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THE MINKOWSKI'S INEQUALITIES UTILIZING NEWLY DEFINED GENERALIZED FRACTIONAL INTEGRAL OPERATORS

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Abstract. Motivated by the recent generalized fractional integral operators proposed by Tunc et. al. [22], we establish a generalization of the reverse Minkowski's inequalities. Within this context, we provide new upper bounds of inequalities utilizing generalized fractional integral operators and show and state other inequalities related to this fractional integral operator.

1. Introduction

Recently, a number of scientist in the field of mathematics have introduced different results about the fractional derivatives and integrals such as Riemann-Liouville fractional derivative, Riemann-Liouville fractional integral operator, Hadamard integral operator, Saigo fractional integral operator and some other, and applied them to some well-know inequalities with applications [1]-[22]. In this paper the authors will provide the some reverse Minkowski's inequalities by means of the generalized fractional integral operators.

The overall structure of the study takes the form of four sections including introduction. The remaining part of the paper proceeds as follows: In Section 2, we introduce generalized k-fractional integrals of a function with respect to the another function which generalizes different types of fractional integrals, including Riemann-Lioville fractional, Hadamard fractional integrals, Katugampola fractional integral, (k, s)-fractional integral operators and many others. In section 3, we provide the main results involving the reverse Minkowski's inequality with the help of fractional integral operators while in section 4 discussing other inequalities using this fractional integral operators. Finally concluding remarks summarize the article.

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2. New Generalized Fractional Integral Operators

In this section we will review the concept of the generalized k-fractional integrals of a function with respect to the another function introduced by Tunc et.al.[22].

Definition 1. *In* [8] *Diaz and Pariguan have defined k -gamma function Γ_k that is generalization of the classical gamma. Γ_k is given by formula*

$$\Gamma_k\left(x\right) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad k > 0.$$

It has shown that Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, clearly given by

$$\Gamma_k\left(\alpha\right) := \int_0^\infty e^{-\frac{t^k}{k}} t^{\alpha-1} dt.$$

Obviously, $\Gamma_k\left(x+k\right) = x\Gamma_k\left(x\right)$, $\Gamma(x) = \lim\limits_{k\to 1} \Gamma_k\left(x\right)$ and $\Gamma_k\left(x\right) = k^{\frac{x}{k}-1}\ \Gamma(\frac{x}{k})$.

Definition 2. *Let define the function*

$$\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left(x\right) := \sum_{m=0}^{\infty} \frac{\sigma\left(m\right)}{k\Gamma_k\left(\rho k m+\lambda\right)} x^m \qquad \left(\rho,\lambda>0;|x|<\mathcal{R}\right),$$

where the coefficients $\sigma\left(m\right)$ $\left(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\right)$ is a bounded sequence of positive real numbers and $\mathcal{R}$ is the set of real numbers.

Definition 3. *For $k>0$, let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). The left and right sided generalized k-fractional integrals of f with respect to the function g on $[a,b]$ are defined, respectively, as follows:*

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} f(x) = \int\limits_a^x \frac{g'(t)}{\left(g(x)-g(t)\right)^{1-\frac{\lambda}{k}}} \mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right] f(t)dt, \quad x>a, \tag{2.1}$$

and

$$\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma,k,g} f(x) = \int\limits_x^b \frac{g'(t)}{\left(g(t)-g(x)\right)^{1-\frac{\lambda}{k}}} \mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(g(t)-g(x)\right)^{\rho}\right] f(t)dt, \quad x<b, \tag{2.2}$$

where λ, $\rho>0$, $\omega\in\mathbb{R}$.

Remark 1. *The significant special cases of the integral operators (2.1) and (2.2) are mentioned below:*

1) For $k=1$, operator in (2.1) leads to generalized fractional integral of f with respect to the function g on $[a,b]$. This relation is given by

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g} f(x) = \int\limits_a^x \frac{g'(t)}{\left(g(x)-g(t)\right)^{1-\lambda}} \mathcal{F}_{\rho,\lambda}^{\sigma}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right] f(t)dt, \quad x>a.$$

2) For $g(t)=t$, operator in (2.1) leads to generalized k-fractional integral of f. This relation is given by

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}f(x)=\int_a^x (x-t)^{\frac{\lambda}{k}-1}\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega(x-t)^{\rho}\right]f(t)dt,\quad x>a.$$

3) For $g(t)=\ln t$, operator in (2.1) leads to generalized Hadamard k-fractional integral of f. This relation is given by

$$\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}f(x)=\int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\lambda}{k}-1}\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(\ln\frac{x}{t}\right)^{\rho}\right]f(t)\frac{dt}{t},\quad x>a.$$

4) For $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ operator in (2.1) leads to generalized (k,s)-fractional integral of f. This relation is given by

$${}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}f(x)=(s+1)^{1-\frac{\lambda}{k}}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\lambda}{k}-1}t^{r}\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(\frac{x^{s+1}-t^{s+1}}{s+1}\right)^{\rho}\right]f(t)dt,\ x>a.$$

Remark 2. *Similarly, all above special cases can also be seen for operator (2.2).*

Remark 3. *For $k=1$ and $g(t)=t$, operators in (2.1) and (2.2) reduce to the following generalized fractional integral operators defined by Raina* [21] *and Agarwal et. al* [1], *respectively:*

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}f(x)=\int_a^x (x-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\left[\omega(x-t)^{\rho}\right]f(t)dt,\quad x>a, \tag{2.3}$$

$$\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}f(x)=\int_x^b (t-x)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\left[\omega(t-x)^{\rho}\right]f(t)dt,\quad x<b, \tag{2.4}$$

Remark 4. *One can obtain other new generalized fractional integral operators with different choices of g.*

Remark 5. *For $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Definition 3, then we have the generalized fractional operators defined by Akkurt et al. in* [3].

Remark 6. *Let $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Definition 3.*

1) Choosing $k=1$, then we have fractional integrals of a function f with respect to function g. [12].

2) Choosing $g(t)=t$, then we have k-fractional integrals [15].

3) Choosing $k=1$ and $g(t)=\ln t$, then we have Hadamard fractional integrals [12].

4) Choosing $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$, then we have (k,s)-fractional integral operators [18].

5) Choosing $k=1$ and $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$, then we have Katugampola fractional integral operators [9].

6) Choosing $k = 1$ and $g(t) = t$, then we have Riemann-Lioville fractional integral operators [12].

3. Reverse Minkowski Fractional Integral Inequality new generalized fractional integral operators

Theorem 1. *Let $u, v \in X_c^p(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x > a, \mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} u^p(x) < \infty$ and $\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} v^p(x) < \infty$. Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). If $0 < m \leq \frac{u(t)}{v(t)} \leq M$ and $\forall t \in [0,x]$, then we have the following reverse Minkowski's inequality associated with the generalized k-fractional integrals with respect to the function g*

$$\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} u^p(x)\right]^{\frac{1}{p}} + \left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} v^p(x)\right]^{\frac{1}{p}} \quad \leq \quad C_1 \left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} (u+v)^p(x)\right]^{\frac{1}{p}}$$

where $C_1 = \frac{M(m+1)+M+1}{(M+1)(m+1)}$ and $p \geq 1$, λ, $\rho > 0$, $\omega \in \mathbb{R}$.

Proof. Since $\frac{u(t)}{v(t)} \leq M$, $t \in [a,x]$, we deduce that

$$u(t) \leq M\left[u(t)+v(t)\right] - Mu(t)$$

which yields

$$u^p(t) \leq \left(\frac{M}{M+1}\right)^p \left[u(t)+v(t)\right]^p. \tag{3.1}$$

Then multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}} \mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (3.1) and integrating on $[a,x]$, we get

$$\begin{aligned} &\int_a^x \frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}} \mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right] u^p(t)dt \\ \leq \quad &\left(\frac{M}{M+1}\right)^p \int_a^x \frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}} \mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right] \left[u(t)+v(t)\right]^p dt. \end{aligned}$$

As a result, we deduce that

$$\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} u^p(x)\right]^{\frac{1}{p}} \leq \frac{M}{M+1}\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} (u+v)^p(x)\right]^{\frac{1}{p}}. \tag{3.2}$$

On the other hand, as $m \leq \frac{u(t)}{v(t)}$, $t \in [a,x]$, we have

$$v^p(t) \leq \left(\frac{1}{m+1}\right)^p \left[u(t)+v(t)\right]^p. \tag{3.3}$$

Similarly, multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (3.3) and integrating on $[a,x]$, we get

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}v^{p}(x)\right]^{\frac{1}{p}} \leq \frac{1}{m+1}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}(u+v)^{p}(x)\right]^{\frac{1}{p}}. \tag{3.4}$$

Then adding the inequalities (3.2) and (3.4), the desired result has been obtained. □

Corollary 1. *We assume that the conditions of Theorem 1 hold.*

1) For $k=1$ in Theorem 1, we have the following reverse Minkowski's inequality associated with the generalized fractional integrals with respect to the function g

$$s\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}u^{p}(x)\right]^{\frac{1}{p}}+\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}v^{p}(x)\right]^{\frac{1}{p}} \leq C_1\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}(u+v)^{p}(x)\right]^{\frac{1}{p}}.$$

2) For $g(t)=t$ in Theorem 1, we have the following reverse Minkowski's inequality associated with the generalized k-fractional integrals

$$s\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}u^{p}(x)\right]^{\frac{1}{p}}+\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}v^{p}(x)\right]^{\frac{1}{p}} \leq C_1\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}(u+v)^{p}(x)\right]^{\frac{1}{p}}.$$

3) For $g(t)=\ln t$ in Theorem 1, we have the following reverse Minkowski's inequality associated with the generalized Hadamard k-fractional integrals

$$s\left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}u^{p}(x)\right]^{\frac{1}{p}}+\left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}v^{p}(x)\right]^{\frac{1}{p}} \leq C_1\left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}(u+v)^{p}(x)\right]^{\frac{1}{p}}.$$

4) For $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ in Theorem 1, we have the following reverse Minkowski's inequality associated with the generalized (k,s)-fractional integrals

$$s\left[{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}u^{p}(x)\right]^{\frac{1}{p}}+\left[{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}v^{p}(x)\right]^{\frac{1}{p}} \leq C_1\left[{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}(u+v)^{p}(x)\right]^{\frac{1}{p}}.$$

Theorem 2. *Let $u,v\in X_c^p(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x>a, \mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}u^{p}(x)<\infty$ and $\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}v^{p}(x)<\infty$. Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). If $0<m\leq\frac{u(t)}{v(t)}\leq M$ and $\forall t\in[0,x]$, then we have the following reverse Minkowski's inequality associated with the generalized k-fractional integrals with respect to the function g*

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}u^{p}(x)\right]^{\frac{2}{p}}+\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}v^{p}(x)\right]^{\frac{2}{p}} \quad \geq \quad C_2\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}u^{p}(x)\right]^{\frac{1}{p}}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}v^{p}(x)\right]^{\frac{1}{p}}$$

where $C_1=\frac{(M+1)(m+1)}{M}-2$ and $p\geq1$, λ, $\rho>0$, $\omega\in\mathbb{R}$.

Proof. From the inequalities (3.2) and (3.4), we have

$$\frac{(M+1)(m+1)}{M}\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}} \leq \left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^p(x)\right]^{\frac{2}{p}}. \tag{3.5}$$

Then, thanks to the Minkowski's inequality, we get

$$\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^p(x)\right]^{\frac{2}{p}} \leq \left(\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}+\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}\right)^2. \tag{3.6}$$

Consequently, by substituting (3.6) into (3.5), we obtain the desired result. □

Corollary 2. *We assume that the conditions of Theorem 2 hold.*
1) For $k=1$ in Theorem 2, we have the following reverse Minkowski's inequality associated with the generalized fractional integrals with respect to the function g

$$\left[\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{2}{p}}+\left[\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{2}{p}} \geq C_2\left[\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}\left[\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}.$$

2) For $g(t)=t$ in Theorem 2, we have the following reverse Minkowski's inequality associated with the generalized k-fractional integrals

$$\left[\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{2}{p}}+\left[\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{2}{p}} \geq C_2\left[\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}\left[\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}.$$

3) For $g(t)=\ln t$ in Theorem 2, we have the following reverse Minkowski's inequality associated with the generalized Hadamard k-fractional integrals

$$\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{2}{p}}+\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{2}{p}} \geq C_2\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}.$$

4) For $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ in Theorem 2, we have the following reverse Minkowski's inequality associated with the generalized (k,s)-fractional integrals

$$\left[{}^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{2}{p}}+\left[{}^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{2}{p}} \geq C_2\left[{}^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}\left[{}^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}.$$

4. Alternative Fractional Integral Inequalities with new generalized fractional integral operators

Theorem 3. *Let $u,v\in X^p_c(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x>a$, $\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x)<\infty$ and $\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^p(x)<\infty$. Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). If $0<m\leq\frac{u(t)}{v(t)}\leq M$ and $\forall t\in[0,x]$, then we have the following inequality associated with the generalized k-fractional integrals with respect to the function g*

$$\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}} \quad\leq\quad C_3\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^p(x)\right]^{\frac{1}{p}}$$

where $C_3=\left(\frac{M}{m}\right)^{\frac{1}{pq}}$, $\frac{1}{p}+\frac{1}{q}=1$, $p\geq 1$ and λ, $\rho>0$, $\omega\in\mathbb{R}$.

Proof. Since $\frac{u(t)}{v(t)} \leq M$, $t \in [a,x]$, we deduce that

$$v^{\frac{1}{q}}(t) \geq \left(\frac{1}{M}\right)^{\frac{1}{q}} u^{\frac{1}{q}}(t)$$

which yields

$$u^{\frac{1}{p}}(t)v^{\frac{1}{q}}(t) \geq \left(\frac{1}{M}\right)^{\frac{1}{q}} u(t). \tag{4.1}$$

Multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (4.1) and integrating on $[a,x]$, we get

$$\left(\frac{1}{M}\right)^{\frac{1}{q}} \mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u(x) \leq \mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u^{\frac{1}{p}}(x)v^{\frac{1}{q}}(x),$$

i.e.

$$\left(\frac{1}{M}\right)^{\frac{1}{pq}} \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u(x)\right]^{\frac{1}{p}} \leq \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u^{\frac{1}{p}}(x)v^{\frac{1}{q}}(x)\right]^{\frac{1}{p}}. \tag{4.2}$$

More over, as $m \leq \frac{u(t)}{v(t)}$, $t \in [a,x]$, we have

$$m^{\frac{1}{p}} v^{\frac{1}{p}}(t) \leq u^{\frac{1}{p}}(t)$$

which gives

$$m^{\frac{1}{p}} v(t) \leq u^{\frac{1}{p}}(t)v^{\frac{1}{q}}(t). \tag{4.3}$$

Then, multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (4.3) and integrating on $[a,x]$, we get

$$m^{\frac{1}{pq}} \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} v(x)\right]^{\frac{1}{p}} \leq \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u^{\frac{1}{p}}(x)v^{\frac{1}{q}}(x)\right]^{\frac{1}{p}}. \tag{4.4}$$

Considering the inequalities (4.2) and (4.4), we obtain the required result. □

Corollary 3. *We assume that the conditions of Theorem 3 hold.*
1) For $k=1$ in Theorem 3, we have the following inequality associated with the generalized fractional integrals with respect to the function g

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g} u^{p}(x)\right]^{\frac{1}{p}} \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g} v^{p}(x)\right]^{\frac{1}{p}} \leq C_3 \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g} (u+v)^{p}(x)\right]^{\frac{1}{p}}.$$

2) For $g(t)=t$ in Theorem 3, we have the following inequality associated with the generalized k-fractional integrals

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} u^{p}(x)\right]^{\frac{1}{p}} \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} v^{p}(x)\right]^{\frac{1}{p}} \leq C_3 \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} (u+v)^{p}(x)\right]^{\frac{1}{p}}.$$

3) For $g(t)=\ln t$ in Theorem 3, we have the following inequality associated with the generalized Hadamard k-fractional integrals

$$\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}\leq C_3\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}(u+v)^p(x)\right]^{\frac{1}{p}}.$$

4) For $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ in Theorem 3, we have the following inequality associated with the generalized (k,s)-fractional integrals

$$\left[{}^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}\left[{}^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}\leq C_3\left[{}^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}(u+v)^p(x)\right]^{\frac{1}{p}}.$$

Theorem 4. *Let $u,v\in X^p_c(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x>a, \mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x)<\infty$ and $\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^p(x)<\infty$. Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). If $0<m\leq\frac{u(t)}{v(t)}\leq M$ and $\forall t\in[0,x]$, then we have the following inequality associated with the generalized k-fractional integrals with respect to the function g*

$$\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u(x)v(x)\quad\leq\quad C_4\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u^p+v^p)(x)+C_5\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u^q+v^q)(x)$$

where $C_4=\frac{2^{p-1}}{p}\left(\frac{M}{M+1}\right)^p$, $C_5=\frac{2^{q-1}}{q}\left(\frac{1}{m+1}\right)^q$, $\frac{1}{p}+\frac{1}{q}=1$, $p\geq 1$ and λ, $\rho>0$, $\omega\in\mathbb{R}$.

Proof. Multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}}\mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (3.1) and integrating on $[a,x]$, we get

$$\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x)\leq\left(\frac{M}{M+1}\right)^p\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^p(x). \tag{4.5}$$

As $m\leq\frac{u(t)}{v(t)}$, $t\in[a,x]$, we have

$$v^q(t)\leq\left(\frac{1}{m+1}\right)^q\left[u(t)+v(t)\right]^q. \tag{4.6}$$

Similarly, multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}}\mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (4.6) and integrating on $[a,x]$, we get

$$\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^q(x)\leq\left(\frac{1}{m+1}\right)^q\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^q(x). \tag{4.7}$$

Applying the Young inequality, we have

$$u(t)v(t) \leq \frac{u^p(t)}{p} + \frac{v^q(t)}{q} \tag{4.8}$$

and multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}} \mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (4.8) and integrating on $[a,x]$, we get

$$\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}\left(uv\right)(x) \leq \frac{1}{p}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x) + \frac{1}{q}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^q(x). \tag{4.9}$$

Then, by substituting the inequalities (4.5) and (4.7) into (4.9), we obtain

$${}^{s}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}\left(uv\right)(x) \leq \frac{1}{p}\left(\frac{M}{M+1}\right)^{p}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^p(x) + \frac{1}{q}\left(\frac{1}{m+1}\right)^{q}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^q(x). \tag{4.10}$$

Using the fact that $(a,b)^r \leq 2^{r-1}(a^r+b^r)$, $r>1$, $a,b\geq 0$ in the right hand side of the inequality (4.10), we have

$$\begin{aligned}
{}^{s}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}\left(uv\right)(x) &\leq \frac{1}{p}\left(\frac{M}{M+1}\right)^{p}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^p(x) + \frac{1}{q}\left(\frac{1}{m+1}\right)^{q}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u+v)^q(x) \\
&\leq \frac{1}{p}\left(\frac{M}{M+1}\right)^{p}2^{p-1}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u^p+v^p)(x) + \frac{1}{q}\left(\frac{1}{m+1}\right)^{q}2^{q-1}\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}(u^q+v^q)(x).
\end{aligned}$$

Thus, the proof is completed. □

Corollary 4. *We assume that the conditions of Theorem 4 hold.*
1) For $k=1$ in Theorem 4, we have the following inequality associated with the generalized fractional integrals with respect to the function g

$$\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}u(x)v(x) \leq C_4\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}(u^p+v^p)(x) + C_5\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}(u^q+v^q)(x).$$

2) For $g(t)=t$ in Theorem 4, we have the following inequality associated with the generalized k-fractional integrals

$$\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u(x)v(x) \leq C_4\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}(u^p+v^p)(x) + C_5\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}(u^q+v^q)(x).$$

3) For $g(t)=\ln t$ in Theorem 4, we have the following inequality associated with the generalized Hadamard k-fractional integrals

$$\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}u(x)v(x) \leq C_4\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}(u^p+v^p)(x) + C_5\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}(u^q+v^q)(x).$$

4) For $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ in Theorem 4, we have the following inequality associated with the generalized (k,s)-fractional integrals

$${}^{s}\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u(x)v(x) \leq C_4^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}(u^p+v^p)(x) + C_5^s\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}(u^q+v^q)(x).$$

Theorem 5. *Let $u, v \in X_c^p(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x > a, \mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u^p(x) < \infty$ and $\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} v^p(x) < \infty$. Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). If $0 < n < m \leq \frac{u(t)}{v(t)} \leq M$ and $\forall t \in [0,x]$, then we have the following inequalities associated with the generalized k-fractional integrals with respect to the function g*

$$\begin{aligned}\frac{M+1}{M-n}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u(x)-nv(x)\right)^p\right]^{\frac{1}{p}} &\leq \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u^p(x)\right]^{\frac{1}{p}} + \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} v^q(x)\right]^{\frac{1}{p}} \\ &\leq \frac{m+1}{m-c}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u(x)-nv(x)\right)^p\right]^{\frac{1}{p}}\end{aligned}$$

where $p \geq 1$ and λ, $\rho > 0$, $\omega \in \mathbb{R}$.

Proof. From the assumption $0 < n < m \leq \frac{u(t)}{v(t)} \leq M$, we have

$$m - n \leq \frac{u(t) - nv(t)}{v(t)} \leq M - n$$

which yields

$$\frac{\left(u(t)-nv(t)\right)^p}{(M-n)^p} \leq v^p(t) \leq \frac{\left(u(t)-nv(t)\right)^p}{(m-n)^p}. \tag{4.11}$$

Similarly, we obtain

$$\frac{1}{M} \leq \frac{v(t)}{u(t)} \leq \frac{1}{m} \Rightarrow \frac{1}{M} - \frac{1}{n} \leq \frac{v(t)}{u(t)} - \frac{1}{n} \leq \frac{1}{m} - \frac{1}{n} \Rightarrow \frac{m-n}{mn} \leq \frac{u(t)-nv(t)}{nu(t)} \leq \frac{M-n}{Mn}$$

which yields

$$\frac{M^p}{(M-n)^p}\left(u(t)-nv(t)\right)^p \leq u^p(t) \leq \frac{m^p}{(m-n)^p}\left(u(t)-nv(t)\right)^p. \tag{4.12}$$

Multiplying by

$$\frac{g'(t)}{\left(g(x)-g(t)\right)^{1-\frac{\lambda}{k}}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (4.11) and integrating on $[a,x]$, we get

$$s\,\frac{1}{M-n}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u(x)-nv(x)\right)^p\right]^{\frac{1}{p}} \leq \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} v^q(x)\right]^{\frac{1}{p}} \leq \frac{1}{m-c}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u(x)-nv(x)\right)^p\right]^{\frac{1}{p}}. \tag{4.13}$$

Following the similar steps for (4.12), we obtain

$$s\,\frac{M}{M-n}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u(x)-nv(x)\right)^p\right]^{\frac{1}{p}} \leq \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u^q(x)\right]^{\frac{1}{p}} \leq \frac{m}{m-c}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u(x)-nv(x)\right)^p\right]^{\frac{1}{p}}. \tag{4.14}$$

Considering the inequalities (4.13) and (4.14), we obtain the required result.

In order to validate our result we can show that $\frac{M+1}{M-n} \leq \frac{m+1}{m-n}$. That is, from the assumption $0 < n < m \leq \frac{u(t)}{v(t)} \leq M$, we have

$$mn+m \leq mn+M \leq Mn+M \Rightarrow (M+1)(m-n) \leq (m+1)(M-n) \Rightarrow \frac{M+1}{M-n} \leq \frac{m+1}{m-n}.$$

$\square$

Corollary 5. *We assume that the conditions of Theorem 5 hold.*
1) For $k=1$ in Theorem 5, we have the following inequality associated with the generalized fractional integrals with respect to the function g

$$\begin{aligned}\frac{M+1}{M-n}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}\left(u(x)-nv(x)\right)^{p}\right]^{\frac{1}{p}} &\leq \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}u^{p}(x)\right]^{\frac{1}{p}}+\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}v^{q}(x)\right]^{\frac{1}{p}} \\ &\leq \frac{m+1}{m-c}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}\left(u(x)-nv(x)\right)^{p}\right]^{\frac{1}{p}}.\end{aligned}$$

2) For $g(t)=t$ in Theorem 5, we have the following inequality associated with the generalized k-fractional integrals

$$\begin{aligned}\frac{M+1}{M-n}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u(x)-nv(x)\right)^{p}\right]^{\frac{1}{p}} &\leq \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}u^{p}(x)\right]^{\frac{1}{p}}+\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}v^{q}(x)\right]^{\frac{1}{p}} \\ &\leq \frac{m+1}{m-c}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u(x)-nv(x)\right)^{p}\right]^{\frac{1}{p}}.\end{aligned}$$

3) For $g(t)=\ln t$ in Theorem 5, we have the following inequality associated with the generalized Hadamard k-fractional integrals

$$\begin{aligned}\frac{M+1}{M-n}\left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u(x)-nv(x)\right)^{p}\right]^{\frac{1}{p}} &\leq \left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}u^{p}(x)\right]^{\frac{1}{p}}+\left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}v^{q}(x)\right]^{\frac{1}{p}} \\ &\leq \frac{m+1}{m-c}\left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u(x)-nv(x)\right)^{p}\right]^{\frac{1}{p}}.\end{aligned}$$

4) For $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ in Theorem 5, we have the following inequality associated with the generalized (k,s)-fractional integrals

$$\begin{aligned}\frac{M+1}{M-n}\left[{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u(x)-nv(x)\right)^{p}\right]^{\frac{1}{p}} &\leq \left[{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}u^{p}(x)\right]^{\frac{1}{p}}+\left[{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}v^{q}(x)\right]^{\frac{1}{p}} \\ &\leq \frac{m+1}{m-c}\left[{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u(x)-nv(x)\right)^{p}\right]^{\frac{1}{p}}.\end{aligned}$$

Theorem 6. *Let $u,v\in X_c^p(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x>a$, $\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}u^{p}(x)<\infty$ and $\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}v^{p}(x)<\infty$. Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). If $0\leq\gamma\leq u(t)\leq\Gamma$, $0\leq\varphi\leq v(t)\leq\Phi$ and $\forall t\in[0,x]$, then we have the following inequalities associated with the generalized k-fractional integrals with respect to the function g*

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}u^{p}(x)\right]^{\frac{1}{p}}+\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}v^{q}(x)\right]^{\frac{1}{p}}\leq C_6\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u+v\right)^{p}(x)\right]^{\frac{1}{p}}$$

where $C_6=\frac{\Gamma(\gamma+\Phi)+\Phi(\varphi+\Gamma)}{(\gamma+\Phi)(\varphi+\Gamma)}$ $p\geq 1$ and $\lambda,\ \rho>0$, $\omega\in\mathbb{R}$.

Proof. From the assumptions of $0\leq\gamma\leq u(t)\leq\Gamma$ and $0\leq\varphi\leq v(t)\leq\Phi$, we deduce that

$$\frac{1}{\Phi}\leq\frac{1}{v(t)}\leq\frac{1}{\varphi}\Rightarrow\frac{\gamma}{\Phi}\leq\frac{u(t)}{v(t)}\leq\frac{\Gamma}{\varphi}$$

which yields

$$v^p(t) \leq \left(\frac{\Phi}{\gamma+\Phi}\right)^p (u(t)+v(t))^p \tag{4.15}$$

and

$$u^p(t) \leq \left(\frac{\Gamma}{\varphi+\Gamma}\right)^p (u(t)+v(t))^p . \tag{4.16}$$

Multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}} \mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (4.15) and (4.16), then integrating on $[a,x]$, we get

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} v^q(x)\right]^{\frac{1}{p}} \leq \frac{\Phi}{\gamma+\Phi}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u+v\right)^p(x)\right]^{\frac{1}{p}} \tag{4.17}$$

and

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u^p(x)\right]^{\frac{1}{p}} \leq \frac{\Gamma}{\varphi+\Gamma}\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u+v\right)^p(x)\right]^{\frac{1}{p}}, \tag{4.18}$$

respectively. Adding the inequalities (4.17) and (4.18), we obtain the desired result. $\square$

Corollary 6. *We assume that the conditions of Theorem 6 hold.*
1) For $k=1$ in Theorem 6, we have the following inequality associated with the generalized fractional integrals with respect to the function g

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g} u^p(x)\right]^{\frac{1}{p}} + \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g} v^q(x)\right]^{\frac{1}{p}} \leq C_6\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}\left(u+v\right)^p(x)\right]^{\frac{1}{p}} .$$

2) For $g(t)=t$ in Theorem 6, we have the following inequality associated with the generalized k-fractional integrals

$$\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} u^p(x)\right]^{\frac{1}{p}} + \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} v^q(x)\right]^{\frac{1}{p}} \leq C_6\left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u+v\right)^p(x)\right]^{\frac{1}{p}} .$$

3) For $g(t)=\ln t$ in Theorem 6, we have the following inequality associated with the generalized Hadamard k-fractional integrals

$$\left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k} u^p(x)\right]^{\frac{1}{p}} + \left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k} v^q(x)\right]^{\frac{1}{p}} \leq C_6\left[\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u+v\right)^p(x)\right]^{\frac{1}{p}} .$$

4) For $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ in Theorem 6, we have the following inequality associated with the generalized (k,s)-fractional integrals

$$\left[{}^s\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} u^p(x)\right]^{\frac{1}{p}} + \left[{}^s\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} v^q(x)\right]^{\frac{1}{p}} \leq C_6\left[{}^s\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u+v\right)^p(x)\right]^{\frac{1}{p}} .$$

Theorem 7. *Let $u,v\in X_c^p(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x > a, \mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} u^p(x)<\infty$ and $\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} v^p(x)<\infty$. Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). If*

$0<m\leq\frac{u(t)}{v(t)}\leq M$ *and* $\forall t\in[0,x]$, *then we have the following inequalities associated with the generalized k-fractional integrals with respect to the function g*

$$\frac{1}{M}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(uv\right)(x)\leq\frac{1}{(m+1)(M+1)}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(u+v\right)^{2}(x)\leq\frac{1}{m}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}\left(uv\right)(x)$$

where $\lambda,\ \rho>0,\ \omega\in\mathbb{R}$.

Proof. From the assumption $0<m\leq\frac{u(t)}{v(t)}\leq M$, we get

$$(m+1)v(t)\leq u(t)+v(t)\leq(M+1)v(t). \tag{4.19}$$

Also we have,

$$\frac{M+1}{M}u(t)\leq u(t)+v(t)\leq\frac{m+1}{m}u(t). \tag{4.20}$$

From the inequalities (4.19) and (4.20), we deduce that

$$\frac{1}{M}u(t)v(t)\leq\frac{\left(u(t)+v(t)\right)^{2}}{(m+1)(M+1)}\leq\frac{1}{m}u(t)v(t). \tag{4.21}$$

Multiplying by

$$\frac{g'(t)}{\left(g(x)-g(t)\right)^{1-\frac{\lambda}{k}}}\mathcal{F}_{\rho,\lambda}^{\sigma,k}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (4.21), then integrating on $[a,x]$, we obtain the required result. □

Corollary 7. *We assume that the conditions of Theorem 7 hold.*
1) For $k=1$ *in Theorem 7, we have the following inequality associated with the generalized fractional integrals with respect to the function g*

$$\frac{1}{M}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}\left(uv\right)(x)\leq\frac{1}{(m+1)(M+1)}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}\left(u+v\right)^{2}(x)\leq\frac{1}{m}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g}\left(uv\right)(x).$$

2) For $g(t)=t$ *in Theorem 7, we have the following inequality associated with the generalized k-fractional integrals*

$$\frac{1}{M}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(uv\right)(x)\leq\frac{1}{(m+1)(M+1)}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u+v\right)^{2}(x)\leq\frac{1}{m}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(uv\right)(x).$$

3) For $g(t)=\ln t$ *in Theorem 7, we have the following inequality associated with the generalized Hadamard k-fractional integrals*

$$\frac{1}{M}\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(uv\right)(x)\leq\frac{1}{(m+1)(M+1)}\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u+v\right)^{2}(x)\leq\frac{1}{m}\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(uv\right)(x).$$

4) For $g(t)=\frac{t^{s+1}}{s+1},\ s\in\mathbb{R}-\{-1\}$ *in Theorem 7, we have the following inequality associated with the generalized* (k,s)*-fractional integrals*

$$\frac{1}{M}\,{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(uv\right)(x)\leq\frac{1}{(m+1)(M+1)}\,{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(u+v\right)^{2}(x)\leq\frac{1}{m}\,{}^{s}\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k}\left(uv\right)(x).$$

Theorem 8. *Let $u,v\in X_c^p(a,x)$ two positive functions in $[0,\infty)$, such that $\forall x>a, \mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x)<\infty$ and $\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^p(x)<\infty$. Let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monotone function having a continuous derivative $g'(x)$ on (a,b). If $0<m\leq\frac{u(t)}{v(t)}\leq M$ and $\forall t\in[0,x]$, then we have the following inequality associated with the generalized k-fractional integrals with respect to the function g*

$$\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}+\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}\leq 2\left[\mathcal{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}h^p(u(x),v(x))\right]^{\frac{1}{p}}$$

where

$$h(u(x),v(x))=\max\left\{M\left[\left(\frac{M}{m}+1\right)u(x)-Mv(x)\right],\frac{(m+M)v(x)-u(x)}{m}\right\}$$

with $p\geq 1$ and $\lambda,\ \rho>0,\ \omega\in\mathbb{R}$.

Proof. From the assumption $0<m\leq\frac{u(t)}{v(t)}\leq M$, we have

$$0<m\leq M+m-\frac{u(t)}{v(t)} \tag{4.22}$$

By the inequality (4.22), we get

$$v(t)<\frac{(m+M)v(x)-u(x)}{m}\leq h(u(t),v(t)). \tag{4.23}$$

On the other hand, we have

$$\frac{1}{M}\leq\frac{1}{M}+\frac{1}{m}-\frac{v(t)}{u(t)}$$

which yields

$$u(t)\leq M\left[\left(\frac{M}{m}+1\right)u(x)-Mv(x)\right]\leq h(u(t),v(t)). \tag{4.24}$$

Then, using the inequalities (4.23) and (4.24), we obtain

$$u^p(t)+v^p(t)\leq 2h^p(u(t),v(t)). \tag{4.25}$$

Multiplying by

$$\frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}}\mathcal{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(x)-g(t)\right)^{\rho}\right]$$

both sides of (4.25), then integrating on $[a,x]$, we obtain the desired result. $\square$

Corollary 8. *We assume that the conditions of Theorem 8 hold.*
1) For $k=1$ in Theorem 8, we have the following inequality associated with the generalized fractional integrals with respect to the function g

$$\left[\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}}+\left[\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}}\leq 2\left[\mathcal{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}h^p(u(x),v(x))\right]^{\frac{1}{p}}.$$

2) For $g(t) = t$ in Theorem 8, we have the following inequality associated with the generalized k-fractional integrals

$$\left[\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}} + \left[\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}} \leq 2\left[\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}h^p(u(x),v(x))\right]^{\frac{1}{p}}.$$

3) For $g(t) = \ln t$ in Theorem 8, we have the following inequality associated with the generalized Hadamard k-fractional integrals

$$\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}} + \left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}} \leq 2\left[\mathcal{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}h^p(u(x),v(x))\right]^{\frac{1}{p}}.$$

4) For $g(t) = \frac{t^{s+1}}{s+1}$, $s \in \mathbb{R} - \{-1\}$ in Theorem 8, we have the following inequality associated with the generalized (k,s)-fractional integrals

$$\left[{}^{s}\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}u^p(x)\right]^{\frac{1}{p}} + \left[{}^{s}\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}v^p(x)\right]^{\frac{1}{p}} \leq 2\left[{}^{s}\mathcal{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}h^p(u(x),v(x))\right]^{\frac{1}{p}}.$$

5. Concluding Remarks

In this research we introduced the generalization of the reverse Minkowski's inequalities using generalized fractional integral operator. In order to validate that their generalized behavior, we show the relation of our results with previously published ones.

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