ON A NEW SUBCLASS OF BI-UNIVALENT FUNCTIONS
SATISFYING SUBORDINATE CONDITIONS

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Abstract. The purpose of our present paper is to introduce a new subclass of bi-univalent functions associated with pseudo-starlike function with Sakaguchi type functions and to determine the coefficient estimates $|a_2|$ and $|a_3|$ for functions in each of this newly-defined class. We also highlight some known consequences of our main results.

1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $S$ be the subclass of $A$ consisting of functions which are analytic and univalent in $\mathbb{U}$.

Here, we recall some definitions and concepts of classes of analytic functions. Let $f \in A$. Then $f$ is said to be in the class $S(\alpha, s, t)$ if it satisfies

$$\Re \left( \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right) > \alpha,$$

for some $0 \leq \alpha < 1$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$ and for all $z \in \mathbb{U}$. The class $S(\alpha, s, t)$ was introduced by Frasin [10]. The class $S(\alpha, 1, t)$ was introduced and studied by Owa et al. [18], and by taking $t = -1$, the class $S(\alpha, 1, -1) \equiv S_\alpha(\alpha)$ was introduced by Sakaguchi [19] and is called Sakaguchi function of order $\alpha$, where as $S_\alpha(0) = S_\alpha$ is the class of starlike functions with respect to symmetrical points in $\mathbb{U}$. Also, we note that $S(\alpha, 1, 0) \equiv S^*(\alpha)$ which is the familiar class of starlike functions of order $\alpha$ ($0 \leq \alpha < 1$).
With a view to recalling the principle of subordination between analytic functions, let the functions \( f \) and \( g \) be analytic in \( U \). Given functions \( f, g \in A \), \( f \) is subordinate to \( g \) if there exists a Schwarz function \( w \in \Lambda \), where

\[
\Lambda = \{ w : w(0) = 0, \ |w(z)| < 1, \ z \in U \},
\]

such that

\[
f(z) = g(w(z)) \quad (z \in U).
\]

We denote this subordination by

\[
f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in U).
\]

In particular, if the function \( g \) is univalent in \( U \), the above subordination is equivalent to

\[
f(0) = g(0), \quad f(U) \subset g(U).
\]

The Keobe One-Quarter Theorem [9] states that, the range of every function of the class \( S \) contains the disk \( \{ w : |w| < 1/4 \} \). Therefore, every \( f \in S \) has an inverse function \( f^{-1} \) satisfying

\[
f^{-1}(f(z)) = z, \quad (z \in U)
\]

and

\[
f(f^{-1}(w)) = w, \quad (|w| < r_0(f); r_0(f) \geq 1/4).
\]

The inverse of \( f(z) \) has a series expansion in some disk about the origin of the form

\[
f^{-1}(w) = w + A_2w^2 + A_3w^3 + \cdots.
\]

A function \( f(z) \) univalent in a neighbourhood of the origin and its inverse satisfy the condition \( f(f^{-1}(w)) = w \).

By using (2) yields

\[
w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \cdots
\]

and now by using (3) we get the following results

\[
g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (4)
\]

An analytic function \( f \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). The class of analytic and bi-univalent function in \( U \) is denoted by \( \Sigma \). For a brief history and interesting examples of functions in the class \( \Sigma \), see the pioneering work on this subject by Srivastava et al. [21], which has apparently revived the study of bi-univalent functions in recent years. From the work of Srivastava et al. [21], we choose to recall the following examples of functions in the class \( \Sigma \)

\[
\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2} \log \left( \frac{1+z}{1-z} \right).
\]
and so on. However, the familiar Koebe function is not a member of the bi-univalent function class \( \Sigma \). Such other common examples of functions in \( S \) as \( z \), \( z^2 \), and \( 1/z^2 \) are also not members of \( \Sigma \) (see \[21\]).

Historically, Lewin \[14\] studied the class of bi-univalent functions, obtaining the bound 1.51 for the modulus of the second coefficient \(|a_2|\). Subsequently, Brannan and Clunie \[7\] conjectured that \(|a_2| \leq \sqrt{2}\) for \( f \in \Sigma \). Later on, Netanyahu \[16\] showed that \( \max |a_2| = \frac{4}{3} \) if \( f(z) \in \Sigma \). Braman and Taha \[8\] introduced certain subclasses of the bi-univalent function class \( \Sigma \) similar to the familiar subclasses \( S^*(\beta) \) and \( K(\beta) \) of starlike and convex functions of order \( \beta \) (\( 0 \leq \beta < 1 \)) in \( U \), respectively (see \[16\]). The classes \( S^*_\Sigma(\beta) \) and \( K_\Sigma(\beta) \) of bi-starlike functions of order \( \beta \) in \( U \) and bi-convex functions of order \( \beta \) in \( U \), corresponding to the function classes \( S^*(\beta) \) and \( K(\beta) \), were also introduced analogously. For each of the function classes \( S^*_\Sigma(\beta) \) and \( K_\Sigma(\beta) \), they found non-sharp estimates for the initial coefficients. Recently, motivated substantially by the aforementioned pioneering work on this subject by Srivastava et al. \[21\], many authors investigated the coefficient bounds for various subclasses of bi-univalent functions (see, for example, \[4,12,15,17,22,23,24\]). Not much is known about the bounds on the general coefficient \(|a_n|\) for \( n \geq 4 \). In the literature, there are only a few works determining the general coefficient bounds for \(|a_n|\) for the analytic bi-univalent functions (see, for example, \[4,13,24\]). The coefficient estimate problem for each of the coefficients

\[ |a_n| \quad (n \in \mathbb{N} \setminus \{1,2\}, \mathbb{N} = \{1,2,3,\ldots\}) \]

is still an open problem.

Babalola \[6\] defined the class \( L_\lambda(\beta) \) of \( \lambda \)-pseudo-starlike functions of order \( \beta \) as below:

**Definition 1.** Let \( f \in A \), suppose \( 0 \leq \beta < 1 \) and \( \lambda \geq 1 \) is real. Then \( f(z) \in L_\lambda(\beta) \) of \( \lambda \)-pseudo-starlike functions of order \( \beta \) in the unit disk if and only if

\[ \Re \left( \frac{z[f'(z)]^\lambda}{f(z)} \right) > \beta. \]

Babalola \[6\] proved that, all pseudo-starlike functions are Bazilevic of type \( (1 - \frac{1}{\lambda}) \) order \( \beta \lambda \) and univalent in open unit disk \( U \).
Definition 2. A function $f \in \Sigma$ is said to be in the class $\mathcal{L}_s^\lambda(\phi, s, t)$, if the following subordinations hold
\[
\frac{(s-t)z[f'(z)]^\lambda}{f(sz) - f(tz)} < \phi(z)
\]
and
\[
\frac{(s-t)w[g'(w)]^\lambda}{g(sw) - f(tw)} < \phi(w)
\]
where $g(w) = f^{-1}(w)$, $s, t \in \mathbb{C}$ with $s \neq t$, $|\phi| > 0$, $|t| \leq 1$.

We note that, for suitable choices $\lambda$, $s$, and $t$, the class $\mathcal{L}_s^\lambda(\phi, s, t)$ reduces to the following known classes.

Definition 3. (see [1]) For $\lambda = 1$, a function $f \in \Sigma$ is said to be in the class $S_s^\lambda(\phi, s, t)$ $(s, t \in \mathbb{C}, s \neq t, |t| \leq 1)$

if it satisfies the following conditions respectively:
\[
\frac{(s-t)zf'(z)}{f(sz) - f(tz)} < \phi(z)
\]
and
\[
\frac{(s-t)wg'(w)}{g(sw) - f(tw)} < \phi(w)
\]
where $g = f^{-1}$.

Definition 4. For $\lambda = s = 1$ and $t = -1$, a function $f \in \Sigma$ is said to be in the class $S_s^\lambda(\phi, t)$ $(t \in \mathbb{C}, |t| \leq 1)$

if it satisfies the following conditions respectively:
\[
\frac{(1-t)zf'(z)}{f(z) - f(tz)} < \phi(z)
\]
and
\[
\frac{(1-t)wg'(w)}{g(w) - g(tw)} < \phi(w)
\]
where $g = f^{-1}$. This class studied by Goyal and Goswami [11].

Definition 5. For $\lambda = s = 1$ and $t = -1$, a function $f \in \Sigma$ is said to be in the class $S_s^\lambda(\phi)$ if it satisfies the following conditions respectively:
\[
\frac{2zf'(z)}{f(z) - f(-z)} < \phi(z)
\]
and
\[
\frac{2wg'(w)}{g(w) - g(-w)} < \phi(w)
\]
where $g = f^{-1}$. This class studied by Shamugham et al. [20].
Definition 6. For \( s = 1 \) and \( t = 0 \), a function \( f \in \Sigma \) is said to be in the class \( \Delta_\Sigma^\lambda(\phi) \) if it satisfies the following conditions respectively:

\[
\frac{zf'(z)}{f(z)} < \phi(z)
\]

and

\[
\frac{wg'(w)}{g(w)} < \phi(w)
\]

where \( g = f^{-1} \).

Definition 7. (see [5]) For \( \lambda = s = 1 \) and \( t = 0 \), a function \( f \in \Sigma \) is said to be in the class \( S_\Sigma^\lambda(\phi) \) if it satisfies the following conditions respectively:

\[
\frac{zf'(z)}{f(z)} < \phi(z)
\]

and

\[
\frac{wg'(w)}{g(w)} < \phi(w)
\]

where \( g = f^{-1} \).

The purpose of our present paper is to introduce a new subclass of bi-univalent functions associated with pseudo-starlike function with Sakaguchi type functions and to determine the coefficient estimates \( |a_2| \) and \( |a_3| \) for functions in each of this newly-defined class. We also highlight some known consequences of our main results.

2. Coefficient Estimates

Let \( \phi \) be an analytic function with positive real part in \( U \), with \( \phi(0) = 1 \) and \( \phi'(0) > 0 \). Also, let \( \phi(U) \) be starlike with respect to \( \phi(0) = 1 \) and symmetric with respect to the axis. Thus, \( \phi \) has the Taylor series expansion

\[
\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_1 > 0).
\]

Suppose that \( u(z) \) and \( v(w) \) are analytic in the unit disk \( U \) with \( u(0) = v(0) = 0 \), \( |u(z)| < 1 \), \( |v(w)| < 1 \), and suppose that

\[
u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad v(w) = c_1 w + \sum_{n=2}^{\infty} c_n w^n (|z| < 1, |w| < 1).\]

It is well known that

\[
|b_1| \leq 1, |b_2| \leq 1 - |b_1|^2, |c_1| \leq 1, |c_2| \leq 1 - |c_1|^2.
\]

Next, the equation [6] and [7] lead to

\[
\phi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \cdots, \quad |z| < 1
\]

and

\[
\phi(u(z)) = 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2) z^2 + \cdots, \quad |w| < 1.
\]
For functions in the class $\mathcal{L}_2^\lambda(\phi,s,t)$ the following estimates are obtained.

**Theorem 1.** Let the function $f$ given by (1) be in the class $\mathcal{L}_2^\lambda(\phi,s,t)$. Then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{[3\lambda - 2\lambda(s + t - \lambda + 1) + st]B_1^2 - (2\lambda - s - t)^2B_2}} + |2\lambda - s - t|^2B_1
\]

and

\[
|a_3| \leq \left\{ \begin{array}{ll}
\frac{B_1}{|[3\lambda - s^2 - t^2 - st]|} & b_1 \leq \frac{|2\lambda - s - t|^2}{|[3\lambda - s^2 - t^2 - st]|} \\
\frac{[(3\lambda - 2\lambda(s + t - \lambda + 1) + st)B_1^2 - (2\lambda - s - t)^2B_2][1 - (3\lambda - s^2 - t^2 - st)|B_2]}{|[3\lambda - s^2 - t^2 - st]([3\lambda - 2\lambda(s + t - \lambda + 1) + st]B_1^2 - (2\lambda - s - t)^2B_2)} & b_1 > \frac{|2\lambda - s - t|^2}{|[3\lambda - s^2 - t^2 - st]|}
\end{array} \right.
\]

**Proof.** Let $f \in \mathcal{L}_2^\lambda(\phi,s,t)$. Then, there are analytic functions $u,v : \mathbb{U} \to \mathbb{U}$ given by (7) such that

\[
(s - t)z \frac{[f'(z)]^\lambda}{f(sz) - f(tz)} = \phi(u(z))
\]

and

\[
(s - t)w \frac{[g'(w)]^\lambda}{g(sw) - g(tw)} = \phi(v(w)).
\]

Since

\[
(s - t)z \frac{[f'(z)]^\lambda}{f(sz) - f(tz)} = 1 + (2\lambda - s - t)a_2z + [(3\lambda - s^2 - t^2 - st)a_3 - (2\lambda(s + t - \lambda - 1) - s^2 - t^2 - 2st)a_2^2] z^2 + \cdots
\]

and

\[
(s - t)w \frac{[g'(w)]^\lambda}{g(sw) - g(tw)} = 1 - (2\lambda - s - t)a_2w + [(6\lambda - s^2 - t^2 - 2\lambda(s + t - \lambda + 1))a_2^2 - (3\lambda - s^2 - t^2 - st)a_3] w^2 + \cdots,
\]

it follows from (6), (10), (13) and (14) that

\[
(2\lambda - s - t)a_2 = B_1 b_1
\]

\[
(3\lambda - s^2 - t^2 - st)a_3 - (2\lambda(s + t - \lambda + 1) - s^2 - t^2 - 2st)a_2^2 = B_1 b_2 + B_2 b_1^2
\]

\[
(6\lambda - s^2 - t^2 - 2\lambda(s + t - \lambda + 1))a_2^2 - (3\lambda - s^2 - t^2 - st)a_3 = B_1 c_2 + B_2 c_1^2.
\]

See that, (15) and (17) together yields:

\[
c_1 = -b_1.
\]

By adding (18) to (16), further computation using (15) and (19) leads to

\[
[(6\lambda - 4\lambda(s + t - \lambda + 1) + 2st)B_1^2 - 2(2\lambda - s - t)^2B_2]a_2^2 = B_1^3(b_2 + c_2).
\]
By using (19) and (20), together with (8), we find that
\[ |(3\lambda - 2\lambda(s + t - \lambda + 1) + st)B_1^2 - (2\lambda - s - t)^2 B_2|a_2|^2 \leq B_3^3(1 - |b_1|^2) \] (21)
which gives the desired estimate on $|a_2|$ as asserted in (11).
In order to find the bound on $|a_3|$, by subtracting (18) from (16), we readily obtain
\[ 2(3\lambda - s^2 - t^2 - st)a_3 - 2(3\lambda - s^2 - t^2 - st)a_2^2 = B_1(b_2 - c_2) + B_2(b_1^2 - c_1^2). \] (22)
Then, from (8) and (19), we have
\[ |3\lambda - s^2 - t^2 - st|B_1|a_3| \leq |3\lambda - s^2 - t^2 - st|B_1 - |2\lambda - s - t|^2|a_2|^2 + B_3^2. \]
Using (11), we get the estimate on $|a_3|$ as asserted in (12). \qed

3. Corollaries and consequences

This section is devoted to the presentation of some special cases of the main results. These results are given in the form of corollaries.

**Corollary 1.** If we let
\[ \phi(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1), \]
then the inequalities (11) and (12) becomes
\[ |a_2| \leq \frac{2\alpha}{\sqrt{2(3\lambda - 2\lambda(s + t - \lambda + 1) + st) - (2\lambda - s - t)^2|\alpha + |2\lambda - s - t|^2}} \]
and
\[ |a_3| \leq \begin{cases} 
\frac{2\alpha}{|3\lambda - s^2 - t^2 - st|} & \text{if } 0 < \alpha \leq \frac{|2\lambda - s - t|^2}{2|3\lambda - 2\lambda(s + t - \lambda + 1) + st) - (2\lambda - s - t)^2| + 2|3\lambda - s^2 - t^2 - st||a_2|^2} \\
\frac{|3\lambda - s^2 - t^2 - st|}{2|3\lambda - s^2 - t^2 - st|} & \text{if } \alpha \leq 1.
\end{cases} \]

**Corollary 2.** If we let
\[ \phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)^2 z^2 + \cdots \quad (0 \leq \beta < 1), \]
then the inequalities (11) and (12) becomes
\[ |a_2| \leq \frac{2(1 - \beta)}{\sqrt{2(3\lambda - 2\lambda(s + t - \lambda + 1) + st)(1 - \beta) - (2\lambda - s - t)^2| + |2\lambda - s - t|^2}} \]
and
\[ |a_3| \leq \begin{cases} 
\frac{2(1 - \beta)^2}{|3\lambda - s^2 - t^2 - st|} & \text{if } 0 < \beta < \frac{|2\lambda - s - t|^2}{2|3\lambda - 2\lambda(s + t - \lambda + 1) + st) - (2\lambda - s - t)^2| + 2|3\lambda - s^2 - t^2 - st|(1 - \beta)(1 - \beta)} \\
\frac{|3\lambda - s^2 - t^2 - st|}{2|3\lambda - s^2 - t^2 - st|} & \text{if } 0 \leq \beta < \frac{2|3\lambda - s^2 - t^2 - st|}{|2\lambda - s - t|^2}.
\end{cases} \]
Remark 1. Taking $\lambda = 1$ in the above Theorem 1 and Corollaries 1, 2, we obtain the results of Altinkaya and Yalcin [1].

Corollary 3. Let $f \in S_{\phi}^*(\phi)$, then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{4B_1^2 + 2|B_1^2 - 2B_2|}}$$

and

$$|a_3| \leq \begin{cases} 
\frac{B_1}{2}, & \text{if } B_1 \leq 2 \\
\frac{|B_1^2 - 2B_2|B_1 + B_1^3}{|2B_1^2 - 4B_2| + 4B_1}, & \text{if } B_1 > 2.
\end{cases}$$

Corollary 4. Let $f \in S_{\phi}^*(\phi)$, then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{|B_1^2 - B_2| + B_1}$$

and

$$|a_3| \leq \begin{cases} 
\frac{B_1}{2}, & \text{if } B_1 \leq \frac{1}{2} \\
\frac{|B_1^2 - B_2|B_1 + 2B_1^3}{2||B_1^2 - B_2| + B_1|}, & \text{if } B_1 > \frac{1}{2}.
\end{cases}$$

Remark 2. For $f \in S_{\phi}^*(\phi)$, the function $\phi$ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^a = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1)$$

and so $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. Hence the Corollary 4 reduces to an improved results of Brannan and Taha [8]. On the other hand when

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1),$$

$B_1 = B_2 = 2(1-\beta)$ and thus the Corollary 4 reduces to the estimate in Brannan and Taha [8].

Corollary 5. Let $f \in S_{\lambda}^*(\phi)$, then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{(2\lambda - 1)|\lambda B_1^2 - (2\lambda - 1)B_2| + 2\lambda - 1|B_1|}}$$
and
\[
|a_3| \leq \begin{cases} 
\frac{B_1}{(3\lambda - 1)}, & \text{if } B_1 \leq \frac{[2\lambda - 1]^2}{|3\lambda - 1|} \\
\frac{(2\lambda - 1)|\lambda B_1^2 - (2\lambda - 1)B_2|B_1 + |3\lambda - 1|B_1^3}{(3\lambda - 1)(|2\lambda - 1|)|\lambda B_1^2 - (2\lambda - 1)B_2| + |2\lambda - 1|^2B_1|}, & \text{if } B_1 > \frac{[2\lambda - 1]^2}{|3\lambda - 1|}. 
\end{cases}
\]

**Remark 3.** For \( f \in \mathcal{S}_\phi(\alpha) \), the function \( \phi \) is given by
\[
\phi(z) = \left( \frac{1+z}{1-z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1)
\]
and so \( B_1 = 2\alpha \) and \( B_2 = 2\alpha^2 \). Hence the Corollary 5 reduces to an improved results of Joshi et al. \([12]\) On the other hand when
\[
\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + \cdots \quad (0 \leq \beta < 1),
\]
\( B_1 = B_2 = 2(1 - \beta) \) and thus the Corollary 5 reduces to the estimate in Joshi et al. \([12]\).

**Corollary 6.** Let \( f \in \mathcal{S}_\phi(\alpha, t) \), then
\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{(1-t)[|B_1^2 - (1-t)B_2| + |1-t||B_1|]}}
\]
and
\[
|a_3| \leq \begin{cases} 
\frac{B_1}{(|2-t-t^2|)}, & \text{if } B_1 \leq \frac{|1-t|^2}{|2-t-t^2|} \\
\frac{|(1-t)B_1^2 - (1-t)^2B_2|B_1 + |2-t-t^2|B_1^3}{(|2-t-t^2)||(1-t)B_1^2 - (1-t)^2B_2| + |1-t|^2|B_1|}}, & \text{if } B_1 \leq \frac{|1-t|^2}{|2-t-t^2|}.
\end{cases}
\]

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