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Novel Concept of Cubic Picture Fuzzy Sets

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Abstaract − As a new extension of a cubic set, the notion of a cubic picture fuzzy set is introduced. The propose work is separated into two portions. Firstly, establish the concept of cubic picture fuzzy set and explore associated properties. Secondly, establish internal (external) cubic picture fuzzy sets and define P-order and R-order union and intersection. Deliver some examples to support of established P-order and R-order union and intersection of internal (external) cubic picture fuzzy sets.

Keywords − *Cubic Picture fuzzy set, P-order union and intersection, R-order union and intersection.*

1 Introduction

Since, in 1965 Zadeh established fuzzy set theory, its become an essential tool to grip inaccurate and vagueness material in different area's of prevailing civilization. Such inaccuracies are associated with the membership function that belongs to [0,1]. Through membership function, we obtain information which makes possible for us to reach the conclusion. The fuzzy set theory, becomes a strong area of making observations in different area's like medical science, social sciences, engineering, management sciences, artificial intelligence, robotics, computer networks, decision making and so on. Due to unassociated sorts of unpredictably's occurring in different area's of life like economics, engineering, medical sciences, management sciences, psychology, sociology, decision making and fuzzy set as noted and often effective mathematical instruments have been offered to make, be moving in and grip those unpredictably's.

Since the establishment of fuzzy set, several extensions have been made such as Atanassov's [3] work on intuitionistic fuzzy set (IFSs) was quite remarkable as he

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extended the concept of FSs by assigning non-membership degree say "$N(x)$" along with membership degree say "$P(x)$" with condition that $0 \leq P(x)+N(x) \leq 1$. Form last few decades, the IFS has been explored by many researchers and successfully applied to many practical fields like medical diagnosis, clustering analysis, decision making pattern recognition [3, 4, 5, 6, 16, 19, 20]. Strengthening the concept IFS Yager suggest Pythagorean fuzzy sets [17] which somehow enlarge the space of positive membership and negative membership by introducing some new condition that $0 \leq P^2(x) + N^2(x) \leq 1$.

Atanassov's [3] structure discourses only satisfaction and dissatisfaction degree of elements in a set which is quite insufficient as human nature has some sort of abstain and refusal issues too. Such hitches were considered by Cuong [9] and he proposed picture fuzzy sets (PFS) of the form $(P(x), I(x), N(x))$ where the elements in triplet represent satisfaction, abstain and dissatisfaction degrees respectively, under the condition that $0 \leq P(x) + I(x) + N(x) \leq 1$ and with refusal degree defined as $1 - (P(x), I(x), N(x))$. The structure of Cuong [9] PFS is considerably more close to human nature than that of earlier concepts and is one of the richest research area now. For undefined terms and notions related to these areas one may refer to [7, 8, 9, 10]. Based on the combination of interval fuzzy sets extension and fuzzy sets extension, Jun. work on cubic set was quite remarkable. He establish cubic set with their operational properties and applied to BCK/BCI-algebras.

In this paper, we extend the structure of cubic sets to the picture fuzzy sets. We establish the concepts of positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets and positive-external (neutral -external, negative-external) cubic picture fuzzy sets, and explore associated properties. We illustrate that the P-order union and intersection of positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets are also positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets. We deliver examples to show that the P-order union and intersection of positive-external (neutral-external, negative-external) cubic picture fuzzy sets may not be positive-external (neutral-external, negative-external) cubic picture fuzzy sets, and the R- order union and intersection of positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets may not be positive-internal (neutral-internal, negative-internal) cubic picture fuzzy sets. By providing some more conditions for the R- order union of two positive-internal (resp. neutral-internal and negative-internal) cubic picture fuzzy sets to be a positive-internal (resp. neutral-internal and negative-internal) cubic picture fuzzy set.

2 Preliminary

The paper gives brief discussion on basic ideas associated to FS, IFS, PFS and Cubic set with their operations and operators utilizing triangular norm and conorm. Also discuss, more familiarized ideas which utilized in following analysis.

Definition 2.1. [18] Let the universe set be $R \neq \phi$. Then

$$A = \{\langle r, P_A(r)|\ r \in R\rangle\},$$

is said to be a fuzzy set of R, where $P_A : R \to [0, 1]$ is said to be the membership degree of r in R.

Definition 2.2. [4] Let Ω be the collection of all closed subintervals of $[0,1]$ and $P_A = [P_{LA}, P_{UA}] \in \Omega$, where P_{LA} and P_{UA} are the lower extreme and the upper extreme, respectively. Let the universe set be $R \neq \phi$. Then

$$A = \{\langle r, \pounds_{P_A}(r)|\ r \in R\rangle\},$$

is said to be an interval valued fuzzy set of R, where $\pounds_{P_A} : R \to \Omega$ is said to be the membership degree of r in R and $\pounds_{P_A} = [\pounds_{P_{LA}}, \pounds_{P_{UA}}]$ is said to be an interval valued fuzzy number.

Definition 2.3. [4] Let the universe set be $R \neq \phi$. Then

$$A = \{\langle r, P_A(r), N_A(r)|\ r \in R\rangle\},$$

is said to be an intuitionistic fuzzy set of R, where $P_A : R \to [0,1]$ and $N_A : R \to [0,1]$ are said to be the degree of positive-membership of r in R and the negative-membership degree of r in R respectively. Also P_A and N_A satisfy the following condition:

$$(\forall r \in R)\ (0 \leq P_A(r) + N_A(r) \leq 1).$$

Definition 2.4. [7] Let the universe set be $R \neq \phi$. Then the set

$$A = \{\langle r, P_A(r), I_A(r), N_A(r)|\ r \in R\rangle\},$$

is said to be a picture fuzzy set of R, where $P_A : R \to [0,1]$, $I_A : R \to [0,1]$ and $N_A : R \to [0,1]$ are said to be the degree of positive-membership of r in R, the neutral-membership degree of r in R and the negative-membership degree of r in R respectively. Also P_A, I_A *and* N_A satisfy the following condition:

$$(\forall r \in R)\ \ (0 \leq P_A(r) + I_A(r) + N_A(r) \leq 1).$$

Definition 2.5. [7] Let the universe set be $R \neq \phi$. Then the set

$$A = \{\langle r, \pounds_{P_A}(r), \pounds_{I_A}(r), \pounds_{N_A}(r)|\ r \in R\rangle\},$$

is said to be an interval valued picture fuzzy set of R, where $\pounds_{P_A} : R \to \Omega$, $\pounds_{I_A} : R \to \Omega$ and $\pounds_{N_A} : R \to \Omega$ are said to be the degree of positive-membership of r in R, the neutral-membership degree of r in R and the negative-membership degree of r in R respectively. Also $\pounds_{P_A}$, $\pounds_{I_A}$ *and* $\pounds_{N_A}$ satisfy the following condition:

$$(\forall r \in R)\ \ (0 \leq Sup\,(\pounds_{P_A}(r)) + Sup\,(\pounds_{I_A}(r)) + Sup\,(\pounds_{N_A}(r)) \leq 1).$$

Definition 2.6. [11] Let the universe set be $R \neq \phi$. Then the set

$$A = \{\langle r, \pounds_A(r), e_A(r)|\ r \in R\rangle\},$$

is said to be a cubic set of R, where $\pounds_A(r)$ is an interval-valued fuzzy set in R and $e_A(r)$ is a fuzzy set in R. For simplicity we denoted the cubic set as, $A = \langle \pounds_A, e_A\rangle$.

Definition 2.7. [11] Let the universe set be $R \neq \phi$. Then the cubic set $A = \langle \pounds_A, e_A\rangle$ is said to be an internal cubic set. if,

$$\pounds_A^-(r) \leq e_A(r) \leq \pounds_A^+(r) \text{ for all } r \in R.$$

Definition 2.8. [11] Let the universe set be $R \neq \phi$. Then the cubic set $A = \langle \pounds_A, e_A \rangle$ is said to be an external cubic set. if,

$$e_A(r) \notin \left(\pounds_A^-(r), \pounds_A^+(r)\right) \text{ for all } r \in R.$$

Example 2.9. Let the universe set be $R \neq \phi$ .Then $A = \{\langle r, \pounds_A(r), e_A(r) | \ r \in R \rangle\}$, be a cubic set of R. If $\pounds_A(r) = [0.3, 0.6]$ and $e_A(r) = 0.5$ for all $r \in R$, Then the set A is called an internal cubic set. If $\pounds_A(r) = [0.3, 0.6]$ and $e_A(r) = 0.8$ for all $r \in R$, Then the set A is called an external cubic set. If $\pounds_A(r) = [0.3, 0.6]$ and $e_A(r) = r$ for all $r \in R$, Then the set A do not belong to both classes of an internal & external cubic sets.

3 Cubic Picture Fuzzy Sets and their Operations

Definition 3.1. Let the universe set be $R \neq \phi$. Then the set

$$A = \{\langle r, \pounds_A(r), e_A(r) | \ r \in R \rangle\},$$

is said to be a cubic picture fuzzy set of R, where

$$\pounds_A(r) = \{\langle r, \pounds_{P_A}(r), \pounds_{I_A}(r), \pounds_{N_A}(r) | \ r \in R \rangle\}$$

is an interval-valued picture fuzzy set in R and

$$e_A(r) = \{\langle r, P_A(r), I_A(r), N_A(r) | \ r \in R \rangle\}$$

is a picture fuzzy set in R. For simplicity we denoted the cubic picture fuzzy set as, $A = \langle \pounds_A, e_A \rangle$.

Example 3.2. Let the universe set be $R = \{r_1, r_2, r_3\}$. Then the cubic picture fuzzy set $A = \langle \pounds_A, e_A \rangle$ with the tabular representation as (below).

cubic picture fuzzy set $A = \langle \pounds_A, e_A \rangle$

R	$\pounds_A$	e_A
r_1	$([0.3, 0.4], [0.0, 0.1], [0.4, 0.4])$	$(0.5, 0.1, 0.3)$
r_2	$([0.2, 0.3], [0.3, 0.4], [0.1, 0.2])$	$(0.4, 0.4, 0.1)$
r_3	$([0.1, 0.3], [0.3, 0.3], [0.2, 0.3])$	$(0.2, 0.1, 0.6)$

Example 3.3. Let the universe set be $R \neq \phi$ and any interval valued picture fuzzy set

$$\pounds_A(r) = \{\langle r, \pounds_{P_A}(r), \pounds_{I_A}(r), \pounds_{N_A}(r) | \ r \in R \rangle\},$$

in R. Assume that $A = \langle \pounds_A, e_{A_1} \rangle$ and $A = \langle \pounds_A, e_{A_0} \rangle$ are cubic picture fuzzy sets, where $e_{A_1} = \{\langle r, 1, 0, 0 | \ r \in R \rangle\}$ and $e_{A_0} = \{\langle r, 0, 0, 0 | \ r \in R \rangle\}$ in R. If we take $P_A(r) = \frac{\pounds_{P_A}^-(r) + \pounds_{P_A}^+(r)}{2}$, $I_A(r) = \frac{\pounds_{I_A}^-(r) + \pounds_{I_A}^+(r)}{2}$ and $N_A(r) = \frac{\pounds_{N_A}^-(r) + \pounds_{N_A}^+(r)}{2}$, then $A = \langle \pounds_A, e_A \rangle$ is a cubic picture fuzzy set in R.

Definition 3.4. Let the universe set be $R \neq \phi$. Then the cubic picture fuzzy set $A = \langle \pounds_A, e_A \rangle$ in R is said to be:

(1) Positive-internal if

$$\pounds^{-}_{P_A}(r) \leq P_A(r) \leq \pounds^{+}_{P_A}(r), \ \forall\ r \in R.$$

(2) Neutral-internal if

$$\pounds^{-}_{I_A}(r) \leq I_A(r) \leq \pounds^{+}_{I_A}(r), \ \forall\ r \in R.$$

(3) Nagative-internal if

$$\pounds^{-}_{N_A}(r) \leq N_A(r) \leq \pounds^{+}_{N_A}(r), \ \forall\ r \in R.$$

If the cubic picture fuzzy set $A = \langle \pounds_A, e_A \rangle$ in R is satisfies all the above properties then the cubic picture fuzzy set is said to be an internal cubic picture fuzzy set in R.

Example 3.5. Let the universe set be $R = \{r_1, r_2, r_3\}$. Then the pair $A = \langle \pounds_A, e_A \rangle$ is said to be an internal cubic picture fuzzy set with the tabular representation as (below).

Internal cubic picture fuzzy set $A = \langle \pounds_A, e_A \rangle$

R	$\pounds_A$	e_A
r_1	$([0.20, 0.33], [0.05, 0.26], [0.39, 0.41])$	$(0.25, 0.15, 0.40)$
r_2	$([0.11, 0.30], [0.30, 0.40], [0.15, 0.25])$	$(0.23, 0.39, 0.24)$
r_3	$([0.33, 0.43], [0.15, 0.28], [0.27, 0.29])$	$(0.42, 0.27, 0.28)$

Definition 3.6. Let the universe set be $R \neq \phi$. Then the cubic picture fuzzy set $A = \langle \pounds_A, e_A \rangle$ in R is said to be:

(1) Positive-external if

$$P_A(r) \notin \left(\pounds^{-}_{P_A}(r), \pounds^{+}_{P_A}(r)\right), \ \forall\ r \in R.$$

(2) Neutral-external if

$$I_A(r) \notin \left(\pounds^{-}_{I_A}(r), \pounds^{+}_{I_A}(r)\right), \ \forall\ r \in R.$$

(3) Nagative-external if

$$N_A(r) \notin \left(\pounds^{-}_{N_A}(r), \pounds^{+}_{N_A}(r)\right), \ \forall\ r \in R.$$

If the cubic picture fuzzy set $A = \langle \pounds_A, e_A \rangle$ in R is satisfies all the above properties then the cubic picture fuzzy set is said to be an external cubic picture fuzzy set in R.

Proposition 3.7. Let the universe set be $R \neq \phi$ and the pair $A = \langle \pounds_A, e_A \rangle$ be a cubic picture fuzzy set in R which is not external. Then there exists $r \in R$ such that, $P_A(r) \in \left(\pounds^{-}_{P_A}(r), \pounds^{+}_{P_A}(r)\right)$, $I_A(r) \in \left(\pounds^{-}_{I_A}(r), \pounds^{+}_{I_A}(r)\right)$ and $N_A(r) \in \left(\pounds^{-}_{N_A}(r), \pounds^{+}_{N_A}(r)\right)$.

Proof. Straightforward. □

Proposition 3.8. Let the universe set be $R \neq \phi$ and the pair $A = \langle \pounds_A, e_A \rangle$ be cubic picture fuzzy set in R. If the pair $A = \langle \pounds_A, e_A \rangle$ be both positive internal and external, then $\forall\ r \in R$

$$P_A(r) \in \left\{ \pounds^-_{P_A}(r) | r \in R \right\} \cup \left\{ \pounds^+_{P_A}(r) | r \in R \right\}.$$

Proof. By utilizing the definition 3.4, we have

$$\pounds^-_{P_A}(r) \leq P_A(r) \leq \pounds^+_{P_A}(r),\ \forall\ r \in R$$

and utilizing the definition 3.6, we have

$$P_A(r) \notin \left(\pounds^-_{P_A}(r), \pounds^+_{P_A}(r) \right),\ \forall\ r \in R$$

It follows that

$$P_A(r) = \pounds^-_{P_A}(r) \text{ or } P_A(r) = \pounds^+_{P_A}(r)$$

and so that

$$P_A(r) \in \left\{ \pounds^-_{P_A}(r) | r \in R \right\} \cup \left\{ \pounds^+_{P_A}(r) | r \in R \right\}.$$

□

Similarly, we have following proposition for other membership degrees.

Proposition 3.9. Let the universe set be $R \neq \phi$ and the pair $A = \langle \pounds_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle \pounds_A, e_A \rangle$ be both neutral internal and external, then $\forall\ r \in R$

$$I_A(r) \in \left\{ \pounds^-_{I_A}(r) | r \in R \right\} \cup \left\{ \pounds^+_{I_A}(r) | r \in R \right\}.$$

Proposition 3.10. Let the universe set be $R \neq \phi$ and the pair $A = \langle \pounds_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle \pounds_A, e_A \rangle$ be both negative internal and external, then $\forall\ r \in R$

$$N_A(r) \in \left\{ \pounds^-_{N_A}(r) | r \in R \right\} \cup \left\{ \pounds^+_{N_A}(r) | r \in R \right\}.$$

Now we define some operations on the cubic picture fuzzy sets and properties of defined operations are also examined. These operations will be of great use like in defining aggregation operators for cubic picture fuzzy sets.

In our further discussion R plays the role of universal set and three pairs $A = \langle \pounds_A, e_A \rangle, B = \langle \pounds_B, e_B \rangle$ & $C = \langle \pounds_C, e_C \rangle$ are the cubic picture fuzzy sets of the form

$A = \langle \pounds_A, e_A \rangle = \{r, \langle \pounds_{P_A}(r), \pounds_{I_A}(r), \pounds_{N_A}(r) \rangle, \langle P_A(r), I_A(r), N_A(r) \rangle | r \in R\}$, with the condition that

$0 \leq Sup\left[\pounds^-_{P_A}(r), \pounds^+_{P_A}(r)\right] + Sup\left[\pounds^-_{I_A}(r), \pounds^+_{I_A}(r)\right] + Sup\left[\pounds^-_{N_A}(r), \pounds^+_{N_A}(r)\right] \leq 1$ and $0 \leq P_A(r) + I_A(r) + N_A(r) \leq 1$.

$B = \langle \pounds_B, e_B \rangle = \{r, \langle \pounds_{P_B}(r), \pounds_{I_B}(r), \pounds_{N_B}(r) \rangle, \langle P_B(r), I_B(r), N_N(r) \rangle | r \in R\}$, with the condition that

$0 \leq Sup\left[\pounds^-_{P_B}(r), \pounds^+_{P_B}(r)\right] + Sup\left[\pounds^-_{I_B}(r), \pounds^+_{I_B}(r)\right] + Sup\left[\pounds^-_{N_B}(r), \pounds^+_{N_B}(r)\right] \leq 1$ and $0 \leq P_B(r) + I_B(r) + N_B(r) \leq 1$.

and

$C = \langle £_C, e_C \rangle = \{r, \langle £_{P_C}(r), £_{I_C}(r), £_{N_C}(r) \rangle, \langle P_C(r), I_C(r), N_C(r) \rangle | r \in R\}$, with the condition that

$0 \leq Sup\left[£^-_{P_C}(r), £^+_{P_C}(r)\right] + Sup\left[£^-_{I_C}(r), £^+_{I_C}(r)\right] + Sup\left[£^-_{N_C}(r), £^+_{N_C}(r)\right] \leq 1$ and $0 \leq P_C(r) + I_C(r) + N_C(r) \leq 1$.

Definition 3.11. Let for two cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$, we define equality as

$$A = B \Leftrightarrow £_A = £_B \text{ and } e_A = e_B.$$

Definition 3.12. Let for two cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$, we define P-order as

$$A \subseteq_p B \Leftrightarrow £_A \subseteq £_B \text{ and } e_A \leq e_B.$$

Definition 3.13. Let for two cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$, we define R-order as

$$A \subseteq_R B \Leftrightarrow £_A \subseteq £_B \text{ and } e_A \geq e_B.$$

Definition 3.14. Let $A_j = \left\langle £_{A_j}, e_{A_j} \right\rangle$ be collection of cubic picture fuzzy sets in $R \neq 0$. Then we define P-order union as

$$\bigcup_{j\in I}^{P} A_j = \left(\bigcup_{j\in I} £_{A_j}, \bigvee_{j\in I} e_{A_j} \right),$$

where I be an index set and

$$\bigcup_{j\in I} £_{A_j} = \left\{ \left\langle r; \left(\bigcup_{j\in I} £_{P_{A_j}} \right)(r), \left(\bigcup_{j\in I} £_{I_{A_j}} \right)(r), \left(\bigcup_{j\in I} £_{N_{A_j}} \right)(r) \right\rangle | r \in R \right\},$$

$$\bigvee_{j\in I} e_{A_j} = \left\{ \left\langle r; \left(\bigvee_{j\in I} P_{A_j} \right)(r), \left(\bigvee_{j\in I} I_{A_j} \right)(r), \left(\bigvee_{j\in I} N_{A_j} \right)(r) \right\rangle | r \in R \right\}.$$

Definition 3.15. Let $A_j = \left\langle £_{A_j}, e_{A_j} \right\rangle$ be collection of cubic picture fuzzy sets in $R \neq 0$. Then we define R-order union as

$$\bigcup_{j\in I}^{R} A_j = \left(\bigcup_{j\in I} £_{A_j}, \bigwedge_{j\in I} e_{A_j} \right),$$

where

$$\bigcup_{j\in I} £_{A_j} = \left\{ \left\langle r; \left(\bigcup_{j\in I} £_{P_{A_j}} \right)(r), \left(\bigcup_{j\in I} £_{I_{A_j}} \right)(r), \left(\bigcup_{j\in I} £_{N_{A_j}} \right)(r) \right\rangle | r \in R \right\},$$

$$\bigwedge_{j\in I} e_{A_j} = \left\{ \left\langle r; \left(\bigwedge_{j\in I} P_{A_j} \right)(r), \left(\bigwedge_{j\in I} I_{A_j} \right)(r), \left(\bigwedge_{j\in I} N_{A_j} \right)(r) \right\rangle | r \in R \right\}.$$

Definition 3.16. Let $A_j = \left\langle £_{A_j}, e_{A_j} \right\rangle$ be collection of cubic picture fuzzy sets in $R \neq 0$. Then we define P-order intersection as

$$\bigcap_{j\in I}^{P} A_j = \left(\bigcap_{j\in I} £_{A_j}, \bigwedge_{j\in I} e_{A_j} \right),$$

where

$$\bigcap_{j\in I} \pounds_{A_j} = \left\{ \left\langle r; \left(\bigcap_{j\in I} \pounds_{P_{A_j}} \right)(r), \left(\bigcap_{j\in I} \pounds_{I_{A_j}} \right)(r), \left(\bigcap_{j\in I} \pounds_{N_{A_j}} \right)(r) \right\rangle |r \in R \right\},$$

$$\bigwedge_{j\in I} e_{A_j} = \left\{ \left\langle r; \left(\bigwedge_{j\in I} P_{A_j} \right)(r), \left(\bigwedge_{j\in I} I_{A_j} \right)(r), \left(\bigwedge_{j\in I} N_{A_j} \right)(r) \right\rangle |r \in R \right\}.$$

Definition 3.17. Let $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ be collection of cubic picture fuzzy sets in $R \neq 0$. Then we define R-order intersection as

$$\bigcap_{j\in I}^{P} A_j = \left(\bigcap_{j\in I} \pounds_{A_j}, \bigvee_{j\in I} e_{A_j} \right),$$

where

$$\bigcap_{j\in I} \pounds_{A_j} = \left\{ \left\langle r; \left(\bigcap_{j\in I} \pounds_{P_{A_j}} \right)(r), \left(\bigcap_{j\in I} \pounds_{I_{A_j}} \right)(r), \left(\bigcap_{j\in I} \pounds_{N_{A_j}} \right)(r) \right\rangle |r \in R \right\},$$

$$\bigvee_{j\in I} e_{A_j} = \left\{ \left\langle r; \left(\bigvee_{j\in I} P_{A_j} \right)(r), \left(\bigvee_{j\in I} I_{A_j} \right)(r), \left(\bigvee_{j\in I} N_{A_j} \right)(r) \right\rangle |r \in R \right\}.$$

The complement of $A = \langle \pounds_A, e_A \rangle$ is define as $A^c = \langle \pounds_A^c, e_A^c \rangle$ be also cubic picture fuzzy set, where $\pounds_A^c = \{\langle r, \pounds_{P_A}(r), \pounds_{I_A}(r), \pounds_{N_A}(r) \rangle \,|r \in R\}$ be an complement of interval picture fuzzy set and $e_A^c = \{\langle r, P_A^c(r), I_A^c(r), N_A^c(r) \rangle \,|r \in R\}$ be an complement of picture fuzzy set in R.

Obviously,

$\left(A_j^c\right)^c = A_j$, $\left(\bigcup_{j\in I}^{P} A_j \right)^c = \bigcap_{j\in I}^{P} A_j^c$, $\left(\bigcap_{j\in I}^{P} A_j \right)^c = \bigcup_{j\in I}^{P} A_j^c$, $\left(\bigcup_{j\in I}^{R} A_j \right)^c = \bigcap_{j\in I}^{R} A_j^c$ and $\left(\bigcap_{j\in I}^{R} A_j \right)^c = \bigcup_{j\in I}^{R} A_j^c$.

Proposition 3.18. Let for any cubic picture fuzzy sets $A = \langle \pounds_A, e_A \rangle$, $B = \langle \pounds_B, e_B \rangle$ and $C = \langle \pounds_C, e_C \rangle$ in R. For P-order we have

(1) If $A \subseteq_p B$ and $B \subseteq_p C$ then $A \subseteq_p C$.
(2) If $A \subseteq_p B$ then $B^C \subseteq_p A^C$.
(3) If $A \subseteq_p B$ and $A \subseteq_p C$ then $A \subseteq_p B \overset{P}{\bigcap} C$.
(3) If $A \subseteq_p B$ and $C \subseteq_p B$ then $A \overset{P}{\bigcup} C \subseteq_P B$.

Proposition 3.19. Let for any cubic picture fuzzy sets $A = \langle \pounds_A, e_A \rangle$, $B = \langle \pounds_B, e_B \rangle$ and $C = \langle \pounds_C, e_C \rangle$ in R. For R-order we have

(1) If $A \subseteq_R B$ and $B \subseteq_R C$ then $A \subseteq_R C$.
(2) If $A \subseteq_R B$ then $B^C \subseteq_R A^C$.
(3) If $A \subseteq_R B$ and $A \subseteq_R C$ then $A \subseteq_R B \overset{R}{\bigcap} C$.
(3) If $A \subseteq_R B$ and $C \subseteq_R B$ then $A \overset{R}{\bigcup} C \subseteq_R B$.

Theorem 3.20. Let the universe set be $R \neq \phi$ and the pair $A = \langle \pounds_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle \pounds_A, e_A \rangle$ be an positive internal (resp. positive external), then the complement $A^c = \langle \pounds_A^c, e_A^c \rangle$ of $A = \langle \pounds_A, e_A \rangle$ be an positive internal (resp. positive external) cubic picture fuzzy set in R.

Proof. If the pair $A = \langle \pounds_A, e_A \rangle$ be an positive internal (resp. positive external) cubic picture fuzzy set in R. Then by the definition3.4, we have

$\pounds^-_{P_A}(r) \leq P_A(r) \leq \pounds^+_{P_A}(r)$ $\left(resp., \ P_A(r) \notin \left(\pounds^-_{P_A}(r), \pounds^+_{P_A}(r)\right)\right)$, $\forall r \in R$ this implies that $\forall r \in R$

$1-\pounds^+_{P_A}(r) \leq 1-P_A(r) \leq 1-\pounds^-_{P_A}(r)$ $\left(resp., \ 1 - P_A(r) \notin \left(1 - \pounds^+_{P_A}(r), 1 - \pounds^-_{P_A}(r)\right)\right)$.

Therefore

$A^c = \langle \pounds^c_A, e^c_A \rangle$ be an positive internal (resp. positive external) cubic picture fuzzy set in R. □

Theorem 3.21. Let the universe set be $R \neq \phi$ and the pair $A = \langle \pounds_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle \pounds_A, e_A \rangle$ be an neutral internal (resp. neutral external), then the complement $A^c = \langle \pounds^c_A, e^c_A \rangle$ of $A = \langle \pounds_A, e_A \rangle$ be an neutral internal (resp. neutral external) cubic picture fuzzy set in R.

Theorem 3.22. Let the universe set be $R \neq \phi$ and the pair $A = \langle \pounds_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle \pounds_A, e_A \rangle$ be an negative internal (resp. negative external), then the complement $A^c = \langle \pounds^c_A, e^c_A \rangle$ of $A = \langle \pounds_A, e_A \rangle$ be an negative internal (resp. negative external) cubic picture fuzzy set in R.

Corollary 3.23. Let the universe set be $R \neq \phi$ and the pair $A = \langle \pounds_A, e_A \rangle$ be a cubic picture fuzzy set in R. If the pair $A = \langle \pounds_A, e_A \rangle$ be an internal (resp., external), then the complement $A^c = \langle \pounds^c_A, e^c_A \rangle$ of $A = \langle \pounds_A, e_A \rangle$ be an internal (resp., external) cubic picture fuzzy set in R.

Theorem 3.24. Let $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ be the collection of positive internal cubic picture fuzzy sets in $R \neq 0$. Then the P-order union and intersection of $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ are also be an positive internal cubic picture fuzzy sets in R.

Proof. Since, $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ be collection of positive internal cubic picture fuzzy sets in R,we have by definition3.4,

$$\pounds^-_{P_{A_j}}(r) \leq P_{A_j}(r) \leq \pounds^+_{P_{A_j}}(r), \forall r \in R, \ j \in I.$$

Then, it follows that

$$\left(\bigcup_{j\in I} \pounds_{P_{A_j}}\right)^- (r) \leq \left(\bigvee_{j\in I} P_{A_j}\right)(r) \leq \left(\bigcup_{j\in I} \pounds_{P_{A_j}}\right)^+ (r),$$

and

$$\left(\bigcap_{j\in I} \pounds_{P_{A_j}}\right)^- (r) \leq \left(\bigwedge_{j\in I} P_{A_j}\right)(r) \leq \left(\bigcap_{j\in I} \pounds_{P_{A_j}}\right)^+ (r).$$

Therefore

$\overset{P}{\bigcup_{j\in I}} A_j = \left(\bigcup_{j\in I} \pounds_{A_j}, \bigvee_{j\in I} e_{A_j}\right)$ and $\overset{P}{\bigcap_{j\in I}} A_j = \left(\bigcap_{j\in I} \pounds_{A_j}, \bigwedge_{j\in I} e_{A_j}\right)$ are positive internal cubic picture fuzzy sets in R. □

Similarly,

Theorem 3.25. Let $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ be the collection of neutral internal cubic picture fuzzy sets in $R \neq 0$. Then the P-order union and intersection of $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ are also neutral internal cubic picture fuzzy sets in R.

Theorem 3.26. Let $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ be the collection of negative internal cubic picture fuzzy sets in $R \neq 0$. Then the P-order union and intersection of $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ are also negative internal cubic picture fuzzy sets in R.

Corollary 3.27. Let $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ be collection of internal cubic picture fuzzy sets in $R \neq 0$. Then the P-order union and intersection of $A_j = \left\langle \pounds_{A_j}, e_{A_j} \right\rangle$ are also internal cubic picture fuzzy sets in R.

In below example we seen that every P-order union and intersection of negative external (resp., positive external, neutral external) cubic picture fuzzy sets may not be negative external (resp., positive external, neutral external) cubic picture fuzzy sets.

Example 3.28. Let for two cubic picture fuzzy sets $A = \langle \pounds_A, e_A \rangle$ and $B = \langle \pounds_B, e_B \rangle$ in R, define as

$A = \{r, \langle [0.32, 0.53], [0.09, 0.15], [0.22, 0.31] \rangle, \langle 0.48, 0.13, 0.36 \rangle \,| r \in [0,1]\}$ and
$B = \{r, \langle [0.29, 0.43], [0.24, 0.38], [0.06, 0.18] \rangle, \langle 0.37, 0.29, 0.33 \rangle \,| r \in [0,1]\}$.

Then A and B are negative external cubic picture fuzzy sets in $[0,1]$, and $A \overset{p}{\bigcup} B = (\pounds_A \cup \pounds_B, e_A \vee e_B)$ with

$\pounds_A \cup \pounds_B = \{r, \langle [0.32, 0.53], [0.24, 0.38], [0.22, 0.31] \rangle \,| r \in [0,1]\}$
$e_A \vee e_B = \{r, \langle 0.48, 0.29, 0.36 \rangle \,| r \in [0,1]\}$

is not an negative external cubic picture fuzzy sets in $[0,1]$.

also, $A \overset{p}{\bigcap} B = (\pounds_A \cap \pounds_B, e_A \wedge e_B)$ with

$\pounds_A \cap \pounds_B = \{r, \langle [0.29, 0.43], [0.09, 0.15], [0.06, 0.18] \rangle \,| r \in [0,1]\}$
$e_A \wedge e_B = \{r, \langle 0.37, 0.13, 0.33 \rangle \,| r \in [0,1]\}$

is not an negative external cubic picture fuzzy sets in $[0,1]$.

Example 3.29. Let the universe set be $R = \{r_1, r_2, r_3\}$. Then the pairs $A = \langle \pounds_A, e_A \rangle$ and $B = \langle \pounds_B, e_B \rangle$ are said to be a cubic picture fuzzy set with the tabular representation as (below).

cubic picture fuzzy set A

R	$\pounds_A$	e_A
r_1	$([0.20, 0.30], [0.15, 0.26], [0.39, 0.41])$	$(0.51, 0.08, 0.40)$
r_2	$([0.14, 0.28], [0.29, 0.39], [0.18, 0.28])$	$(0.13, 0.43, 0.24)$
r_3	$([0.41, 0.43], [0.09, 0.18], [0.27, 0.33])$	$(0.26, 0.27, 0.28)$

cubic picture fuzzy set B

R	$\pounds_B$	e_B
r_1	$([0.20, 0.33], [0.05, 0.16], [0.39, 0.41])$	$(0.35, 0.19, 0.40)$
r_2	$([0.11, 0.29], [0.30, 0.40], [0.15, 0.25])$	$(0.32, 0.29, 0.24)$
r_3	$([0.33, 0.39], [0.15, 0.23], [0.27, 0.29])$	$(0.42, 0.27, 0.28)$

$A \overset{p}{\bigcup} B = (\pounds_A \cup \pounds_B, e_A \vee e_B)$

R	$\pounds_A \cup \pounds_B$	$e_A \vee e_B$
r_1	$([0.20, 0.33], [0.15, 0.26], [0.39, 0.41])$	$(0.51, 0.19, 0.40)$
r_2	$([0.14, 0.29], [0.30, 0.40], [0.18, 0.28])$	$(0.32, 0.43, 0.24)$
r_3	$([0.41, 0.43], [0.15, 0.23], [0.27, 0.33])$	$(0.42, 0.27, 0.28)$

$$A \overset{p}{\bigcap} B = (\pounds_A \cap \pounds_B, e_A \wedge e_B)$$

R	$\pounds_A \cap \pounds_B$	$e_A \wedge e_B$
r_1	$([0.20, 0.30], [0.05, 0.16], [0.39, 0.41])$	$(0.35, 0.08, 0.40)$
r_2	$([0.11, 0.28], [0.29, 0.39], [0.15, 0.25])$	$(0.13, 0.29, 0.24)$
r_3	$([0.33, 0.39], [0.09, 0.18], [0.27, 0.29])$	$(0.26, 0.27, 0.28)$

Then $A = \langle \pounds_A, e_A \rangle$ and $B = \langle \pounds_B, e_B \rangle$ are both positive and neutral external cubic picture fuzzy sets in $[0, 1]$, and tabular representation of $A \overset{p}{\bigcup} B$, $A \overset{p}{\bigcap} B$ are neither an positive external nor neutral external cubic picture fuzzy sets.

Similarly, we seen that every P-order union and intersection of positive internal (resp., neutral internal, negative internal) cubic picture fuzzy sets may not be positive internal (resp., neutral internal, negative internal) cubic picture fuzzy sets.

Example 3.30. Let for two cubic picture fuzzy sets $A = \langle \pounds_A, e_A \rangle$ and $B = \langle \pounds_B, e_B \rangle$ in R, define as

$A = \{r, \langle [0.32, 0.53], [0.09, 0.15], [0.22, 0.31] \rangle, \langle 0.48, 0.18, 0.33 \rangle \, | r \in [0, 1]\}$ and
$B = \{r, \langle [0.29, 0.43], [0.24, 0.35], [0.06, 0.18] \rangle, \langle 0.37, 0.37, 0.33 \rangle \, | r \in [0, 1]\}$.

Then A and B are positive internal cubic picture fuzzy sets in $[0, 1]$, and $A \overset{p}{\bigcup} B = (\pounds_A \cup \pounds_B, e_A \vee e_B)$ with

$\pounds_A \cup \pounds_B = \{r, \langle [0.32, 0.53], [0.24, 0.35], [0.22, 0.31] \rangle \, | r \in [0, 1]\}$
$e_A \vee e_B = \{r, \langle 0.48, 0.37, 0.33 \rangle \, | r \in [0, 1]\}$

is not an positive internal cubic picture fuzzy sets in $[0, 1]$.

also, $A \overset{p}{\bigcap} B = (\pounds_A \cap \pounds_B, e_A \wedge e_B)$ with

$\pounds_A \cap \pounds_B = \{r, \langle [0.29, 0.43], [0.09, 0.15], [0.06, 0.18] \rangle \, | r \in [0, 1]\}$
$e_A \wedge e_B = \{r, \langle 0.37, 0.18, 0.33 \rangle \, | r \in [0, 1]\}$

is not an positive internal cubic picture fuzzy sets in $[0, 1]$.

Theorem 3.31. Let the universe set be $R \neq \phi$ and for two positive internal cubic picture fuzzy sets $A = \langle \pounds_A, e_A \rangle$ and $B = \langle \pounds_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$\max \left\{ \pounds^-_{P_A}(r), \pounds^-_{P_B}(r) \right\} \leq (P_A \wedge P_B)(r).$$

Then R-order union of A and B is a posative internal cubic picture fuzzy set in R.

Proof. Since $A = \langle \pounds_A, e_A \rangle$ and $B = \langle \pounds_B, e_B \rangle$ are two positive internal cubic picture fuzzy sets in R. Then by definition 3.4 we have $\forall \; r \in R$,

$\pounds^-_{P_A}(r) \leq P_A(r) \leq \pounds^+_{P_A}(r)$, and $\pounds^-_{P_B}(r) \leq P_B(r) \leq \pounds^+_{P_B}(r)$,

and so $(P_A \wedge P_B)(r) \leq (\pounds_{P_A} \cup \pounds_{P_B})^+ (r)$,

it follows that

$(\pounds_{P_A} \cup \pounds_{P_B})^- (r) = \max \left\{ \pounds^-_{P_A}(r), \pounds^-_{P_B}(r) \right\} \leq (P_A \wedge P_B)(r) \leq (\pounds_{P_A} \cup \pounds_{P_B})^+ (r)$.

Hence

$A \overset{R}{\bigcup} B = (\pounds_A \cup \pounds_B, e_A \wedge e_B)$ is a R-order union of A and B is a positive internal cubic picture fuzzy set in R. ☐

Theorem 3.32. Let the universe set be $R \neq \phi$ and for two neutral internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$\max\left\{£^-_{I_A}(r), £^-_{I_B}(r)\right\} \leq (I_A \wedge I_B)(r).$$

Then R-order union of A and B is a neutral internal cubic picture fuzzy set in R.

Theorem 3.33. Let the universe set be $R \neq \phi$ and for two negative internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$\max\left\{£^-_{N_A}(r), £^-_{N_B}(r)\right\} \leq (N_A \wedge N_B)(r).$$

Then R-order union of A and B is a negative internal cubic picture fuzzy set in R.

Corollary 3.34. Let the universe set be $R \neq \phi$ and for two internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. If R-order union of A and B is satisfies all conditions of positive, neutral and negative internal cubic picture fuzzy sets. Then R-order union of A and B is said to be internal cubic picture fuzzy set in R.

Theorem 3.35. Let the universe set be $R \neq \phi$ and for two positive internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$(P_A \vee P_B)(r) \leq \min\left\{£^+_{P_A}(r), £^+_{P_B}(r)\right\}.$$

Then R-order intersection of A and B is a positive internal cubic picture fuzzy set in R.

Proof. Since $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ are two positive internal cubic picture fuzzy sets in R. Then by definition 3.4 we have $\forall\ r \in R$,

$£^-_{P_A}(r) \leq P_A(r) \leq £^+_{P_A}(r)$, and $£^-_{P_B}(r) \leq P_B(r) \leq £^+_{P_B}(r)$,
and so $\left(£_{P_A} \cap £_{P_B}\right)^-(r) \leq (P_A \vee P_B)(r)$,
it follows that

$\left(£_{P_A} \cap £_{P_B}\right)^-(r) \leq (P_A \vee P_B)(r) \leq \min\left\{£^+_{P_A}(r), £^+_{P_B}(r)\right\} = \left(£_{P_A} \cap £_{P_B}\right)^+(r)$.
Hence

$A \overset{R}{\bigcap} B = (£_A \cap £_B, e_A \vee e_B)$ is a R-order intersection of A and B is a positive internal cubic picture fuzzy set in R. □

Theorem 3.36. Let the universe set be $R \neq \phi$ and for two neutral internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$(I_A \vee I_B)(r) \leq \min\left\{£^+_{I_A}(r), £^+_{I_B}(r)\right\}.$$

Then R-order intersection of A and B is a neutral internal cubic picture fuzzy set in R.

Theorem 3.37. Let the universe set be $R \neq \phi$ and for two negative internal cubic picture fuzzy sets $A = \langle £_A, e_A \rangle$ and $B = \langle £_B, e_B \rangle$ in R. Then for all $r \in R$ such that

$$(N_A \vee N_B)(r) \leq \min\left\{£^+_{N_A}(r), £^+_{N_B}(r)\right\}.$$

Then R-order intersection of A and B is a negative internal cubic picture fuzzy set in R.

Corollary 3.38. Let the universe set be $R \neq \phi$ and for two internal cubic picture fuzzy sets $A = \langle \pounds_A, e_A \rangle$ and $B = \langle \pounds_B, e_B \rangle$ in R. If R-order intersection of A and B is satisfies all conditions of positive, neutral and negative internal cubic picture fuzzy sets. Then R-order intersection of A and B is said to be internal cubic picture fuzzy set in R.

References

[1] S. Ashraf, T. Mahmood, S. Abdullah and Q. Khan, *Different Approaches to Multi-Criteria Group Decision Making Problems for Picture Fuzzy Environment*, Bulletin of the Brazilian Mathematical Society, New Series, 2018. https://doi.org/10.1007/s00574-018-0103-y.

[2] S. Ashraf, T. Mahmood and Q. Khan, *Picture Fuzzy Linguistic Sets and Their Applications for Multi-Attribute Group Decision Making Problems*, The Nucleus 55(2), 66-73, 2018.

[3] K. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 61(2), 137-142, 1994.

[4] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20, 87-96, 1986.

[5] K. Atanassov, G. Gargov, *Interval valued intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 31, 343-349, 1989.

[6] K. Atanassov, *Remark on intuitionistic fuzzy numbers*, Notes on intuitionistic fuzzy sets, 13, 29-32, 2007.

[7] B. C. Cuong, V. H. Phan, *some fuzzy logic operations for picture fuzzy sets*, In preceding of seventh international conference on knowledge and system engineering (IEEE), 2015.

[8] B. C. Cuong, T. N. Roan, *A classification of representable t-norm operators for picture fuzzy sets*, In preceding of eight international conference on knowledge and system engineering (IEEE), 2016.

[9] B. C. Cuong, *Picture Fuzzy Sets- a new concept for computational intelligence problems*, In Proceedings of the Third World Congress on Information and Communication Technologies, 1-6, 2013.

[10] B. C. Cuong, *Picture fuzzy sets*, journal of computer science and cybernetics, 30(4) 409-420, 2001.

[11] Y. B. Jun, C. S. Kim and K. O. Yang, *Cubic sets*, Ann. Fuzzy Math. Inform. 4(1), 83–98, 2012.

[12] Y. B. Jun, C. S. Kim and M. S. Kang, *Cubic subalgebras and ideals of BCK/BCI-algebras*, Far East. J. Math. Sci. 44, 239–250, 2010.

[13] Y. B. Jun, C. S. Kim and J. G. Kang, *Cubic q-ideals of BCI-algebras*, Ann. Fuzzy Math. Inf. 1, 25–34, 2011.

[14] T. Mahmood, F. Mehmood, Q. Khan, *Cubic Hesitant Fuzzy Sets and Their Applications to Multi Criteria Decision Making*, International Journal of Algebra and Statistics, 5(1), 19–51, 2016.

[15] T. Mahmood, S. Abdullah, S. Rashid, M. Bilal, *Multicriteria Decision Making Based On Cubic Sets*, Journal of New Theory, 16, 01-09, 2017.

[16] H. W. Liu, G. J. Wang, *Multi-criteria methods based on intuitionistic fuzzy sets*, European Journal Operational Research, 179(1), 220–233, 2007.

[17] R. R. Yager, *Pythagorean fuzzy subsets*, In Proc Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, June 24-28, 57-61, 2013.

[18] L. A. Zadeh *Fuzzy sets*, Information and Control, 8(3), 338- 356, 1965.

[19] H. Zhao, Z. S. Xu, M. F. Ni, S. S. Liu, *Generalized aggregation operator for intuitionistic fuzzy sets*, International Journal of Intelligent Systems, 25(1), 1-30, 2010.

[20] L. G. Zhou, Z. F. Tao, H. Y. Chen, J. Liu, *Continuous interval-valued intuitionistic fuzzy aggregation operators and their applications to group decision making*. Appl Math Model, 38, 2190-2205, 2014.