.3}$$

known as inhomogeneous Bessel differential equations. Both equations (1.2) and (1.3) are similar and can be converted into each other by changing z into iz. In the solution of equation (1.2), a function appeared in an article by Struve [31], was later ascribed Struve's name and the special notation H_v. It is defined as

$$H_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{z}{2}\right)^{2n+v+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+v+\frac{3}{2}\right)}. \tag{1.4}$$

The modified Struve functions L_v of order v was introduced by J. W. Nicholson in 1911. It is defined as

$$L_v(z)=-ie^{-iv\frac{\pi}{2}}H_v(iz)=\sum_{n=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2n+v+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+v+\frac{3}{2}\right)}, \tag{1.5}$$

where $\Gamma(z)$ is the gamma function. Applications of Struve functions occur in water-wave and surface-wave problems, unsteady aerodynamics, resistive MHD instability theory and optical diffraction. More recently, Struve functions have appeared in many particle quantum dynamical studies of spin decoherence and nanotubes. For some details see [1, 23].

Now consider the second order inhomogeneous differential equation

$$z^2w''(z)+bzw'(z)+\left[cz^2-v^2+(1-b)v\right]w(z)=\frac{4\left(\frac{z}{2}\right)^{v+1}}{\sqrt{\pi}\Gamma\left(v+\frac{b}{2}\right)}, \tag{1.6}$$

where $b,\ c,\ v\in\mathbb{C}$. The equation (1.6) generalizes the equation (1.2) and (1.3). In particular for $b=1,\ c=1$, we obtain (1.2). For $b=1,\ c=-1$ we get (1.3). Its particular solution has the series form

$$w_v(z)=\sum_{n=0}^{\infty}\frac{(-1)^n c^n\left(\frac{1}{2}z\right)^{2n+v+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+v+\frac{b+2}{2}\right)}. \tag{1.7}$$

It is known as generalized Struve functions of order v. Consider the transformation

$$\begin{aligned} u_{v,b,c}(z) &= 2^v\sqrt{\pi}\Gamma(v+(b+2)/2)z^{(-v-1)/2}w_{v,b,c}(\sqrt{z}) \\ &= \sum_{k=0}^{\infty}\frac{(-c/4)^n z^n}{(3/2)_n\,(k)_n}, \end{aligned} \tag{1.8}$$

where $k=v+(b+2)/2\neq 0,-1,-2,-3,...$and

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases} 1, & n=0,\ \gamma\in\mathbb{C}\backslash\{0\}, \\ \gamma(\gamma+1)\ldots(\gamma+n-1), & n\in\mathbb{N},\ \gamma\in\mathbb{C}. \end{cases}$$

The function $u_{v,b,c}$ is analytic in $\mathcal{U}$ and is the solution of the differential equation

$$4z^2u''(z)+2(2v+b+3)zu'(z)+(cz+2v+b)u(z)=2v+b. \tag{1.9}$$

The function $u_{v,b,c}$ is introduced and studied by Orhan and Yagmur [24] and further investigated in [3, 30, 32].

Recently, many mathematicians have set the univalence criteria of several those integral operators which preserve the class $\mathcal{S}$. By using a variety of different analytic techniques, operators and special functions, several authors have studied univalence criterion, a few of them are as mentioned below.

Kanas and Srivastava [20], and Deniz and Orhan [13, 14] studied univalence criteria for analytic functions defined in $\mathcal{U}$ by using the Loewner chains method. Kiryakova, Saigo and Srivastava [21] obtained some univalence criteria for certain generalized fractional integral and derivatives, accompanying all the linear integro-differential operators. In 2010 Baricz and Frasin [4] studied some integral operators involving Bessel functions. These integral operators were defined by using the normalized Bessel functions of the first kind. Frasin [18] and Arif and Raza [2] studied the convexity and strongly convexity of the integral operators defined in [4]. Recently Deniz et al. [16] and Deniz [10] studied the integral operator defined by generalized Bessel functions of order v. For further details of these univalence criterion, we refer the readers to [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 25, 26, 27, 28]. Motivated by the work of these above authors, we contribute to this univalence theory by studying the univalence of integral operators involving generalized Struve functions. These operators are defined as follows:

$$\mathcal{W}_{\alpha_1,\ldots,\alpha_n,\beta}(z)=\left[\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{u_{v_i,b,c}(z)}{t}\right)^{\frac{1}{\alpha_i}}dt\right]^{\frac{1}{\beta}}, \tag{1.10}$$

$$\mathcal{X}_{n,\gamma}(z)=\left[(n\gamma+1)\int_0^z\prod_{i=1}^{n}\left\{u_{v_i,b,c}(z)\right\}^{\gamma}dt\right]^{\frac{1}{n\gamma+1}}, \tag{1.11}$$

and

$$\mathcal{Z}_{\lambda}(z)=\left[\lambda\int_0^z t^{\lambda-1}\left(e^{u_{v_i,b,c}(z)}\right)^{\lambda}dt\right]^{1/\lambda}. \tag{1.12}$$

In order to derive our main results, we need the following lemmas.

1.1. Lemma. [26] *Let β and d be complex numbers such that $Re(\beta)>0$ and $|d|\leq 1$, where $d\neq -1$. If the function $f\in\mathcal{A}$ satisfies the following inequality*

$$\left|d|z|^{2\beta}+\left(1-|z|^{2\beta}\right)\frac{zf''(z)}{\beta f'(z)}\right|\leq 1 \qquad (z\in\mathcal{U}),$$

then the function $\mathcal{F}_\beta$ defined by

$$\mathcal{F}_\beta(z)=\left(\beta\int_0^z t^{\beta-1}f'(t)dt\right)^{1/\beta}, \tag{1.13}$$

is in class $\mathcal{S}$ of normalized univalent functions in $\mathcal{U}$.

1.2. Lemma. [29] *If $f\in\mathcal{A}$ satisfies the following inequality*

$$\left(\frac{1-|z|^{2Re(\alpha)}}{Re(\alpha)}\right)\left|\frac{zf''(z)}{f'(z)}\right|\leq 1 \qquad (Re(\alpha)>0),$$

for all $\beta\in\mathbb{C}$ such that $Re(\beta)\geq Re(\alpha)$, then the function $\mathcal{F}_\beta$ defined by (1.13) is in the class $\mathcal{S}$ of normalized univalent functions in $\mathcal{U}$.

1.3. Lemma. [25] *Let the parameter $\lambda \in \mathbb{C}$ and $\theta \in \mathbb{R}$ be so constrained that $Re(\lambda) \geq 1$, $\theta > 1$ and $2\theta|\lambda| \leq 3\sqrt{3}$. If the function $q \in \mathcal{A}$ satisfies the following inequality*

$$|zq'(z)| \leq \theta,$$

then the function $\mathcal{Q}_\lambda : \mathcal{U} \to \mathbb{C}$ defined by

$$\mathcal{Q}_\lambda(z) = \left(\lambda \int_0^z t^{\lambda-1} \left(e^{q(t)}\right)^\lambda dt\right)^{1/\lambda},$$

is in the class $\mathcal{S}$ of normalized univalent functions in $\mathcal{U}$.

1.4. Lemma. [22] *(Schwarz lemma) Let $f(z)$ be the function regular in the open unit disk $\mathcal{U}$ with $|f(z)| < M$, M is fixed. If $f(z)$ has one zero with multiplicity order greater than m for $z = 0$, then*

$$|f(z)| < \frac{M|z|^m}{R^m} \qquad (z \in \mathcal{U}), \tag{1.14}$$

the equality can hold only if $f(z) = e^{i\theta}\frac{Mz^m}{R^m}$, where θ is constant.

2. Inequalities involving in our Main Results

2.1. Lemma. [24] *If $b, v \in \mathbb{R}$, and $c \in \mathbb{C}$, $k = v + \frac{b+2}{2}$ are so constrained that*

$$k > \max\left\{0, \frac{7|c|}{24}\right\},$$

then the function $u_{v,b,c} : \mathcal{U} \to \mathbb{C}$ defined by (1.8) satisfies the following inequalities:

i. $\left|u'_{v,b,c}(z) - \frac{u_{v,b,c}(z)}{z}\right| \leq \frac{2|c|}{3(4k-|c|)} \qquad (z \in \mathcal{U}),$

ii. $\left|\frac{u_{v,b,c}(z)}{z}\right| \geq \frac{6k-2|c|}{6k-|c|} \qquad (z \in \mathcal{U}),$

iii. $\left|\frac{zu'_{v,b,c}(z)}{u_{v,b,c}(z)} - 1\right| \leq \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)} \qquad (z \in \mathcal{U}),$

iv. $\left|zu'_{v,b,c}(z)\right| \leq \frac{12k+|c|}{12k-3|c|} \qquad (z \in \mathcal{U}).$

Proof. For the proof of the first three inequalities, see [24]. In order to prove the assertion (iv) of Lemma 2.1, we will use the well-known triangle inequality and the following inequalities

$$\begin{aligned} (3/2)_n &\geq \frac{3(n+1)}{4} \qquad (n \in \mathbb{N}), \\ (k)_n &\geq k^n \qquad (n \in \mathbb{N}). \end{aligned}$$

Now, consider that

$$\begin{aligned} \left|zu'_{v,b,c}(z)\right| &= \left|z + \sum_{n=1}^{\infty} \frac{(-c)^n (n+1) z^{n+1}}{(3/2)_n 4^n (k)_n}\right| \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{|c|^n (n+1)}{(3/2)_n 4^n k^n} \\ &\leq 1 + \frac{|c|}{3k} \sum_{n=1}^{\infty} \left(\frac{|c|}{4k}\right)^{n-1} \\ &= \frac{12k+|c|}{12k-3|c|} \qquad \left(k > \frac{|c|}{4}\right). \end{aligned}$$

□

3. Univalence of Integral Operators Involving the Generalized Struve Functions

The inequalities established in Lemma 2.1, will be used to find the sufficient conditions for univalence of the integral operators defined in (1.10) when the functions $u_{v_i,b,c}(z)$ $(i=1,2,...,n)$ belong to the class $\mathcal{A}$ and the parameters $\alpha_i \in \mathbb{C}/\{0\}$ $(i=1,2,...,n)$ and $\beta \in \mathbb{C}$ are so constrained that the integral operator in (1.10) is well defined.

3.1. Theorem. *Let $v_1,\ldots,v_n$, $b\in\mathbb{R}$, $c\in\mathbb{C}$ and $k_i > \frac{7|c|}{24}$ with $k_i = v_i + (b+2)/2$, $i=1,\ldots,n$. Let $u_{v_i,b,c}:\mathcal{U}\to\mathbb{C}$ be defined as*

$$u_{v_i,b,c}(z) = 2^{v_i}\sqrt{\pi}\,\Gamma\left(v_i+\frac{b+2}{2}\right) z^{\frac{(-1-v_i)}{2}} w_{v_i,b,c}(\sqrt{z}).$$

Suppose $k=\min\{k_1,k_2,\ldots,k_n\}$, $Re(\beta)>0, c\in\mathbb{C}/\{-1\}$ and $\alpha_i\in\mathbb{C}/\{0\}$ $(i=1,2,...,n)$ and these numbers satisfy the relation

$$|d| + \frac{|c|\,(6k-|c|)}{3\,(4k-|c|)\,(3k-|c|)}\sum_{i=1}^{n}\frac{1}{|\beta\alpha_i|} \le 1,$$

then the function $\mathcal{W}_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,\beta}:\mathcal{U}\to\mathbb{C}$ defined by (1.10) is in the class $\mathcal{S}$ of normalized univalent functions in $\mathcal{U}$.

Proof. By setting $\beta=1$ in (1.10) so that

$$\mathcal{W}_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{u_{v_i,b,c}(t)}{t}\right)^{\frac{1}{\alpha_i}} dt.$$

First of all, we observe that, since $u_{v_i,b,c}(z)\in\mathcal{A}\quad(i=1,2,...,n)$,

$$u_{v_i,b,c}(0) = u'_{v_i,b,c}(0) - 1 = 0,$$

therefore, we observe that $\mathcal{W}_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(z)\in\mathcal{A}$, that is,

$$\mathcal{W}_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(0) = \mathcal{W}'_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(0) - 1 = 0.$$

On the other hand, it is easy to see that

$$\mathcal{W}'_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(z) = \prod_{i=1}^{n}\left(\frac{u_{v_i,b,c}(z)}{z}\right)^{\frac{1}{\alpha_i}} \tag{3.1}$$

From (3.1) , we obtain

$$\frac{z\mathcal{W}''_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(z)}{\mathcal{W}'_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(z)} = \sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{zu'_{v_i,b,c}(z)}{u_{v_i,b,c}(z)} - 1\right).$$

By using the assertion (iii) of Lemma 2.1, for each $v_i\quad(i=1,2,...,n)$, we obtain

$$\begin{aligned}\left|\frac{z\mathcal{W}''_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(z)}{\mathcal{W}'_{v_1,...,v_n,b,c,\alpha_1,...,\alpha_n,1}(z)}\right| &\le \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left|\frac{zu'_{v_i,b,c}(z)}{u_{v_i,b,c}(z)} - 1\right| \\ &\le \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\frac{|c|\,(6k_i-|c|)}{3\,(4k_i-|c|)\,(3k_i-|c|)} \\ &\quad \left(z\in\mathcal{U};\quad k,k_i = v_i+\frac{b+2}{2} > \frac{7\,|c|}{24}\quad(i=1,2,...,n)\right).\end{aligned}$$

Now as it is shown that the function

$$\tau : \left(\frac{7|c|}{24}, \infty\right) \to \mathbb{R},$$

defined by

$$\tau(k) = \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)},$$

is decreasing function. Therefore

$$\frac{|c|(6k_i-|c|)}{3(4k_i-|c|)(3k_i-|c|)} \leq \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}.$$

Hence

$$\left|\frac{z\mathcal{W}''_{v_1,\ldots,v_n,b,c,\alpha_1,\ldots,\alpha_n,1}(z)}{\mathcal{W}'_{v_1,\ldots,v_n,b,c,\alpha_1,\ldots,\alpha_n,1}(z)}\right| \leq \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}\sum_{i=1}^{n}\frac{1}{|\alpha_i|}.$$

Now using the Lemma 1.1 and the triangle inequality, we get

$$\left|d|z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{z\mathcal{W}''_{v_1,\ldots,v_n,b,c,\alpha_1,\ldots,\alpha_n,1}(z)}{\beta\mathcal{W}'_{v_1,\ldots,v_n,b,c,\alpha_1,\ldots,\alpha_n,1}(z)}\right|$$
$$\leq |d| + \frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)}\sum_{i=1}^{n}\frac{1}{|\beta\alpha_i|} \leq 1,$$

This implies that $\mathcal{W}_{v_1,\ldots,v_n,b,c,\alpha_1,\ldots,\alpha_n,\beta} \in \mathcal{S}$, which completes the proof of Theorem 3.1. □

By setting $\alpha_1 = \alpha_2 = ... = \alpha_n = \alpha$ in Theorem 3.1 we will obtain the result given below

3.2. Corollary. *Let $v_1, \ldots v_n, b \in \mathbb{R}$ and $c \in \mathbb{C}$ and $k_i > \frac{7|c|}{24}$ with $k_i = v_i + (b+2)/2$, $i = 1, \ldots, n$. Let $u_{v_i,b,c} : \mathcal{U} \to \mathbb{C}$ be defined as*

$$u_{v_i,b,c}(z) = 2^{v_i}\sqrt{\pi}\Gamma\left(v_i + \frac{b+2}{2}\right) z^{\frac{(-1-v_i)}{2}} w_{v_i,b,c}(\sqrt{z}).$$

Suppose $k = \min\{k_1, k_2, \ldots, k_n\}$, $Re(\beta) > 0$ and $\alpha \in \mathbb{C}\setminus\{0\}$ $(i = 1, 2, ..., n)$ and these numbers satisfy the relation

$$|d| + \frac{n}{|\beta\alpha|}\frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)} \leq 1,$$

then the function $\mathcal{W}_{v_1,\ldots,v_n,b,c,\alpha_1,\ldots,\alpha_n,\beta} : \mathcal{U} \to \mathbb{C}$ defined by (1.10) is in the class $\mathcal{S}$ of normalized univalent functions in $\mathcal{U}$.

Our second main result, to find the sufficient univalence conditions for an integral operator of type (1.11). The key tools in the proof are Lemma 1.2 and the inequality (*iii*) of Lemma 2.1.

3.3. Theorem. *Let $v_1, \ldots, v_n, b \in \mathbb{R}$ and $c \in \mathbb{C}$ and $k_i > \frac{7|c|}{24}$ with $k_i = v_i + (b+2)/2$, $i = 1, \ldots, n$. Let $u_{v_i,b,c} : \mathcal{U} \to \mathbb{C}$ be defined as*

$$u_{v_i,b,c}(z) = 2^{v_i}\sqrt{\pi}\Gamma\left(v_i + \frac{b+2}{2}\right) z^{\frac{(-1-v_i)}{2}} w_{v_i,b,c}(\sqrt{z}).$$

Suppose $k = \min\{k_1, k_2, \ldots, k_n\}$ and $Re(\gamma) > 0$ and these numbers satisfy the relation

$$\frac{n|\gamma|}{Re(\gamma)}\frac{|c|(6k-|c|)}{3(4k-|c|)(3k-|c|)} \leq 1,$$

then the function $\mathcal{X}_{v_1,\ldots,v_n,b,c,n,\gamma}:\mathcal{U}\to\mathbb{C}$ *defined by* (1.11) *is in the class* $\mathcal{S}$ *of normalized univalent functions in* $\mathcal{U}$.

Proof. Consider the function $\widetilde{\mathcal{X}}_{v_1,\ldots,v_n,b,c,n,\gamma}(z):\mathcal{U}\to\mathbb{C}$ defined by

$$\widetilde{\mathcal{X}}_{v_1,\ldots,v_n,b,c,n,\gamma}(z)=\int_0^z\prod_{i=1}^n\left(\frac{u_{v_i,b,c}(t)}{t}\right)^{\gamma}dt.$$

We observe that $\widetilde{\mathcal{X}}_{v_1,\ldots,v_n,b,c,n,\gamma}(z)\in\mathcal{A}$, that is

$$\widetilde{\mathcal{X}}_{v_1,\ldots,v_n,b,c,n,\gamma}(0)=\widetilde{\mathcal{X}}'_{v_1,\ldots,v_n,b,c,n,\gamma}(0)-1=0.$$

By using the assertion (iii) of Lemma 2.1 and the fact that

$$\frac{|c|\,(6k_i-|c|)}{3\,(4k_i-|c|)\,(3k_i-|c|)}\leq\frac{|c|\,(6k-|c|)}{3\,(4k-|c|)\,(3k-|c|)},$$

we have

$$\begin{aligned}\left(\frac{1-|z|^{2\Re(\alpha)}}{Re\,(\gamma)}\right)\left|\frac{z\widetilde{\mathcal{X}}''_{v_1,\ldots,v_n,b,c,n,\gamma}(z)}{\widetilde{\mathcal{X}}'_{v_1,\ldots,v_n,b,c,n,\gamma}(z)}\right| &\leq \frac{|\gamma|}{Re\,(\gamma)}\sum_{i=1}^n\left|\frac{zu'_{v_i,b,c}(z)}{u_{v_i,b,c}(z)}-1\right|\\ &\leq \frac{n\,|\gamma|}{Re\,(\gamma)}|c|\,(6k-|c|)3\,(4k-|c|)\,(3k-|c|)\leq 1.\end{aligned}$$

Now, since $Re\,(n\gamma+1)>Re\,(\gamma)$ and the function can be written in the form of

$$\mathcal{X}_{v_1,\ldots v_n,b,c,n,\gamma}=\left[(n\gamma+1)\int_0^z t^{n\gamma}\prod_{i=1}^n\left(\frac{u_{v_i,b,c}(t)}{t}\right)^{\gamma}dt\right]^{\frac{1}{(n\gamma+1)}}.$$

From Lemma 1.2 implies that $\mathcal{X}_{v_1,\ldots v_n,b,c,n,\gamma}\in\mathcal{S}$, which completes the proof of Theorem 3.3. □

By setting $n=1$ in Theorem 3.3, we get the following result.

3.4. Corollary. *Let* $v,b\in\mathbb{R}$ *and* $c\in\mathbb{C}$ *and* $k>\frac{7|c|}{24}$ *with* $k=v+(b+2)/2$, *Let* $u_{v,b,c}:\mathcal{U}\to\mathbb{C}$ *be defined as*

$$u_{v,b,c}(z)=2^v\sqrt{\pi}\Gamma\left(v+\frac{b+2}{2}\right)z^{\frac{(-1-v)}{2}}w_{v,b,c}(\sqrt{z}).$$

Suppose $Re\,(\gamma)>0$ *and these numbers satisfy the relation*

$$\frac{|\gamma|}{Re\,(\gamma)}\frac{|c|\,(6k-|c|)}{3\,(4k-|c|)\,(3k-|c|)}\leq 1,$$

then the function $\mathcal{X}_{v,b,c,\gamma}:\mathcal{U}\to\mathbb{C}$ *defined by*

$$\mathcal{X}_{v,b,c,\gamma}=\left[(\gamma+1)\int_0^z\left(u_{v,b,c}(t)\right)^{\gamma}dt\right]^{\frac{1}{\gamma+1}},$$

is in the class $\mathcal{S}$ *of normalized univalent functions in* $\mathcal{U}$.

Next, by applying the Lemma 1.4 and the inequality (iv) of Lemma 2.1, we get the following result.

3.5. Theorem. *Let $q(t) \in \mathcal{A}$, λ be a complex number such that $Re(\lambda) \geq 1$, M be a real number and $M > 1$, here $M = \frac{4k+|c|}{4k-|c|}$. If*

$$|zu'_{v,b,c}(z)| \leq \frac{12k+|c|}{12k-3|c|}, \qquad z \in \mathcal{U} \tag{3.2}$$

and

$$|\lambda| \leq \frac{3\sqrt{3}}{2}\left(\frac{12k-3|c|}{12k+|c|}\right), \tag{3.3}$$

then the function (1.12) *is in the class* $\mathcal{S}$.

Proof. Let us consider the function

$$f(z) = \int_0^z \left(e^{u_{v,b,c}(z)}\right)^{\lambda} dz, \tag{3.4}$$

which is regular in $\mathcal{U}$. Consider the function

$$h(z) = \frac{1}{|\lambda|}\frac{zf''(z)}{f'(z)}, \tag{3.5}$$

where the constant $|\lambda|$ satisfies the inequality (3.3). From (3.4) and (3.5), it follows that

$$h(z) = \frac{\lambda}{|\lambda|} zu'_{v,b,c}(z). \tag{3.6}$$

With the help of (3.2), (3.6) becomes

$$|h(z)| \leq \frac{12k+|c|}{12k-3|c|} \qquad \text{for all } z \in \mathcal{U}. \tag{3.7}$$

From (3.6), we obtain $h(0) = 0$ and by using the Schwarz-Lemma we have

$$\frac{1}{|\lambda|}\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{12k+|c|}{12k-3|c|}|z|, \tag{3.8}$$

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq |\lambda|\left(\frac{12k+|c|}{12k-3|c|}\right)|z|\left(1-|z|^2\right). \tag{3.9}$$

Consider the function $H : [0,1] \to \mathbb{R}$, $H(x) = x(1-x^2)$ where $x = |z|$.

We have

$$H(x) \leq \frac{2}{3\sqrt{3}}, \tag{3.10}$$

for all $x \in [0,1]$. From (3.3) and (3.10), (3.9) becomes

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1.$$

Hence from Lemma 1.2, it is clear that for $Re(\lambda) = 1$ the integral operator (1.12) is in $\mathcal{S}$. □

Now, we discuss some special cases on the behalf of the above mentioned theorems.

4. Some Special Cases of Struve Functions and Modified Struve Functions

Struve Functions

We obtain the Struve function of first kind of order v, denoted by $H_v(z)$, defined by (1.4) by setting $b = c = 1$ in (1.8). Let $\mathcal{H}_v : \mathcal{U} \to \mathbb{C}$ be defined as

$$\mathcal{H}_v(z) = 2^v \sqrt{\pi}\Gamma\left(v + \frac{b+2}{2}\right) z^{\frac{(-1-v)}{2}} H_v(\sqrt{z}).$$

We observe that

$$\begin{aligned} \mathcal{H}_{-1/2}(z) &= \sqrt{z}\sin\left(\sqrt{z}\right), \\ \mathcal{H}_{1/2}(z) &= 2\left(1-\cos\sqrt{z}\right), \\ \mathcal{H}_{3/2}(z) &= 4\left(1+\frac{2}{z}\right) - 8\left(\frac{\sin\left(\sqrt{z}\right)}{\sqrt{z}} + \frac{\cos\left(\sqrt{z}\right)}{\sqrt{z}}\right). \end{aligned}$$

By making use of these particular values, we have the following assertions.

(1) Let $v_1, v_2, ..., v_n > -1.75$ $(n \in \mathbb{N})$. Consider the function $\mathcal{H}_{v_i}(z) : \mathcal{U} \to \mathbb{C}$ defined by

$$\mathcal{H}_{v_i}(z) = 2^{v_i}\sqrt{\pi}\Gamma\left(v_i + \frac{3}{2}\right) z^{\frac{(-1-v_i)}{2}} H_{v_i}(\sqrt{z}), \tag{4.1}$$

let $v = \min\{v_1, v_2, ..., v_n\}$ and let the parameters d, β, α_i $(i = 1, 2, ..., n)$ be defined as in Theorem 3.1. Now consider the function $\mathcal{W}_{v_1,...,v_n,\alpha_1,...,\alpha_n,\beta}(z) : \mathcal{U} \to \mathbb{C}$, defined by

$$\mathcal{W}_{v_1,...,v_n,\alpha_1,...,\alpha_n,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{\mathcal{H}_{v_i}(z)}{t}\right)^{\frac{1}{\alpha_i}} dt\right]^{\frac{1}{\beta}}. \tag{4.2}$$

This function is in class $\mathcal{S}$, when the following inequality is satisfied.

$$|d| + \frac{4(3v+4)}{3(24v^2+58v+35)} \sum_{i=1}^{n} \frac{1}{|\beta\alpha_i|} \leq 1.$$

In particular, we have the followings.
At $v = \frac{-1}{2}$, the function $\mathcal{W}_{v,\alpha,\beta}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.2), reduces to

$$\mathcal{W}_{-1/2,\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{\sin\left(\sqrt{t}\right)}{\sqrt{t}}\right)^{\frac{1}{\alpha}} dt\right]^{\frac{1}{\beta}},$$

belongs to the class $\mathcal{S}$ when $|d| + \frac{5}{18}\frac{1}{|\beta\alpha|} \leq 1$.
At $v = \frac{1}{2}$, the function $\mathcal{W}_{v,\alpha,\beta}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.2), takes the form

$$\mathcal{W}_{1/2,\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{2\left(1-\cos\sqrt{t}\right)}{t}\right)^{\frac{1}{\alpha}} dt\right]^{\frac{1}{\beta}},$$

belongs to the class $\mathcal{S}$ when $|d| + \frac{11}{105}\frac{1}{|\beta\alpha|} \leq 1$.
At $v = \frac{3}{2}$, the function $\mathcal{W}_{v,\alpha,\beta}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.2), implies to

$$\mathcal{W}_{3/2,\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{4\left(1+\frac{2}{t}\right) - 8\left(\frac{\sin\left(\sqrt{t}\right)}{\sqrt{t}} + \frac{\cos\left(\sqrt{t}\right)}{\sqrt{t}}\right)}{t}\right)^{\frac{1}{\alpha}} dt\right]^{\frac{1}{\beta}},$$

belongs to the class $\mathcal{S}$ when $|d| + \frac{17}{264}\frac{1}{|\beta\alpha|} \leq 1$.

(2) Let $v_1, v_2, ..., v_n > -1.75$ $(n \in \mathbb{N})$. Consider the function $\mathcal{H}_{v_i}(z) : \mathcal{U} \to \mathbb{C}$ defined by (4.1). Let $v = \min\{v_1, v_2, ..., v_n\}$ and the parameters γ be defined as in Theorem 3.3. The function $\mathcal{X}_{v_1,v_2,...,v_n,n,\gamma}(z) : \mathcal{U} \to \mathbb{C}$, defined by

$$\mathcal{X}_{v_1,v_2,...,v_n,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \prod_{i=1}^{n} \{\mathcal{H}_{v_i}(z)\}^{\gamma}\, dt\right]^{\frac{1}{n\gamma+1}}, \tag{4.3}$$

belongs to the class $\mathcal{S}$ when the following inequality holds.

$$\frac{n\,|\gamma|}{\mathcal{R}(\gamma)}\frac{4\,(3v+4)}{3\,(24v^2+58v+35)} \leq 1.$$

In particular, we have the followings.
At $v = \frac{-1}{2}$, the function $\mathcal{X}_{v,n,\gamma}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.3), reduces to

$$\mathcal{X}_{-1/2,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \left\{\sqrt{t}\sin\left(\sqrt{t}\right)\right\}^{\gamma} dt\right]^{\frac{1}{n\gamma+1}},$$

belongs to the class $\mathcal{S}$ when $\frac{n|\gamma|}{Re(\gamma)}\frac{5}{18} \leq 1$.
At $v = \frac{1}{2}$, the function $\mathcal{X}_{v,n,\gamma}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.3), takes the form

$$\mathcal{X}_{1/2,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \left\{2\left(1-\cos\sqrt{t}\right)\right\}^{\gamma} dt\right]^{\frac{1}{n\gamma+1}},$$

belongs to the class $\mathcal{S}$ when $\frac{n|\gamma|}{Re(\gamma)}\frac{11}{105} \leq 1$.
At $v = \frac{3}{2}$, the function $\mathcal{X}_{v,n,\gamma}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.3), implies to

$$\mathcal{X}_{3/2,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \left\{4\left(1+\frac{2}{t}\right) - 8\left(\frac{\sin\left(\sqrt{t}\right)}{\sqrt{t}} + \frac{\cos\left(\sqrt{t}\right)}{\sqrt{t}}\right)\right\}^{\gamma} dt\right]^{\frac{1}{n\gamma+1}},$$

belongs to the class $\mathcal{S}$ when $\frac{n|\gamma|}{Re(\gamma)}\frac{17}{264} \leq 1$.

(3) Let $v_1, v_2, ..., v_n > -1.75$ $(n \in \mathbb{N})$. Consider the function $\mathcal{H}_{v_i}(z) : \mathcal{U} \to \mathbb{C}$ as defined by (4.1). Let $v = \min\{v_1, v_2, ..., v_n\}$ and the parameters λ be defined as in Theorem 3.5. Let us define a function $\mathcal{Z}_{v_1,v_2,...,v_n,\lambda}(z) : \mathcal{U} \to \mathbb{C}$ by

$$\mathcal{Z}_{v_1,v_2,...,v_n,\lambda}(z) = \left[\lambda\int_0^z t^{\lambda-1}\left(e^{\mathcal{H}_{v_i}}\right)^{\lambda} dt\right]^{1/\lambda}. \tag{4.4}$$

This function lies from the class $\mathcal{S}$, when the following inequality is satisfied.

$$|\lambda| \leq \frac{3\sqrt{3}}{2}\left(\frac{12v+15}{12v+19}\right).$$

In particular, we have the followings.
At $v = \frac{-1}{2}$, the function $\mathcal{Z}_{v,\lambda}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.4), reduces to

$$\mathcal{Z}_{-1/2,\lambda}(z) = \left[\lambda\int_0^z t^{\lambda-1}\left(e^{\sqrt{t}\sin\left(\sqrt{t}\right)}\right)^{\lambda} dt\right]^{1/\lambda},$$

belongs to the class $\mathcal{S}$ when $|\lambda| \leq \frac{27\sqrt{3}}{26}$.
At $v = \frac{1}{2}$, the function $\mathcal{Z}_{v,\lambda}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.4), takes the form

$$\mathcal{Z}_{1/2,\lambda}(z) = \left[\lambda \int_0^z t^{\lambda-1} \left(e^{2\left(1-\cos\sqrt{t}\right)}\right)^{\lambda} dt\right]^{1/\lambda},$$

belongs to the class $\mathcal{S}$ when $|\lambda| \leq \frac{63\sqrt{3}}{50}$.
At $v = \frac{3}{2}$, the function $\mathcal{Z}_{v,\lambda}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.4), implies to

$$\mathcal{Z}_{3/2,\lambda}(z) = \left[\lambda \int_0^z t^{\lambda-1} \left(e^{4\left(1+\frac{2}{t}\right)-8\left(\frac{\sin\left(\sqrt{t}\right)}{\sqrt{t}}+\frac{\cos\left(\sqrt{t}\right)}{\sqrt{t}}\right)}\right)^{\lambda} dt\right]^{1/\lambda},$$

belongs to the class $\mathcal{S}$ when $|\lambda| \leq \frac{99\sqrt{3}}{74}$.

Modified Struve Functions

We obtain the modified struve function of first kind of order v, denoted by $L_v(z)$, defined by (1.5), by putting $b = -c = 1$ in (1.8). Define a function $\mathcal{L}_v(z) : \mathcal{U} \to \mathbb{C}$ by

$$\mathcal{L}_v(z) = 2^v \sqrt{\pi}\Gamma\left(v + \frac{3}{2}\right) z^{\frac{(-1-v)}{2}} L_v(\sqrt{z}).$$

We observe that

$$\mathcal{L}_{1/2}(z) = 2\left(\cosh\sqrt{z} - 1\right).$$

By making use of this particular value, we have the following assertions.

(1) Let $v_1, v_2, ..., v_n > -1.75$ $(n \in \mathbb{N})$. Consider the function $\mathcal{L}_{v_i}(z) : \mathcal{U} \to \mathbb{C}$ defined by

$$\mathcal{L}_{v_i}(z) = 2^v \sqrt{\pi}\Gamma\left(v + \frac{3}{2}\right) z^{\frac{(-1-v)}{2}} L_{v_i}(\sqrt{z}) \tag{4.5}$$

Let $v = \min\{v_1, v_2, ..., v_n\}$ and the parameters d, β, α_i $(i = 1, 2, ..., n)$ be defined as in Theorem 3.1. Consider the function $\mathcal{W}_{v_1,...,v_n,\alpha_1,...,\alpha_n,\beta}(z) : \mathcal{U} \to \mathbb{C}$, defined by

$$\mathcal{W}_{v_1,...,v_n,\alpha_1,...,\alpha_n,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{\mathcal{L}_{v_i}(z)}{t}\right)^{\frac{1}{\alpha_i}} dt\right]^{\frac{1}{\beta}}. \tag{4.6}$$

This function is in class $\mathcal{S}$, when the following inequality is satisfied.

$$|d| + \frac{4\left(3v+4\right)}{3\left(24v^2+58v+35\right)} \sum_{i=1}^{n} \frac{1}{|\beta\alpha_i|} \leq 1.$$

In particular, at $v = \frac{1}{2}$, the function $\mathcal{W}_{v,\alpha,\beta}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.6), reduces to

$$\mathcal{W}_{1/2,\alpha,\beta}(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{2\left(\cosh\sqrt{t} - 1\right)}{t}\right)^{\frac{1}{\alpha}} dt\right]^{\frac{1}{\beta}},$$

belongs to the class $\mathcal{S}$ when $|d| + \frac{11}{105}\frac{1}{|\beta\alpha|} \leq 1$.

(2) Let $v_1, v_2, ..., v_n > -1.75$ $(n \in \mathbb{N})$, $v = \min\{v_1, v_2, ..., v_n\}$ and the parameters γ be defined as in Theorem 3.3. Consider the function $\mathcal{X}_{v_1,v_2,...,v_n,n,\gamma}(z) : \mathcal{U} \to \mathbb{C}$, defined by

$$\mathcal{X}_{v_1,v_2,...,v_n,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \prod_{i=1}^{n} \{\mathcal{L}_{v_i}(z)\}^{\gamma}\, dt\right]^{\frac{1}{n\gamma+1}}, \tag{4.7}$$

where $\mathcal{L}_{v_i}(z)$ is defined by (4.5). This function $\mathcal{X}_{v_1,v_2,...,v_n,n,\gamma}(z)$ is in class $\mathcal{S}$, when the following inequality holds.

$$\frac{n\,|\gamma|}{Re\,(\gamma)}\,\frac{4\,(3v+4)}{3\,(24v^2+58v+35)} \leq 1.$$

In particular, at $v = \frac{1}{2}$, the function $\mathcal{X}_{v,n,\gamma}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.7), implies to

$$\mathcal{X}_{1/2,n,\gamma}(z) = \left[(n\gamma+1)\int_0^z \left\{2\left(\cosh\sqrt{t}-1\right)\right\}^{\gamma} dt\right]^{\frac{1}{n\gamma+1}},$$

belongs to the class $\mathcal{S}$ when $\frac{n|\gamma|}{Re(\gamma)}\,\frac{11}{105} \leq 1$.

(3) Let $v_1, v_2, ..., v_n > -1.75$ $(n \in \mathbb{N})$, $v = \min\{v_1, v_2, ..., v_n\}$ and the parameters λ be defined as in Theorem(3.5). Consider the function $\mathcal{Z}_{v_1,v_2,...,v_n,\lambda}(z) : \mathcal{U} \to \mathbb{C}$ defined by

$$\mathcal{Z}_{v_1,v_2,...,v_n,\lambda}(z) = \left[\lambda\int_0^z t^{\lambda-1}\left(e^{\mathcal{L}_{v_i}}\right)^{\lambda} dt\right]^{1/\lambda}, \tag{4.8}$$

where $\mathcal{L}_{v_i}(z)$ is defined by (4.5). This function $\mathcal{Z}_{v_1,v_2,...,v_n,\lambda}(z)$ is in class $\mathcal{S}$ of normalized univalent functions in $\mathcal{U}$, when the following inequality is satisfied

$$|\lambda| \leq \frac{3\sqrt{3}}{2}\left(\frac{12v+15}{12v+19}\right).$$

In particular, at $v = \frac{1}{2}$, the function $\mathcal{Z}_{v,\lambda}(z) : \mathcal{U} \to \mathbb{C}$, defined by (4.8), takes the form

$$\mathcal{Z}_{1/2,\lambda}(z) = \left[\lambda\int_0^z t^{\lambda-1}\left(e^{2\left(\cosh\sqrt{t}-1\right)}\right)^{\lambda} dt\right]^{1/\lambda},$$

belongs to the class $\mathcal{S}$ when $|\lambda| \leq \frac{63\sqrt{3}}{50}$.

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