Research Article GU J Sci 31(3): 967-983 (2018)

Gazi University

Journal of Science

http://dergipark.gov.tr/gujs

Log-Gamma - Rayleigh Distribution: Properties and Applications

Samir ASHOUR [1], Darwish DARWISH [1], Ahmad SAAD [1,*]

[1] *Institute of Statistical Studies & Research, Cairo University, 5 Ahmed Zowil St., Orman, Giza, Egypt*

Article Info

Received: 14/08/2017
Accepted: 20/04/2018

Keywords

T-X method
Gamma distribution
Rayleigh distribution
Quantile function
Hazard function
Moments
Maximum likelihood estimation

Abstract

In this paper, for the first time we defined and studied a new two parameter lifetime model by using the *T-X* method, called the Log-Gamma Rayleigh distribution. This distribution can be considered as a new generalization of the Gamma distribution and the Rayleigh distribution. We obtain some of its mathematical properties. Some structural properties of the new distribution are studied. Maximum likelihood estimation method is used for estimating the model parameters. An application to real data set is given to show the flexibility and potentiality of the new model.

1. INTRODUCTION

Eugene et al. [6] for the first time introduced the beta-generated family of distributions. They noted that the probability density function pdf of the beta random variable and the cumulative distribution function CDF of any distribution are between 0 and 1. The beta-generated random variable X is defined with the following CDF and pdf

$$G(x)=\int_{0}^{F(x)} b(t)\,dt \tag{1}$$

and

$$g(x)=\frac{1}{B(\alpha,\beta)} f(x)\,F^{\alpha-1}(x)[1-F(x)]^{\beta-1}$$

where b(t) is the pdf of the beta random variable with parameters α and β, F(x) and f(x) are the CDF and the pdf of any random variable.

Many authors derived and studied many beta-generated distributions in the literature, for example beta-Gumbel (Nadarajah and Kotz, [12]), beta-Weibull (Famoye et al. [7]), beta-exponential (Nadarajah and Kotz, [13]), beta-gamma (Kong et al., [8]), beta-Pareto (Akinsete et al., [1]), beta-generalized exponential (Barreto-Souza et al., [5]), beta-generalized Pareto (Mahmoudi, [10]), and beta-Cauchy (Alshawarbeh, et al., [2]).

*Corresponding author, e-mail: ahmed_issr2010@yahoo.com

Alzaatreh et al. [3] presented a new general method for generating new distributions, called T-X family of distributions. This method depending on replace the beta pdf in (1) with a pdf of any continuous random variable and applying a function W(F(x)) that satisfies the following conditions:
1- “W(F(x)) $\in$ [a, b].”
2- “W(F(x)) is differentiable and monotonically non-decreasing.”
3- “W(F(x)) $\rightarrow$ a as x $\rightarrow -\infty$ and W(F(x)) $\rightarrow$ b as x $\rightarrow \infty$.”

Let X be a random variable with pdf f (x) and CDF F(x), and let T be a continuous random variable with pdf r(t) and CDF R(t) defined on [a, b] for $-\infty < a < b < \infty$. Alzaatreh et al. [3] defined the CDF and pdf of a new family of distributions as

$$G(x)=\int_{a}^{W(F(x))} r(t)\,dt = R\{W(F(x))\} \tag{2}$$

and

$$g(x)=[\frac{d}{dx}W(F(x))]\; r[W(F(x))]$$

Recently, Amini et al. [4] introduced two new general families of continuous distributions called log gamma- generated families(LG-G) of distributions as follows:
For any continuous parent distribution F(x) of a random variable X with corresponding parent pdf f(x), the two new LG-G families are given with the following two pdfs:

$$f_1(x)=\frac{q^p}{\Gamma(p)}\left[-\log\left(1-F(x)\right)\right]^{p-1}\left(1-F(x)\right)^{q-1} f(x), \quad p,q>0$$

and

$$f_2(x)=\frac{q^p}{\Gamma(p)}\left[-\log\left(F(x)\right)\right]^{p-1}\left(F(x)\right)^{q-1} f(x), \quad p,q>0$$

where $\Gamma(.)$ is the complete gamma function.

Note that the log gamma- generated families which introduced by Amini et al. [4] are two special cases from T-X method defined by Alzaatreh et al. [3]. Take the W(F(x)) = (1 − F(x)) in (2), and let the random variable T follows the log-gamma distribution with the following CDF and pdf

$$R(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}\int_0^x(-\log x)^{\alpha-1}x^{\beta-1}, \quad 0<x<1 \quad , \quad \beta,\alpha>0 \tag{3}$$

$$r(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}(-\log x)^{\alpha-1}x^{\beta-1}, \quad 0<x<1 \quad , \quad \beta,\alpha>0 \tag{4}$$

From (2), (3) and (4) we will define the log-gamma -X family based on T-X method with the following CDF and pdf

$$G(x)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\int_{0}^{(1-F(x))}(-\log x)^{\alpha-1}x^{\beta-1},\quad \infty<x<-\infty \quad , \quad \beta,\alpha>0$$

$$=1-\frac{\gamma\left[\alpha,-\beta\log(1-F(x))\right]}{\Gamma(\alpha)} \tag{5}$$

and

$$g(x)=\frac{\beta^{\alpha}f(x)}{\Gamma(\alpha)}\left[-\log(1-F(x))\right]^{\alpha-1}\left[1-F(x)\right]^{\beta-1},\quad \infty<x<-\infty \quad , \quad \beta,\alpha>0 \tag{6}$$

where $\gamma(.,.)$ is the lower incomplete gamma function defined by

$$\gamma(\alpha,x)=\int_{0}^{x}e^{-u}\,u^{\alpha-1}\,du$$

Merovci and Elbatal [11] defined and studied a generalization of the Rayleigh distribution called the Weibull Rayleigh distribution (WR). In this paper we present a new generalization of the Gamma distribution and the Rayleigh distribution called the Log-Gamma Rayleigh distribution.

2. THE LOG-GAMMA - RAYLEIGH DISTRIBUTION (LGR)

If the random variable X have the Rayleigh distribution with pdf and CDF given, respectively, by

$$\text{“}f(x)=\frac{x}{\sigma^{2}}e^{-\frac{x^{2}}{2\sigma^{2}}} \quad , \quad x>0 \quad , \quad \sigma>0\text{”} \tag{7}$$

and

$$F(x)=1-e^{-\frac{x^{2}}{2\sigma^{2}}},\quad x>0 \quad , \quad \sigma>0 \tag{8}$$

then using (5), (6), (7) and (8) the log-gamma - Rayleigh distribution (LGR) is defined with CDF

$$G(x)=1-\frac{\gamma(\alpha,\frac{\beta x^{2}}{2\sigma^{2}})}{\Gamma(\alpha)},\quad x>0 \quad , \quad \alpha,\beta,\sigma>0$$

The corresponding pdf of the LGR is

$$\text{“}g(x)=\frac{\beta^{\alpha}x}{\sigma^{2}\Gamma(\alpha)}\left(\frac{x^{2}}{2\sigma^{2}}\right)^{\alpha-1}\left(e^{-\frac{x^{2}}{2\sigma^{2}}}\right)^{\beta},\quad x>0 \quad , \quad \alpha,\beta,\sigma>0\text{”}$$

The LGR distribution has the following special cases:

1) When $\beta=1$, the gamma distribution is obtained as

$$G(y)=\frac{\gamma(\alpha,y)}{\Gamma(\alpha)},\quad y>0\quad,\quad \alpha,\sigma>0$$

with

$$g(y)=\frac{1}{\Gamma(\alpha)}y^{\alpha-1}e^{-y},\quad y>0\quad,\quad \alpha,\sigma>0$$

where $$y=\frac{x^2}{2\sigma^2}$$

2) When $\alpha=\beta=1$, the Rayleigh distribution in (7) and (8) is obtained.
Hence the log-gamma - Rayleigh distribution can be considered as a generalization of the gamma and Rayleigh distributions.

If we set $\lambda=\frac{\beta}{\sigma^2}$, then the CDF of the LGR can be written as

$$G(x)=1-\frac{\gamma(\alpha,\frac{\lambda x^2}{2})}{\Gamma(\alpha)},\quad x>0\quad,\quad \alpha,\lambda>0 \tag{9}$$

and, the corresponding pdf of the LGR is given by

$$\text{“}g(x)=\frac{\lambda x}{\Gamma(\alpha)}(\frac{\lambda x^2}{2})^{\alpha-1}e^{-\frac{\lambda x^2}{2}},\quad x>0\quad,\quad \alpha,\lambda>0\text{”} \tag{10}$$

“Figures 1 and 2 illustrates some of the possible shapes of the pdf and CDF of LGR for selected values of the parameters α and λ, respectively”

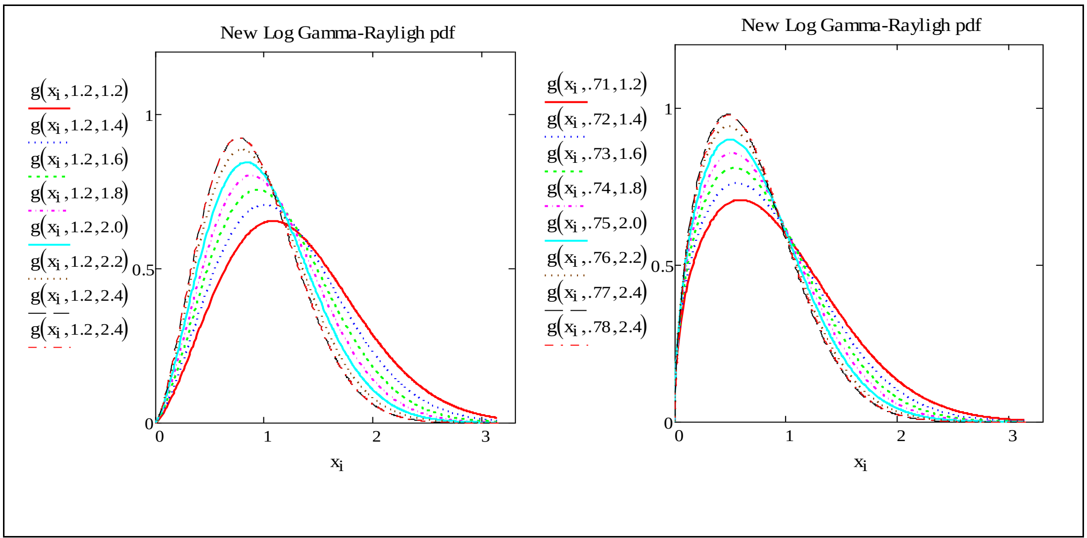

Figure 1. *Density function* $g(x;\alpha,\lambda)$ *of the LGR*

The graphs in Figure 1 indicate that the LGR is unimodal.

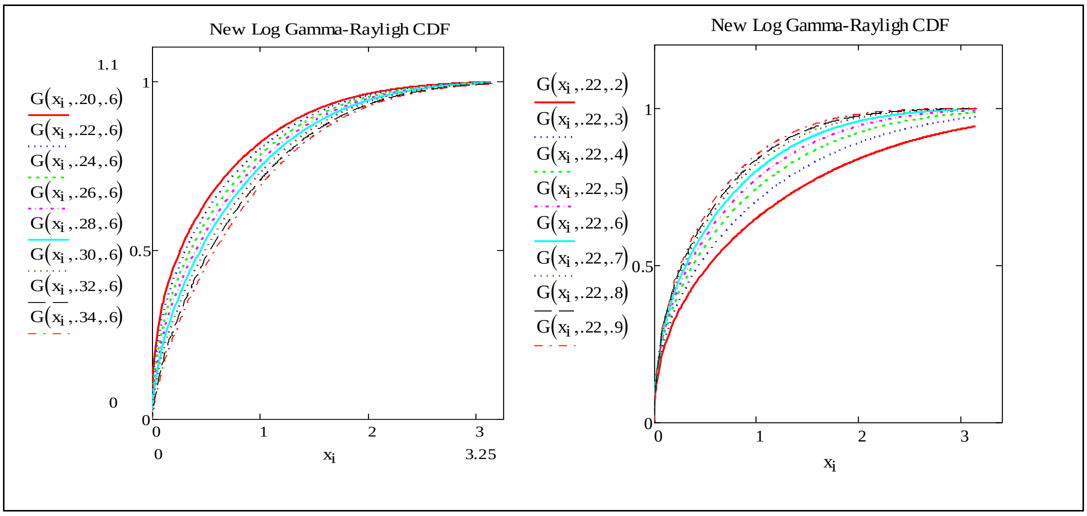

Figure 2. *Cumulative function $G(x;\alpha,\lambda)$ of the LGR*

Some remarks of LGR:

1) If a random variable Y follows the log-gamma distribution with parameters α and β, then the random variable $X=\sqrt{\frac{2}{\lambda}\log(\frac{1}{Y})}$ follows the LGR.

2) If a random variable Z follows the gamma distribution with parameters α and β, then the random variable $X=\sqrt{\frac{2}{\lambda}Z}$ follows the LGR.

2.1. Survival and Hazard Functions

The survival function of the LGR is

$$S(x)=\frac{\gamma(\alpha,\frac{\lambda x^{2}}{2})}{\Gamma(\alpha)}$$

“The hazard rate function and reversed hazard rate function of the LGR will be”

$$H(x)=\frac{g(x)}{1-G(x)}=\frac{\lambda x(\frac{\lambda x^{2}}{2})^{\alpha-1}e^{-\frac{\lambda x^{2}}{2}}}{\gamma(\alpha,\frac{\lambda x^{2}}{2})}$$

and

$$\tau(x)=\frac{g(x)}{G(x)}=\frac{\lambda x(\frac{\lambda x^{2}}{2})^{\alpha-1}e^{-\frac{\lambda x^{2}}{2}}}{\Gamma(\alpha)-\gamma(\alpha,\frac{\beta x^{2}}{2\sigma^{2}})}$$

Figures 3 and 4 illustrates some of the possible shapes of the survival and hazard rate function of LGR for selected values of the parameters α and λ, respectively

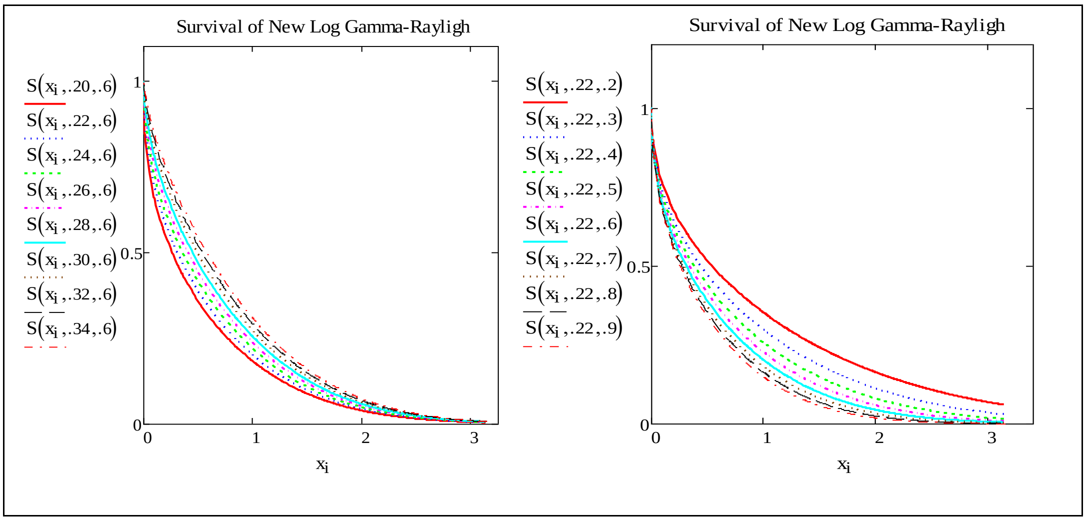

Figure 3. *Survival function* $S(x;\alpha,\lambda)$ *of the LGR*

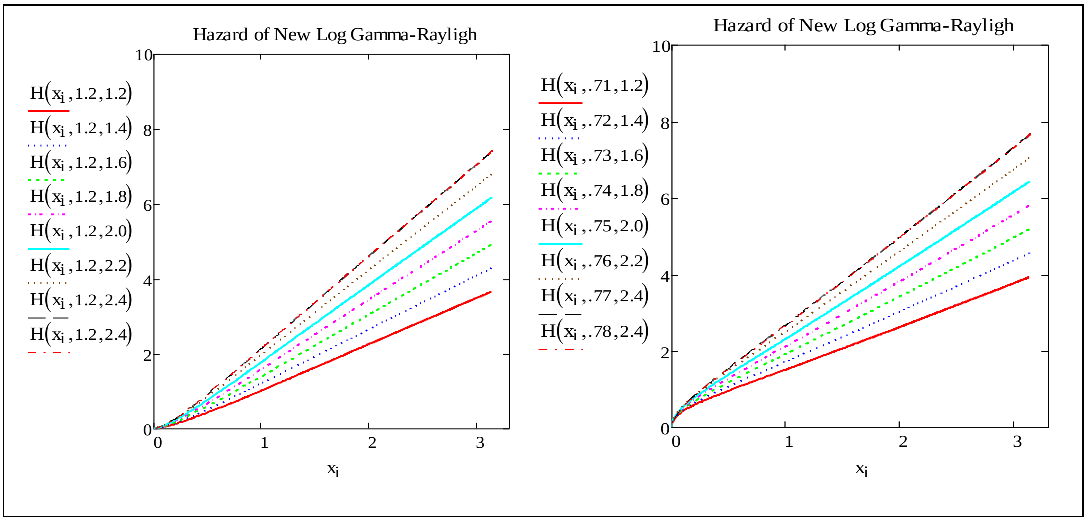

Figure 4. *Hazard function* $H(x;\alpha,\lambda)$ *of the LGR*

2.2. Quantile Function, Median, and Simulation

The quantile function for the LGR distribution is given as

$$Q(p)=\left[\frac{2}{\lambda}\gamma^{-1}\left(\alpha,(1-p)\Gamma(\alpha)\right)\right]^{\frac{1}{2}}$$

where $\gamma^{-1}(.,.)$ is the inverse incomplete gamma function implemented in most used mathematical software's (see Pinho et al. [14]).

Proof: By inverting (9)

$$G(x)=1-\frac{\gamma(\alpha,\frac{\lambda x^2}{2})}{\Gamma(\alpha)},$$

$$1-G(x)=\frac{\gamma(\alpha,\frac{\lambda x^2}{2})}{\Gamma(\alpha)},$$

$$\gamma(\alpha,\frac{\lambda x^2}{2})=\Gamma(\alpha)[1-G(x)],$$

$$\gamma^{-1}\left(\alpha,\Gamma(\alpha)[1-G(x)]\right)=\frac{\lambda x^2}{2},$$

$$\frac{2}{\lambda}\gamma^{-1}\left(\alpha,\Gamma(\alpha)[1-G(x)]\right)=x^2,$$

$$\left[\frac{2}{\lambda}\gamma^{-1}\left(\alpha,\Gamma(\alpha)[1-G(x)]\right)\right]^{\frac{1}{2}}=x$$

Let $G(x)=p$ where $0<p<1$. Then the quantile function is

$$Q(p)=\left[\frac{2}{\lambda}\gamma^{-1}\left(\alpha,(1-p)\Gamma(\alpha)\right)\right]^{\frac{1}{2}}$$

Consequently, the median of LGR will be

$$Q(0.5)=\left[\frac{2}{\lambda}\gamma^{-1}\left(\alpha,\frac{\Gamma(\alpha)}{2}\right)\right]^{\frac{1}{2}}$$

“Let U be a uniform variate on the unit interval (0,1). Thus by means of the inverse transformation method, we consider the random variable X given by”:

$$X=\left[\frac{2}{\lambda}\gamma^{-1}\left(\alpha,(1-U)\Gamma(\alpha)\right)\right]^{\frac{1}{2}} \tag{11}$$

This follows the LGR.

3. MOMENTS AND MOMENT GEENERATING FUNCTION

In this section, the non-central moments, the central moments, incomplete moments, and moment generating function of the LGR are computed.

Theorem 3.1: If X is a random variable distributed as a LGR, then the non-central moment is given by

$$E(X^r)=\left(\frac{2}{\lambda}\right)^{\frac{r}{2}}\frac{\Gamma(\alpha+\frac{r}{2})}{\Gamma(\alpha)}$$

Proof:

$$E(X^r)=\int_0^\infty x^r\,\frac{\lambda x}{\Gamma(\alpha)}\left(\frac{\lambda x^2}{2}\right)^{\alpha-1}e^{-\frac{\lambda x^2}{2}}dx$$

$$u=\frac{\lambda x^2}{2}\quad\rightarrow\quad x=\left(\frac{2}{\lambda}\right)^{\frac{1}{2}}u^{\frac{1}{2}}$$

$$du=\lambda x dx\quad\rightarrow\quad dx=\frac{1}{\lambda x}du$$

$$x:0\rightarrow\infty\quad,\quad u:0\rightarrow\infty$$

Therefore

$$E(X^r)=\frac{1}{\Gamma(\alpha)}\int_0^\infty\left(\frac{2}{\lambda}\right)^{\frac{r}{2}}u^{\frac{r}{2}}\,\lambda x u^{\alpha-1}e^{-u}\,\frac{1}{\lambda x}du$$

$$=\frac{\left(\frac{2}{\lambda}\right)^{\frac{r}{2}}}{\Gamma(\alpha)}\int_0^\infty u^{\alpha+\frac{r}{2}-1}e^{-u}du$$

$$=\left(\frac{2}{\lambda}\right)^{\frac{r}{2}}\frac{\Gamma(\alpha+\frac{r}{2})}{\Gamma(\alpha)}$$

If $r=1$,

$$E(X)=\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\left(\frac{2}{\lambda}\right)^{\frac{1}{2}}$$

If $r=2$,

$$E(X^2)=\frac{2\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)}=\frac{2\alpha}{\lambda}$$

Therefore, the variance of LGR is given by

$$Var(X)=\frac{2}{\lambda}\left\{\alpha-\left(\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\right)^2\right\}$$

Using the relations, the skewness and kurtosis can be calculated as

$$\text{“}skewness = \frac{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3}{[Var(X)]^{\frac{3}{2}}}\text{”}$$

and

$$\text{“}kurtosis = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)[E(X)]^2 + 3[E(X)]^4}{[Var(X)]^2}\text{”}$$

Table 1. *First four non-central moments, skewness and kurtosis of the LGR distribution for various values of parameters*

α	λ	μ_1'	μ_2'	μ_3'	μ_4'	Skewness	Kurtosis
0.5	0.00025	7.507	66.689	663.642	7084	0.237	1.66
	0.0005	10.617	133.379	1877	28330		
	0.00075	13.003	200.068	3448	63750		
	0.001	15.014	266.758	5309	113300		
0.6	0.00025	7.766	71.754	747.467	8394	0.321	1.734
	0.0005	10.983	143.508	2114	33570		
	0.00075	13.451	215.262	3884	75540		
	0.001	15.532	287.016	5980	134300		
0.7	0.00025	7.993	78.294	882.021	10830	0.472	1.898
	0.0005	11.303	156.588	2495	43310		
	0.00075	13.844	234.882	4583	97440		
	0.001	15.985	313.175	7056	173200		
0.8	0.00025	8.011	84.139	1058	14780	0.722	2.289
	0.0005	11.33	168.278	2993	59130		
	0.00075	13.876	252.417	5499	133100		
	0.001	16.023	336.556	8466	236500		
0.9	0.00025	7.386	84.025	1251	21460	1.215	3.542
	0.0005	10.445	168.049	3537	85830		
	0.00075	12.793	252.074	6499	193100		
	0.001	14.772	336.099	10010	343300		

Table 1 reveals that the skewness and kurtosis depend on the shape parameter α.

Theorem 3.2: The moment about the mean of the LGR is as follows:

$$E(X-\mu)^r = \left(\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)} \left(\frac{2}{\lambda}\right)^{\frac{1}{2}} \right)^r \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \left(\Gamma(\alpha)\right)^{k+1} \left(\Gamma(\alpha+\frac{1}{2})\right)^{-k} \Gamma(\alpha+\frac{k}{2})$$

Proof:

Using the binomial expansion for $(X-\mu)^r$, the central moments $E(X-\mu)^r$ for any random variable X can be written as

$$E(X-\mu)^r=\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\mu^{r-k}E(X^k)$$

Therefore, the central moments for the LGR random variable X can be simplified to

$$E(X-\mu)^r=\left(\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\left(\frac{2}{\lambda}\right)^{\frac{1}{2}}\right)^r\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\left(\Gamma(\alpha)\right)^{k+1}\left(\Gamma(\alpha+\frac{1}{2})\right)^{-k}\Gamma(\alpha+\frac{k}{2})$$

Theorem 3.3: If X is a random variable distributed as a LGR with parameters α and λ, the r^{th} incomplete moment of X is given by:

$$M_r(z)=\frac{\gamma\left(\alpha+\frac{r}{2},\frac{\lambda z^2}{2}\right)}{\Gamma(\alpha)}\left(\frac{2}{\lambda}\right)^{\frac{r}{2}}$$

Proof:

$$M_r(z)=\int_0^z x^r\frac{\lambda x}{\Gamma(\alpha)}(\frac{\lambda x^2}{2})^{\alpha-1}e^{-\frac{\lambda x^2}{2}}dx$$

Let

$$y=\frac{\lambda x^2}{2}\quad\rightarrow\quad x=\left(\frac{2}{\lambda}\right)^{\frac{1}{2}}y^{\frac{1}{2}}$$

$$dy=\lambda x dx\quad\rightarrow\quad dx=\frac{1}{\lambda x}dy$$

$$x:0\rightarrow z\quad,\quad y:0\rightarrow\frac{\lambda z^2}{2}$$

This implies

$$\text{“}M_r(z)=\frac{1}{\Gamma(\alpha)}\int_0^{\frac{\lambda z^2}{2}}\left(\frac{2}{\lambda}\right)^{\frac{r}{2}}y^{\frac{r}{2}}\lambda x\, y^{\alpha-1}e^{-y}\frac{1}{\lambda x}dy\text{”}$$

$$=\frac{\left(\frac{2}{\lambda}\right)^{\frac{r}{2}}}{\Gamma(\alpha)}\int_0^{\frac{\lambda z^2}{2}}y^{\alpha+\frac{r}{2}-1}e^{-y}dy$$

$$=\frac{\gamma\left(\alpha+\frac{r}{2},\frac{\lambda z^2}{2}\right)}{\Gamma(\alpha)}\left(\frac{2}{\lambda}\right)^{\frac{r}{2}}$$

where γ is the lower incomplete gamma function.

Theorem 3.4: "Let X have a LGR. The moment generating function of X denoted by $M_X(t)$ is given by"

$$M_X(t)=\sum_{i=0}^{\infty}\frac{t^i}{i!\,\Gamma(\alpha)}\left(\frac{2}{\lambda}\right)^{\frac{i}{2}}\Gamma(\alpha+\frac{i}{2})$$

Proof:
By definition

$$M_X(t)=E(e^{tx})=\int_0^{\infty}e^{tx}\,\frac{\lambda x}{\Gamma(\alpha)}(\frac{\lambda x^2}{2})^{\alpha-1}e^{-\frac{\lambda x^2}{2}}dx$$

Using Taylor series

$$M_X(t)=\int_0^{\infty}\left(1+\frac{tx}{1!}+\frac{t^2x^2}{2!}+\ldots+\frac{t^nx^n}{\text{n!}}+\ldots\right)g(x)dx$$

$$=\sum_{i=0}^{\infty}\frac{t^iE(X^i)}{\text{i!}}$$

$$=\sum_{i=0}^{\infty}\frac{t^i}{i!\,\Gamma(\alpha)}\left(\frac{2}{\lambda}\right)^{\frac{i}{2}}\Gamma(\alpha+\frac{i}{2})$$

4. MODE AND MEAN DEVIATIONS

The mode of the LGR is obtained by finding the first derivate of log g(x) with respect to x and equating it to zero

$$\text{“}\log\big(g(x)\big)=\log(\lambda)+\log(x)-\log\big(\Gamma(\alpha)\big)+(\alpha-1)\big(\log(\lambda)+2\log(x)-\log(2)\big)-\frac{\lambda x^2}{2}\text{”}$$

$$\text{“}\frac{d}{dx}\log\big(g(x)\big)=\frac{1}{x}+\frac{2(\alpha-1)}{x}-\lambda x=\frac{2\alpha-1}{x}-\lambda x\text{”}$$

When $\frac{d}{dx}\log\big(g(x)\big)=0$ therefore $\frac{2\alpha-1}{x_0}=\lambda x_0$

Then

$$x_0=\left(\frac{(2\alpha-1)}{\lambda}\right)^{\frac{1}{2}}$$

The mean deviation about the mean and the median are useful measures of variation for a population. Let $\mu=E(X)$ and M be the mean and median of the LGR, respectively. The mean deviation about the mean is

$$
\begin{aligned}
E\{|X-\mu|\} &= \int_0^{\infty} |x-\mu|\, g(x)dx \\
&= \int_0^{\mu} (\mu-x) g(x)dx + \int_{\mu}^{\infty} (x-\mu) g(x)dx \\
&= 2\int_0^{\mu} (\mu-x) g(x)dx \\
&= 2\mu G(\mu) - 2\int_0^{\mu} x g(x)dx \\
&= \frac{2\mu}{\Gamma(\alpha)} \gamma(\alpha, \frac{\lambda^2\mu^2}{2}) - \frac{2}{\Gamma(\alpha)} \left(\frac{2}{\lambda}\right)^{\frac{1}{2}} \gamma\left(\alpha+\frac{1}{2}, \frac{\lambda^2\mu^2}{2}\right)
\end{aligned}
$$

In a similar way, the mean deviation from the median is given by

$$
\begin{aligned}
E\{|X-M|\} &= \int_0^{\infty} |x-M|\, g(x)dx \\
&= \int_0^{M} (M-x) g(x)dx + \int_{M}^{\infty} (x-M) g(x)dx \\
&= 2\int_0^{M} (M-x) g(x)dx + \int_0^{\infty} (x-M) g(x)dx \\
&= 2\int_0^{M} (M-x) g(x)dx + E(X) - M \\
&= 2MG(M) + \mu - M - 2\int_0^{M} x g(x)dx \\
&= \mu - 2\int_0^{M} x g(x)dx \\
&= \mu - \frac{2}{\Gamma(\alpha)} \left(\frac{2}{\lambda}\right)^{\frac{1}{2}} \gamma\left(\alpha+\frac{1}{2}, \frac{\lambda^2 M^2}{2}\right)
\end{aligned}
$$

5. DISTRIBUTION OF THE ORDER STATISTICS

In this section, we derive closed form expressions for the pdf of the r^{th} order statistic of the LGR. Let $X_1, X_2, ..., X_n$ be a simple random sample from LGR distribution with CDF and pdf given by (9) and (10), respectively. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the order statistics obtained from this sample. The pdf of the r^{th} order statistic of the LGR is

$$“ g_{X_{(r)}}(x)=\frac{n!x}{(r-1)!\,(n-r)!\lambda\Gamma(\alpha)}(\frac{\lambda x^2}{2})^{\alpha-1}e^{-\frac{\lambda x^2}{2}}\left(1-\frac{\gamma(\alpha,\frac{\lambda x^2}{2})}{\Gamma(\alpha)}\right)^{r-1}\left(\frac{\gamma(\alpha,\frac{\lambda x^2}{2})}{\Gamma(\alpha)}\right)^{n-r} ”$$

The pdf of the largest order statistic $X_{(n)}$ is therefore

$$g_{X_{(n)}}(x)=\frac{n\lambda x}{\Gamma(\alpha)}(\frac{\lambda x^2}{2})^{\alpha-1}e^{-\frac{\lambda x^2}{2}}\left(1-\frac{\gamma(\alpha,\frac{\lambda x^2}{2})}{\Gamma(\alpha)}\right)^{n-1}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$g_{X_{(1)}}(x)=\frac{n\lambda x}{\Gamma(\alpha)}(\frac{\lambda x^2}{2})^{\alpha-1}e^{-\frac{\lambda x^2}{2}}\left(\frac{\gamma(\alpha,\frac{\lambda x^2}{2})}{\Gamma(\alpha)}\right)^{n-1}$$

6. PARAMETERS ESTIMATION

Let $X_1, X_2, ..., X_n$ be a random sample from a LGR with parameters α and λ, then the log-likelihood function from (10) is given by

$$\begin{aligned}\log L(\alpha,\lambda)=\ell=\sum_{i=1}^{n}\log\big(g(x_i)\big)&=n[\log\lambda-\log\Gamma(\alpha)]+\sum_{i=1}^{n}\log(x_i)+2(\alpha-1)\sum_{i=1}^{n}\log(x_i)\\&+n(\alpha-1)\log\lambda-n(\alpha-1)\log 2-\frac{\lambda}{2}\sum_{i=1}^{n}x_i^2\\&=n\alpha\log\lambda+(2\alpha-1)\sum_{i=1}^{n}\log(x_i)-n\log\Gamma(\alpha)-n(\alpha-1)\log 2-\frac{\lambda}{2}\sum_{i=1}^{n}x_i^2\end{aligned} \quad (12)$$

The first partial derivatives of (12) are

$$\frac{\partial\ell}{\partial\alpha}=n\log\lambda+2\sum_{i=1}^{n}\log(x_i)-n\,\psi(\alpha)-n\log 2$$

and

$$\frac{\partial\ell}{\partial\lambda}=\frac{n\alpha}{\lambda}-\frac{1}{2}\sum_{i=1}^{n}x_i^2$$

“The MLE of the parameters α and λ, say $\hat{\alpha}$ and $\hat{\lambda}$ are obtained by solving the equations $\frac{\partial\ell}{\partial\alpha}=\frac{\partial\ell}{\partial\lambda}=0$
Therefore”

$$\hat{\alpha} = \psi^{-1}\left(\log 2 - \log \hat{\lambda} - \frac{2}{n}\sum_{i=1}^{n}\log(x_i)\right)$$

and

$$\hat{\lambda} = \frac{2n\hat{\alpha}}{\sum_{i=1}^{n} x_i^2}$$

The second derivatives with respect to α and λ will be:

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -n\,\psi'(\alpha)$$

$$\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} = \frac{n}{\lambda}$$

$$\frac{\partial^2 \ell}{\partial \lambda^2} = -\frac{n\alpha}{\lambda^2}$$

"Now we can derive the elements of the Fisher information matrix as follows"

"$I_{1,1} = -E\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right)$, $I_{2,2} = -E\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right)$, $I_{1,2} = I_{2,1} = -E\left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \lambda}\right)$
"

then the Fisher information matrix is

$$I = \begin{pmatrix} I_{1,1} & I_{1,2} \\ I_{2,1} & I_{2,2} \end{pmatrix}$$

The variance-covariance matrix of $(\hat{\alpha}, \hat{\lambda})$ is obtained by inverting the Fisher information matrix as follows:

$$I^{-1}(\hat{\alpha}, \hat{\lambda}) = \begin{pmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha}, \hat{\lambda}) \\ Cov(\hat{\alpha}, \hat{\lambda}) & Var(\hat{\lambda}) \end{pmatrix}$$

7. NUMERICAL ILLUSTRATION

In this section, random numbers are generated using the CDF of the LGR distribution, and then the maximum likelihood estimates are obtained.

We will generate 1000 samples of each of sizes 10, 15,…,30 from the LGR distribution for different values of the parameters α and λ, using the CDF of LGR, and then the maximum likelihood estimates for each sample will be obtained, along with the mean, root of the mean square error, bias and standard error of those estimates. The steps of this procedure will be as the following:

1. Set initial values for the parameters α and λ.

2. Generate 1000 samples of each of sizes 10, 15,…, 30, using (11).

3. Obtain the maximum likelihood estimates for α and λ for the different sample sizes.
4. Obtain the mean, biases, root of the mean square error and standard errors for the MLE estimates for the different sample sizes.
5. Repeat steps 1:4 for different values of α and λ (Results are listed in Table (2)).

Table 2. *Means, Biases, Root of the Mean Square Errors and Standard Errors for the MLEs of LGR distribution for different values of parameters*

n		$\alpha=0.5$	$\lambda=0.25$	$\alpha=0.75$	$\lambda=0.5$	$\alpha=0.85$	$\lambda=0.75$
10	Mean	0.474	0.381	1.221	1.027	1.916	154.342
	Biase	-0.026	0.131	0.471	0.527	1.066	79.342
	R.MSE	2.146	0.402	3.237	1.174	3.823	174.71
	S.E	0.215	0.038	0.32	0.105	0.367	15.573
15	Mean	0.791	0.311	1.319	0.761	2.015	132.179
	Biase	0.291	0.061	0.569	0.261	1.165	57.179
	R.MSE	0.8	0.263	1.439	0.704	2.37	129.533
	S.E	0.05	0.017	0.088	0.044	0.138	7.753
20	Mean	0.466	0.276	0.724	0.632	1.371	106.606
	Biase	-0.034	0.026	-0.026	0.132	0.521	31.606
	R.MSE	1.075	0.199	1.67	0.49	1.893	86.92
	S.E	0.054	0.009891	0.084	0.024	0.091	4.051
25	Mean	0.629	0.251	0.999	0.593	1.452	98
	Biase	0.129	0.001491	0.249	0.093	0.602	23
	R.MSE	0.497	0.176	0.856	0.449	1.291	73.624
	S.E	0.019	0.007031	0.033	0.018	0.046	2.799
30	Mean	0.385	0.233	0.683	0.547	1.253	92.286
	Biase	-0.115	-0.017	-0.067	0.047	0.403	17.286
	R.MSE	0.864	0.142	1.252	0.388	1.315	65.814
	S.E	0.029	0.004708	0.042	0.013	0.042	2.118

8. APPLICATION

This section presents application of LGR using real data set. In this application, we obtain the maximum likelihood estimates of the parameters of the fitted distributions. LGR is compared with other distributions (Weibull Rayleigh distribution (WR), Expontiated Weibull distributions (EW) and Expontiated Rayleigh (ER)) based on the maximized log-likelihood, the Kolmogorov-Smirnov (K-S) test along with the corresponding p-value, Akaike Information Criterion (AIC), Bayesian Information Criteria (BIC), Anderson-Darling statistic (AD), and Cramer von Mises statistic (CM).The data set was taken from Crowder [6], which gives the breaking strengths of single carbon fibers of different lengths:

2.247, 2.64, 2.842, 2.908, 3.099, 3.126, 3.245, 3.328, 3.355, 3.383, 3.572, 3.581, 3.681, 3.726, 3.727, 3.728, 3.783, 3.785, 3.786, 3.898, 3.912, 3.964, 4.05, 4.063, 4.082, 4.111, 4.118, 4.141, 4.216, 4.251, 4.262, 4.326, 4.402, 4.457, 4.466, 4.519, 4.542, 4.555, 4.614, 4.632, 4.634, 4.636, 4.678, 4.698, 4.738, 4.832, 4.924, 5.043, 5.099, 5.134, 5.359, 5.473, 5.571, 5.684, 5.721, 5.998, 6.06

Table 3. *Summarized results of fitting different distributions to the data set*

Model	MLEs		-2 log L	AIC	BIC	KS	P value	AD	CM
LGR	$\hat{\alpha}$	6.60549	140.266	144.266	148.353	0.056	0.994	0.155	0.023
	$\hat{\lambda}$	0.70943							
WR	$\hat{c}$	0.0514	144.45	150.45	156.579	0.109	0.508	0.684	0.114
	$\hat{\gamma}$	1.93513							
	$\hat{\sigma}$	2.13298							
EW	$\hat{a}$	280.997	148.962	154.962	161.091	4.419	0	0.776	0.124
	$\hat{\theta}$	0.517							
	$\hat{k}$	0.867							
ER	$\hat{\delta}$	9.97416	141.973	145.973	150.059	0.068	0.953	0.261	0.042
	$\hat{\sigma}$	1.79066							

Because the LGR has the lowest -2logL, AIC, BIC, KS, AD and CM statistics and the largest P value in Table 3, it can be concluded that the LGR is a strong competitor to other distributions used here for fitting data set.

The variance covariance matrix of the MLEs under the LGR for the data set is computed as

$$I^{-1} = \begin{pmatrix} 0.026 & 2.747\times10^{-3} \\ 2.747\times10^{-3} & 3.184\times10^{-4} \end{pmatrix}$$

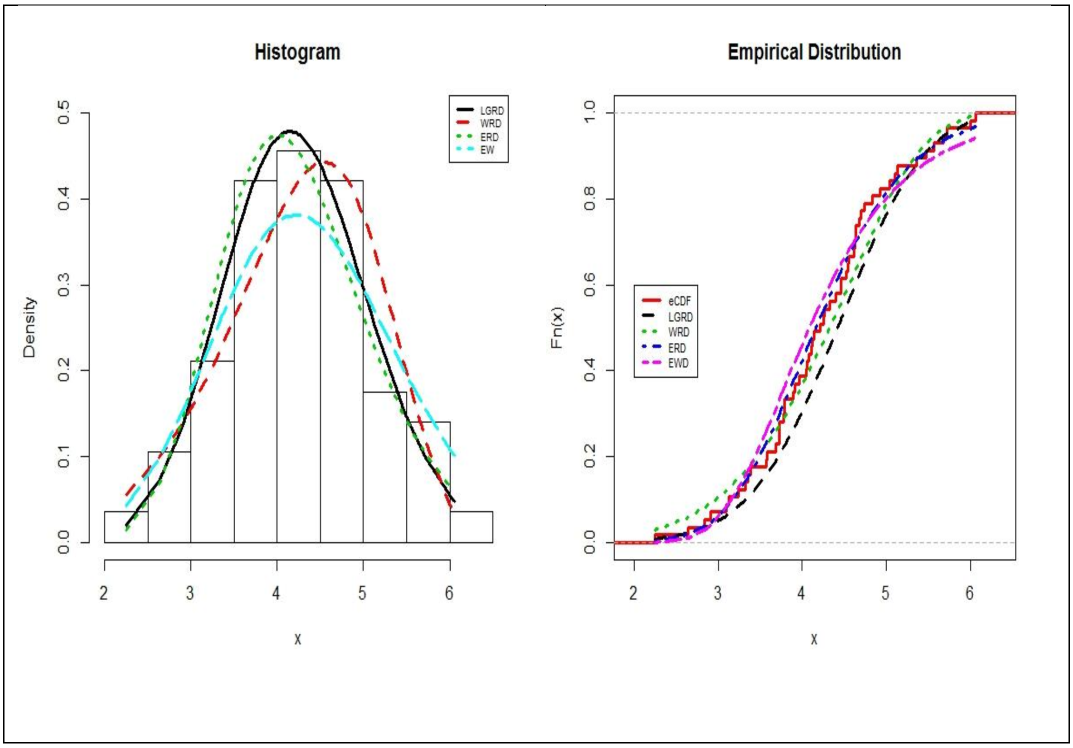

Figure 5. *The fitted densities of distributions for the data set*

9. CONCLUSIONS

This article defined a new generalization of Gamma distribution and Rayleigh distribution using the T-X method, called the log-gamma - Rayleigh distribution (LGR). Various properties of the distribution were studied. The moments, deviations from the mean and median, mode, survival function, hazard function and the maximum likelihood estimates of the parameters, have been investigated. The application of the new distribution has also been demonstrated with real life data. The results, compared with other known distributions, revealed that the LGR provides a better fit for modeling real life data.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Akinsete, A., Famoye, F., Lee, C., "The beta-Pareto distribution", Statistics, 42(6): 547-563, (2008).

[2] Alshawarbeh, E., Lee, C., Famoye, F., "The beta-Cauchy distribution", Journal of Probability and Statistical Science, 10(1): 41-58, (2012).

[3] Alzaatreh, A., Lee, C., Famoye, F., "A new method for generating families of continuous distributions", Metron, 71(1): 63-79, (2013).

[4] Amini, M, MirMostafaee, S. M. T. K. and Ahmadi, J., "Log-gamma-generated families of distributions", A journal of Theoretical and Applied Statistics, 48 (4): 913-932, (2014).

[5] Barreto-Souza, W., Santos, A.H.S., Cordeiro, G.M., "The beta generalized exponential distribution", Journal of Statistical Computation and Simulation, 80(2): 159-172, (2010).

[6] Crowder, M. J., "Classical Competing Risks", Chapman and Hall/CRC, Boca Raton, (2001).

[7] Eugene, N., Lee, C., Famoye, F., "Beta-normal distribution and its applications", Communications in Statistics-Theory and Methods, 31(4): 497-512, (2002).

[8] Famoye, F., Lee, C., Olumolade, O., "The beta-Weibull distribution", Journal of Statistical Theory and Applications, 4(2): 121-136, (2005).

[9] Kong, L., Lee, C., Sepanski, J.H., "On the properties of beta-gamma distribution", Journal of Modern Applied Statistical Methods, 6(1): 187-211, (2007).

[10] Mahmoudi, E., "The beta generalized Pareto distribution with application to lifetime data", Mathematics and Computers in Simulation, 81(11): 2414-2430, (2011).

[11] Merovci,F., Elbatal, I., "Weibull Rayleigh Distribution: Theory and Applications" , Applied Mathematics & Information Sciences, 9(5): 1-11, (2015).

[12] Nadarajah, S., Kotz, S., "The beta Gumbel distribution", Mathematical Problems in Engineering, 4: 323-332, (2004).

[13] Nadarajah, S., Kotz, S., "The beta exponential distribution", Reliability Engineering and System Safety, 91: 689-697, (2006).

[14] Pinho, L. G. B., Cordeiro, G. M., Nobre, J. S., "The gamma- exponentiated Weibull distribution", Journal of Statistical Theory and Applications, 11 (4): 379- 395, (2012).