Tychonoff Objects in the Topological Category of Cauchy Spaces

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Abstract
There are various forms of Tychonoff objects for an arbitrary set-based topological category. In this paper, any explicit characterization of each of the Tychonoff Objects is given in the topological category of Cauchy spaces. Moreover, we characterize each of them for the category of Cauchy spaces and investigate the relationships among the various $T_i$, $i = 0, 1, 2, 3, 4$, $PreT_2$, and $T_2$ (we will refer to it as the usual one) structures are examined in this category.

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1. Introduction

In general topology and analysis, a Cauchy space is a generalization of metric spaces and uniform spaces. The theory of Cauchy spaces was initiated by H. J. Kowalsky [20]. Cauchy spaces were introduced by H. Keller [17] in 1968.

In 1970, the study of regular Cauchy completions was initiated by J. Ramaley and O. Wyler [31]. Later, D. C. Kent and G. D. Richardson ([18, 19]) characterized the $T_3$ Cauchy spaces which have $T_3$ completions and constructed a regular completion functor.

In 1968, Keller [17] introduced the axiomatic definition of Cauchy spaces, which is given briefly in the preliminaries section.

Filter spaces are generalizations of Cauchy spaces. If we exclude the last of three Keller’s [17] axioms for a Cauchy space, then the resulting space is what we call a filter space. In [13], it is shown that the category FIL of filter spaces is isomorphic to the category of filter meretopic spaces which were introduced by Katětov [16]. The category of Cauchy spaces is also known to be a bireflective, finally dense subcategory of FIL [30].

All our preliminary information on Cauchy spaces and more information can be found in [24].

The notions of "closedness" and "strong closedness" in set based topological categories are introduced by Baran [2, 4] and it is shown in [9] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [14] in some well-known topological categories. Moreover, various generalizations of each of $T_i$, $i = 0, 1, 2$ separation properties for an arbitrary topological category over $Set$, the category of sets are given and the relationship among various forms of each of these notions are investigated by Baran in [2, 7, 8, 10, 11].

The main goal of this paper is

1. to give the characterization of each of the Tychonoff objects in the topological category of Cauchy spaces,

2. to examine how these generalizations are related, and
3. to show that specific relationships that arise among the various $T_i$, $i = 0, 1, 2, 3, 4$, $PreT_2$, and $T_2$ (we will refer to it as the usual one) structures are examined in the topological category of Cauchy spaces.

2. Preliminaries

The followings are some basic definitions and notations which we will use throughout the paper.

Let $E$ and $B$ be any categories. The functor $U : E \rightarrow B$ is said to be topological or that $E$ is a topological category over $B$ if $U$ is concrete (i.e., faithful, amnestic and transportable), has small (i.e., sets) fibers, and for which every $U$-source has an initial lift or, equivalently, for which each $U$-sink has a final lift [1].

Note that a topological functor $U : E \rightarrow B$ is said to be normalized if constant objects, i.e., subterminals, have a unique structure [1, 5, 10, 26, 29].

Recall in [1, 29], that an object $X \in E$ (where $X \in E$ stands for $X \in Ob E$), a topological category, is discrete iff every map $U(X) \rightarrow U(Y)$ lifts to a map $X \rightarrow Y$ for each object $Y \in E$ and an object $X \in E$ is indiscrete iff every map $U(Y) \rightarrow U(X)$ lifts to a map $Y \rightarrow X$ for each object $Y \in E$.

Let $E$ be a topological category and $X \in E$. A is called a subspace of $X$ if the inclusion map $i : A \rightarrow X$ is an initial lift (i.e., an embedding) and we denote it by $A \subset X$.

A filter on a set $X$ is a collection of subsets of $X$, containing $X$, which is closed under finite intersection and formation of supersets (it may contain $\emptyset$). Let $F(X)$ denote the set of filters on $X$. If $\alpha, \beta \in F(X)$, then $\beta \geq \alpha$ if and only if for each $U \in \alpha$, $\exists V \in \beta$ such that $V \subseteq U$, that is equivalent to $\beta \supseteq \alpha$. This defines a partial order relation on $F(X). \ x = \{\{x\}\}$ is the filter generated by the singleton set $\{x\}$ where $\{\cdot\}$ means generated filter and $\alpha \cap \beta = \{\{U \cup V | U \in \alpha, V \in \beta\}\}$. If $U \cup V \neq \emptyset$, for all $U \in \alpha$ and $V \in \beta$, then $\alpha \cup \beta$ is the filter $\{\{U \cap V | U \in \alpha, V \in \beta\}\}$. If $\exists U \in \alpha$ and $V \in \beta$ such that $U \cap V = \emptyset$, then we say that $\alpha \cup \beta$ fails to exist.

Let $A$ be a set and $q$ be a function on $A$ that assigns to each point $x$ of $A$ a set of filters (proper or not, where a filter $\delta$ is proper iff $\delta$ does not contain the empty set, i.e., $\delta \neq \emptyset$) (the filters converging to $x$) is called a convergence structure on $A$. ($A, q$) a convergence space (in [29], it is called a convergence space) iff it satisfies the following three conditions ([28] p. 1374 or [29] p. 142):

1. $[x] = [\{x\}] \in q(x)$ for each $x \in A$ (where $|F| = \{B \subset A : F \subset B\}$).
2. $\beta \geq \alpha \in q(x)$ implies $\beta \in q(x)$ for any filter $\beta$ on $A$.
3. $\alpha \in q(x) \Rightarrow \alpha \cap q(x) \in q(x)$.

A map $f : (A, q) \rightarrow (B, s)$ between two convergence spaces is called continuous iff $\alpha \in q(x)$ implies $f(\alpha) \in s(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$). The category of convergence spaces and continuous maps is denoted by Con (in [29] Conv).

For filters $\alpha$ and $\beta$ we denote by $\alpha \cup \beta$ the smallest filter containing both $\alpha$ and $\beta$.

**Definition 2.1.** (cf. [17]) Let $A$ be a set and $K \subset F(A)$ be subject to the following axioms:

1. $[x] = [\{x\}] \in K$ for each $x \in A$ (where $[x] = \{B \subset A : x \in B\}$);
2. $\alpha \in K$ and $\beta \geq \alpha$ implies $\beta \in K$ (i.e., $\beta \supseteq \alpha \in K$ implies $\beta \in K$ for any filter $\beta$ on $A$);
3. if $\alpha, \beta \in K$ and $\alpha \cup \beta$ exists (i.e., $\alpha \cup \beta$ is proper), then $\alpha \cap \beta \in K$.

Then $K$ is a precauchy (Cauchy) structure if it obeys 1-2 (resp. 1-3) and the pair $(A, K)$ is called a precauchy space (Cauchy space), resp. Members of $K$ are called Cauchy filters. A map $f : (A, K) \rightarrow (B, L)$ between Cauchy spaces is said to be Cauchy continuous (Cauchy map) iff $\alpha \in K$ implies $f(\alpha) \in L$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$). The concrete category whose objects are the precauchy (Cauchy) spaces and whose morphisms are the Cauchy continuous maps is denoted by $\text{PCHY}$ (CHY), respectively.

**Definition 2.2.** A source $\{f_i : (A, K) \rightarrow (A_i, K_i), i \in I\}$ in $\text{CHY}$ is an initial lift iff $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$ [24, 30, 32].

**Definition 2.3.** An epimorphism $f : (A, K) \rightarrow (B, L)$ in $\text{CHY}$ (equivalently, $f$ is surjective) is a final lift iff $\alpha \in L$ implies that there exists a finite sequence $\alpha_1, ..., \alpha_n$ of Cauchy filters in $K$ such that every member of $\alpha_i$ intersects every member of $\alpha_{i+1}$ for all $i < n$ and such that $\bigcap_{i=1}^n f(\alpha_i) \subset \alpha$ [24, 30, 32].

**Definition 2.4.** Let $B$ be set and $p \in B$. Let $B \vee p, B$ be the wedge at $p$ ([2] p. 334), i.e., two disjoint copies of $B$ identified at $p$, i.e., the pushout of $p : 1 \rightarrow B$ along itself (where $1$ is the terminal object in Set). An epi sink $\{i_1, i_2 : (B, K) \rightarrow (B \vee_p B, L)\}$, where $i_1, i_2$ are the canonical injections, in $\text{CHY}$ is a final lift if and only if the following statement holds. For any filter $\alpha$ on the wedge $B \vee_p B$, where either $\alpha \supseteq \iota_k(\alpha_1)$ for some $k = 1, 2$ and some $\alpha_1 \in K$, or $\alpha \in L$, we have that there exist Cauchy filters $\alpha_1, \alpha_2 \in K$ such that every member of $\alpha_1$ intersects every member of $\alpha_2$ (i.e., $\alpha_1 \cup \alpha_2$ is proper) and $\alpha \supseteq i_1 \alpha_1 \cap i_2 \alpha_2$. This is a special case of Definition 2.3.
Definition 2.5. The discrete structure \((A, K)\) on \(A\) in \(\text{CHY}\) is given by \(K = \{[a] \mid a \in A\} \cup \{[\emptyset]\}\) [24, 30].

Definition 2.6. The indiscrete structure \((A, K)\) on \(A\) in \(\text{CHY}\) is given by \(K = F(A)\) [24, 30].

\(\text{CHY}\) is a normalized topological category. The category of Cauchy spaces is cartesian closed, and contains the category of uniform spaces as a full subcategory [30].

3. \(T_2\)-Objects

Recall, in [2, 11], that there are various ways of generalizing the usual \(T_2\) separation axiom to topological categories. Moreover, the relationships among various forms of \(T_2\)-objects are established in [11].

Let \(B\) be a nonempty set, \(B^2 = B \times B\) be the cartesian product of \(B\) with itself and \(B^2 \vee B^2\) be two distinct copies of \(B^2\) identified along the diagonal. A point \((x, y)\) in \(B^2 \vee B^2\) will be denoted by \((x, y)_1\) (or \((x, y)_2\)) if \((x, y)\) is in the first (or second) component of \(B^2 \vee B^2\), respectively. Clearly, \((x, y)_1 = (x, y)_2\) iff \(x = y\) [2].

The principal axis map \(A : B^2 \vee B^2 \to B^3\) is given by \(A(x, y)_1 = (x, y, x)\) and \(A(x, y)_2 = (x, x, y)\). The skewed axis map \(S : B^2 \vee B^2 \to B^3\) is given by \(S(x, y)_1 = (x, x, y)\) and \(S(x, y)_2 = (x, y, y)\) and the fold map, \(\nabla : B^2 \vee B^2 \to B^2\) is given by \(\nabla(x, y)_i = (x, y)\) for \(i = 1, 2\). Note that \(\pi_1S = \pi_1\pi_1A, \pi_2S = \pi_2\pi_2A, \pi_1A = \pi_1\pi_1A,\) and \(\pi_3S = \pi_3\pi_2A\), where \(\pi_k : B^3 \to B\) the \(k\)-th projection \(k = 1, 2, 3\) and \(\pi_{ij} = \pi_i + \pi_j : B^2 \vee B^2 \to B\), for \(i, j \in \{1, 2\}\) [2].

Definition 3.1. (cf. [2, 4, 10, 11]) Let \(\mathcal{U} : \mathcal{E} \to \text{Set}\) be a topological functor, \(X\) an object in \(\mathcal{E}\) with \(\mathcal{U}(X) = B\).

1. \(X\) is \(T_0\) iff the initial lift of the \(\mathcal{U}\)-source \(\{A : B^2 \vee B^2 \to \mathcal{U}(X^3) = B^3\) and \(\nabla : B^2 \vee B^2 \to \mathcal{U}D(B^2) = B^2\}\) is discrete, where \(D\) is the discrete functor which is a left adjoint to \(\mathcal{U}\).

2. \(X\) is \(T_0'\) iff the initial lift of the \(\mathcal{U}\)-source \(\{id : B^2 \vee B^2 \to \mathcal{U}(B^2 \vee B^2) = B^2 \vee B^2\) and \(\nabla : B^2 \vee B^2 \to \mathcal{U}D(B^2) = B^2\}\) is discrete, where \(B^2 \vee B^2\) is the final lift of the \(\mathcal{U}\)-sink \(\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee B^2\}\) and \(\mathcal{U}D(B^2)\) is the discrete subobject on \(B^2\). Here, \(i_1\) and \(i_2\) are the canonical injections.

3. \(X\) is \(T_0''\) iff \(X\) does not contain an indiscrete subspace with (at least) two points [25, 34].

4. \(X\) is \(T_1\) iff the initial lift of the \(\mathcal{U}\)-source \(\{S : B^2 \vee B^2 \to \mathcal{U}(X^3) = B^3\) and \(\nabla : B^2 \vee B^2 \to \mathcal{U}D(B^2) = B^2\}\) is discrete.

5. \(X\) is \(\text{Pre}T_2\) iff the initial lifts of the \(\mathcal{U}\)-source \(\{A : B^2 \vee B^2 \to \mathcal{U}(X^3) = B^3\}\) and \(\{S : B^2 \vee B^2 \to \mathcal{U}(X^3) = B^3\}\) coincide.

6. \(X\) is \(\text{Pre}T_2'\) iff the initial lift of the \(\mathcal{U}\)-source \(\{S : B^2 \vee B^2 \to \mathcal{U}(X^3) = B^3\}\) and the final lift of the \(\mathcal{U}\)-sink \(\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee B^2\}\) coincide, where \(i_1\) and \(i_2\) are the canonical injections.

7. \(X\) is \(T_2\) iff \(X\) is \(T_0\) and \(\text{Pre}T_2\).

8. \(X\) is \(T_2'\) iff \(X\) is \(T_0'\) and \(\text{Pre}T_2'\).

9. \(X\) is \(ST_2\) iff \(\delta\), the diagonal, is strongly closed in \(X^2\).

10. \(X\) is \(\Delta T_2\) iff \(\delta\), the diagonal, is closed in \(X^2\).

11. \(X\) is \(KT_2\) iff \(X\) is \(T_0''\) and \(\text{Pre}T_2\).

12. \(X\) is \(LT_2\) iff \(X\) is \(T_0''\) and \(\text{Pre}T_2'\).

13. \(X\) is \(MT_2\) iff \(X\) is \(T_0\) and \(\text{Pre}T_2'\).

14. \(X\) is \(NT_2\) iff \(X\) is \(T_0\) and \(\text{Pre}T_2\).

Remark 3.1. Note that for the category \(\text{Top}\) of topological spaces, \(T_0, T_0', T_0'', T_1, T_1', T_2, T_2'\), or all of the \(T_i\)'s in Definition 3.1 reduce to the usual \(T_0\), or \(T_1\), or \(T_2\) (where a topological space is called \(T_2\) if for any two distinct points, if there is a neighbourhood of one missing the other, then the two points have disjoint neighbourhoods), or \(T_2\) separation axioms, respectively [2].

Definition 3.2. A Cauchy space \((A, K)\) is said to be \(T_2\) (we will refer to it as the usual one) if and only if \(x = y\), whenever \([x] \cap [y] \in K\) [33].
Theorem 3.1. (cf. [21]) Let \((A, K)\) be a Cauchy space. Then,

1. \((A, K)\) in \(\text{CHY}\) is \(\tilde{T}_0\) iff it is \(T_0\) iff it is \(T_1\) iff for each distinct pair \(x\) and \(y\) in \(A\), we have \([x] \cap [y] \notin K\).
2. All objects \((A, K)\) in \(\text{CHY}\) are \(\tilde{T}_0\).
3. All objects \((A, K)\) in \(\text{CHY}\) are \(\text{PreT}_2\).
4. \((A, K)\) is \(\text{PreT}_2^\ast\) iff for each pair of distinct points \(x\) and \(y\) in \(A\), we have \([x] \cap [y] \in K\) (equivalently, for each finite subset \(F\) of \(A\), we have \([F] \in K\)).
5. \((A, K)\) is \(\tilde{T}_2\) iff for each distinct pair \(x\) and \(y\) in \(A\), we have \([x] \cap [y] \notin K\).
6. \((A, K)\) is \(T_2^\ast\) iff for each distinct points \(x\) and \(y\) in \(A\), we have \([x] \cap [y] \in K\) (equivalently, for each finite subset \(F\) of \(A\), we have \([F] \in K\)).

Remark 3.2. (cf. [21])

1. If a Cauchy space \((A, K)\) is \(\tilde{T}_0\) or \(T_0\) then it is \(T_0^\ast\). However, the converse is not true generally. For example, let \(A = \{x, y\}\) and \(K = \{[x], [y], \{[x, y]\}, \emptyset\}\). Then \((A, K)\) is \(T_0^\ast\) but it is not \(\tilde{T}_0\) or \(T_0\) for \((T_1)\).
2. If a Cauchy space \((A, K)\) is \(\text{PreT}_2\) then its converse is not true. For example, let \(A = \{x, y\}\) and \(K = \{[x], [y], \emptyset\}\). Then \((A, K)\) is \(\text{PreT}_2\) but it is not \(\text{PreT}_2^\ast\).

Corollary 3.1. \((A, K)\) be in \(\text{CHY}\). \((A, K)\) is \(ST_2\) iff it is \(\Delta T_2\) iff for each pair of distinct points \(x\) and \(y\) in \(A\) and for any \(\alpha, \beta \in K, \alpha \cup \beta\) is improper if \(\alpha \subset [x]\) and \(\beta \subset [y]\) [21].

Remark 3.4. (A, K) be in \(\text{CHY}\). By Remark 4.5 (2) of [22], \((A, K)\) is \(\tilde{T}_2\) iff \((A, K)\) is \(ST_2\) or \(\Delta T_2\).

Remark 3.5. ([3], p. 106) Let \(\alpha\) and \(\beta\) be filters on \(A\). If \(f : A \to B\) is a function, then \(f(\alpha \cap \beta) = f\alpha \cap f\beta\).

Let \((A, K)\) be in \(\text{CHY}\), and \(F\) be a nonempty subset of \(A\). Let \(q : (A, K) \to (A/F, L)\) be the quotient map that identifying \(F\) to a point, \(\ast\) [2].

Theorem 3.2. (cf. [23])

1. If \((A, K)\) is \(T_3^\ast\), then \((A/F, L)\) is \(T_3^\ast\).
2. If \((A, K)\) is \(\tilde{T}_2\), then \((A/F, L)\) is \(\tilde{T}_2\).
3. If \((A, K)\) is \(\text{PreT}_2\), then \((A/F, L)\) is \(\text{PreT}_2\).
4. If \((A, K)\) is \(\text{PreT}_2^\ast\), then \((A/F, L)\) is \(\text{PreT}_2^\ast\).

Theorem 3.3. \((A, K)\) be in \(\text{CHY}\). \(\emptyset \neq F \subset A\) is closed iff for each \(\alpha \in A\) with \(\alpha \notin F\) and for all \(\alpha \in K, \alpha \cup [F] \) is improper or \(\alpha \notin [a]\) [21].

Theorem 3.4. \((A, K)\) be in \(\text{CHY}\). \(\emptyset \neq F \subset A\) is strongly closed iff for each \(\alpha \in A\) with \(\alpha \notin F\) and for all \(\alpha \in K, \alpha \cup [F] \) is improper or \(\alpha \notin [a]\) [21].

Theorem 3.5. (cf. [23])

1. If \((A, K)\) is \(ST_2\) or \(\Delta T_2\) and \(F\) is (strongly) closed, then \((A/F, L)\) is \(ST_2\) or \(\Delta T_2\).
2. All objects \((A, K)\) in \(\text{CHY}\) are \(KT_2\).
3. \((A, K)\) in \(\text{CHY}\) is \(LT_2\) iff \(A\) is a point or the empty set.
4. \((A, K)\) in \(\text{CHY}\) is \(MT_2\) iff \(A\) is a point or the empty set.
5. \((A, K)\) in \(\text{CHY}\) is \(NT_2\) iff for each distinct pair \(x\) and \(y\) in \(A\), \([x] \cap [y] \notin K\).

Remark 3.6. (cf. [23])

1. If a Cauchy space \((A, K)\) is \(LT_2(2MT_2)\) then it is \(KT_2\). However, the converse is not true, in general. For example, let \(A = \{x, y\}\) and \(K = \{[x], [y], [\{x, y\}], \emptyset\}\). Then \((A, K)\) is \(KT_2\) but it is not \(LT_2(2MT_2)\).
2. If a Cauchy space \((A, K)\) is \(NT_2\) then it is \(KT_2\). However, the converse is not true, in general. For example, let \(A = \{x, y\}\) and \(K = \{[x], [y], [\{x, y\}], \emptyset\}\). Then \((A, K)\) is \(KT_2\) but it is not \(NT_2\).
3. If a Cauchy space \((A, K)\) is \(LT_2(2MT_2)\) then it is \(NT_2\). However, the converse is not true, in general. For example, let \(A = \{x, y\}\) and \(K = \{[x], [y], [\{x, y\}], \emptyset\}\). Then \((A, K)\) is \(NT_2\) but it is not \(LT_2(2MT_2)\).
Theorem 4.6. Let \((A, K)\) be a Cauchy space and \(B \subset A\).

(1) If \((A, K)\) is \(\text{Pre} T_2\), then \((B, K_1)\) is also \(\text{Pre} T_2\).
(2) If \((A, K)\) is \(\text{Pre} T'_2\), then \((B, K_1)\) is also \(\text{Pre} T'_2\).
(3) If \((A, K)\) is \(\text{\bar{T}}_2\), then \((B, K_1)\) is also \(\text{\bar{T}}_2\).
(4) If \((A, K)\) is \(T_2\), then \((B, K_1)\) is also \(T_2\).

Proof. Let \(f : B \rightarrow A\) be the inclusion map defined by \(f(x) = x\) for \(x \in B\) and \(K_1\) be the initial lift of \(f : B \rightarrow (A, K)\).

(1) Suppose that \((A, K)\) is \(\text{Pre} T_2\) and \(x \in B\). By Definition 2.2 and Theorem 3.1(3), \((B, K_1)\) is also \(\text{Pre} T_2\).
(2) Let \((A, K)\) is \(\text{Pre} T'_2\) and \(x, y\) be any two distinct points of \(B\). Since \(B \subset A\) and \((A, K)\) is \(\text{Pre} T'_2\), by Theorem 3.1(4), we have \([x] \cap [y] \in K\) and \(f([x]) \cap f([y]) = f([x]) \cap f([y]) = [x] \cap [y] \in K\). Hence by Definition 2.2, \([x] \cap [y] \in K_1\) and by Theorem 3.1(4), \((B, K_1)\) is \(\text{Pre} T'_2\).
(3) Suppose that \((A, K)\) is \(\text{\bar{T}}_2\) and \(x, y\) be any two distinct points of \(B\). Since \(B \subset A\) and \((A, K)\) is \(\text{\bar{T}}_2\), by Theorem 3.1(5), we have \([x] \cap [y] \notin K\) and \(f([x]) \cap f([y]) = f([x]) \cap f([y]) = [x] \cap [y] \notin K\). Hence by Definition 2.2, \([x] \cap [y] \notin K_1\) and by Theorem 3.1(5), \((B, K_1)\) is \(\text{\bar{T}}_2\).

The proof (4) is similar to the proof of (2) by using Theorem 3.1(6).

4. \(T_3\)-Objects

We now recall, ([2, 7, 12]), various generalizations of the usual \(T_3\) separation axiom to arbitrary set based topological categories and characterize each of them for the topological categories \(\text{CHY}\).

Definition 4.1. (cf. [2, 7, 12]) Let \(U : \mathcal{E} \rightarrow \text{Set}\) be a topological functor, \(X\) an object in \(\mathcal{E}\) with \(U(X) = B\). Let \(F\) be a non-empty subset of \(B\).

1. \(X\) is \(S\overline{T}_3\) iff \(X\) is \(T_1\) and \(X/F\) is \(\text{Pre} T_2\) for all strongly closed \(F \neq \emptyset\) in \(U(X)\).
2. \(X\) is \(S\overline{T}_3^2\) iff \(X\) is \(T_1\) and \(X/F\) is \(\text{Pre} T'_2\) for all strongly closed \(F \neq \emptyset\) in \(U(X)\).
3. \(X\) is \(\overline{T}_3\) iff \(X\) is \(T_1\) and \(X/F\) is \(\text{Pre} T_2\) for all closed \(F \neq \emptyset\) in \(U(X)\).
4. \(X\) is \(\overline{T}_3^2\) iff \(X\) is \(T_1\) and \(X/F\) is \(\text{Pre} T'_2\) for all closed \(F \neq \emptyset\) in \(U(X)\).
5. \(X\) is \(K\overline{T}_3\) iff \(X\) is \(T_1\) and \(X/F\) is \(\text{Pre} T_2\) if it is \(T_1,\) where \(F \neq \emptyset\) in \(U(X)\).
6. \(X\) is \(L\overline{T}_3\) iff \(X\) is \(T_1\) and \(X/F\) is \(\text{Pre} T'_2\) if it is \(T_1,\) where \(F \neq \emptyset\) in \(U(X)\).
7. \(X\) is \(S\overline{T}_3^3\) iff \(X\) is \(T_1\) and \(X/F\) is \(\text{Pre} T_3\) if it is \(T_1,\) where \(F \neq \emptyset\) in \(U(X)\).
8. \(X\) is \(\Delta\overline{T}_3\) iff \(X\) is \(T_1\) and \(X/F\) is \(\Delta T_2\) if it is \(T_1,\) where \(F \neq \emptyset\) in \(U(X)\).

Remark 4.1. 1. For the category \(\text{Top}\) of topological spaces, all of the \(T_3\)'s reduce to the usual \(T_3\) separation axiom (cf. [2, 12?]).
2. If \(U : \mathcal{E} \rightarrow B\), where \(B\) is a topos [15], then Parts (1), (2), and (5)-(8) of Definition 4.1 still make sense since each of these notions requires only finite products and finite colimits in their definitions. Furthermore, if \(B\) has infinite products and infinite wedge products, then Definition 4.1 (4), also, makes sense.

Theorem 4.1. (cf. [23])

(1) \((A, K)\) in \(\text{CHY}\) is \(\overline{T}_3\) iff for each distinct pair \(x\) and \(y\) in \(A\), \([x] \cap [y] \notin K\).
(2) \((A, K)\) in \(\text{CHY}\) is \(\overline{T}_3^2\) iff \(A\) is a point or the empty set.
(3) \((A, K)\) in \(\text{CHY}\) is \(\overline{T}_3\) iff for each distinct pair \(x\) and \(y\) in \(A\), \([x] \cap [y] \notin K\).
(4) \((A, K)\) in \(\text{CHY}\) is \(\overline{T}_3^2\) iff \(A\) is a point or the empty set.
(5) \((A, K)\) in \(\text{CHY}\) is \(K\overline{T}_3\) iff for each distinct pair \(x\) and \(y\) in \(A\), \([x] \cap [y] \notin K\).
(6) \((A, K)\) in \(\text{CHY}\) is \(L\overline{T}_3\) iff \(A\) is a point or the empty set.
(7) \((A, K)\) in \(\text{CHY}\) is \(\overline{T}_3^3\) iff for each pair of distinct points \(x\) and \(y\) in \(A\) and for any \(\alpha, \beta \in K, \alpha \cup \beta\) is improper if \(\alpha \subset [x]\) and \(\beta \subset [y]\).
(8) \((A, K)\) in \(\text{CHY}\) is \(\Delta\overline{T}_3\) iff for each pair of distinct points \(x\) and \(y\) in \(A\) and for any \(\alpha, \beta \in K, \alpha \cup \beta\) is improper if \(\alpha \subset [x]\) and \(\beta \subset [y]\).
Theorem 4.2. If $(A, K)$ is $KT_3$, then $(A/F, L)$ is $KT_3$.

Proof. Suppose $(A, K)$ is $KT_3$. Let $a$ and $b$ be any distinct pair of points in $A/F$. By Theorem 4.1 (5), we only need to show that $[a] \cap [b] \neq L$, where $L$ is the structure on $A/F$ induced by $q$. Suppose that $a \neq *$ and $[a], [b] \in L$ implies $\exists [a], [y] \in K$ such that $[a] \supseteq q([a]), [b] \supseteq q([y])$, and $x = qx = a, qy = *$ for any $y \in F$. If $[a] \cap [b] \in L$, then $[a] \cap [y] \in K$, by definition of the quotient map and Remark 3.5. But $[a] \cap [y] \notin K$ since $(A, K)$ is $KT_3$. Hence $[a] \cap [y] \notin L$. Similarly, if $a \neq b \neq *$ and $[a], [b] \in L$ implies $\exists [a], [b] \in K$ such that $[a] \supseteq q([a]), [b] \supseteq q([b])$, and $x = qx = a, qb = b$. If $[a] \cap [b] \in L$, then $[a] \cap [b] \in K$, by definition of the quotient map and Remark 3.5. But $[a] \cap [b] \notin K$ since $(A, K)$ is $KT_3$. Hence $[a] \cap [b] \notin L$.

Consequently for each distinct points $a$ and $b$ in $A/F$, we have $[a] \cap [b] \notin L$. Hence by Theorem 4.1 (5), $(A/F, L)$ is $KT_3$.

\[ \square \]

Theorem 4.3. If $(A, K)$ is $\Delta T_3$, then $(A/F, L)$ is $\Delta T_3$.

Proof. It follows from Theorem 4.2.

\[ \square \]

5. $T_4$-Objects

We now recall various generalizations of the usual $T_4$ separation axiom to arbitrary set based topological categories that are defined in [2, 7, 12], and characterize each of them for the topological categories CHY.

Definition 5.1. (cf. [2, 7, 12]) Let $\mathcal{U} : \mathcal{E} \to \textbf{Set}$ be a topological functor and $X$ an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let $F$ be a non-empty subset of $B$.

1. $X$ is $ST_4$ iff $X$ is $T_1$ and $X/F$ is $ST_3$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
2. $X$ is $ST_4^*$ iff $X$ is $T_1$ and $X/F$ is $ST_3^*$ for all strongly closed $F \neq \emptyset$ in $U(X)$.
3. $X$ is $T_4^*$ iff $X$ is $T_1$ and $X/F$ is $T_3^*$ for all closed $F \neq \emptyset$ in $U(X)$.
4. $X$ is $T_4^*$ iff $X$ is $T_1$ and $X/F$ is $T_3$ for all closed $F \neq \emptyset$ in $U(X)$.
5. $X$ is $\Delta T_4$ iff $X$ is $T_1$ and $X/F$ is $\Delta T_3$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$.
6. $X$ is $KT_4$ iff $X$ is $T_1$ and $X/F$ is $KT_3$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$.
7. $X$ is $LT_4$ iff $X$ is $T_1$ and $X/F$ is $LT_2$ if it is $T_1$, where $F \neq \emptyset$ in $U(X)$.

Remark 5.1. 1. For the category $\textbf{Top}$ of topological spaces, all of the $T_4$’s reduce to the usual $T_4$ separation axiom ([2, 7, 12]).

2. If $\mathcal{U} : \mathcal{E} \to B$, where $B$ is a topos [15], then Definition 5.1 still makes sense since each of these notions requires only finite products and finite colimits in their definitions.

Theorem 5.1. (cf. [23])

1. $(A, K)$ in CHY is $ST_4$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.
2. $(A, K)$ in CHY is $ST_4^*$ iff $A$ is a point or the empty set.
3. $(A, K)$ in CHY is $T_4^*$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.
4. $(A, K)$ in CHY is $T_4$ iff $A$ is a point or the empty set.

Theorem 5.2. $(A, K)$ in CHY is $\Delta T_4$ iff for each pair of distinct points $x$ and $y$ in $A$ and for any $\alpha, \beta \in K$, $\alpha \cup \beta$ is improper if $\alpha \subset [x]$ and $\beta \subset [y]$.

Proof. It follows from Definition 5.1 (5), Theorem 3.1 (1) and Theorem 4.3.

\[ \square \]

Theorem 5.3. $(A, K)$ in CHY is $KT_4$ iff for each distinct pair $x$ and $y$ in $A$, $[x] \cap [y] \notin K$.

Proof. It follows from Definition 5.1 (6), Theorem 3.1 (1) and Theorem 4.2.

\[ \square \]

Theorem 5.4. $(A, K)$ in CHY is $LT_4$ iff $A$ is a point or the empty set.

Proof. It follows from Definition 5.1 (7) and Theorem 3.5 (3).

\[ \square \]
Remark 5.2. Let \((A, K)\) be a Cauchy space. It follows from Theorem 3.5, Theorem 4.1, Theorem 5.1, Theorem 5.2 and Theorem 5.3 that \((A, K)\) is \(NT2\) iff \((A, K)\) is \(ST3\) iff \((A, K)\) is \(T3\) iff \((A, K)\) is \(ST4\) iff \((A, K)\) is \(T4\) iff \((A, K)\) is \(AT4\) iff for each distinct pair \(x\) and \(y\) in \(A\), \([x] \cap [y] \notin K\).

Remark 5.3. Let \((A, K)\) be a Cauchy space. It follows from Theorem 3.5, Theorem 4.1, Theorem 5.1 and Theorem 5.4 that \((A, K)\) is \(ST4\) iff \((A, K)\) is \(T4\) iff \((A, K)\) is \(LT2\) iff \((A, K)\) is \(MT2\) iff \((A, K)\) is \(ST3\) iff \((A, K)\) is \(ST4\) iff \((A, K)\) is \(T4\) iff \((A, K)\) is \(LT4\) iff \(A\) is a point or the empty set.

### 6. Tychonoff objects

In this section, the characterization of Tychonoff objects in this category is given. Furthermore, we investigate the relationships between Tychonoff objects and \(ST2, \Delta T2, ST3, \Delta T3\), generalized separation properties and separation properties at a point \(p\) in this category.

**Definition 6.1.** (cf. [7, 12]) Let \(U : E \rightarrow \text{Set}\) be a topological functor and \(X\) an object in \(E\) with \(U(X) = B\).

1. \(X\) is \(\Delta T_{3\frac{1}{2}}\) iff \(X\) is a subspace of \(\Delta T_3\).
2. \(X\) is \(ST_{3\frac{1}{2}}\) iff \(X\) is a subspace of \(ST_4\).
3. \(X\) is \(T'_{3\frac{1}{2}}\) iff \(X\) is a subspace of \(T'_{4}\).
4. \(X\) is \(ST'_{3\frac{1}{2}}\) iff \(X\) is a subspace of \(ST'_{4}\).
5. \(X\) is \(C\Delta T_{3\frac{1}{2}}\) iff \(X\) is a subspace of a compact \(\Delta T_2\).
6. \(X\) is \(CST_{3\frac{1}{2}}\) iff \(X\) is a subspace of a compact \(ST_2\).
7. \(X\) is \(LT_{3\frac{1}{2}}\) iff \(X\) is a subspace of a compact \(T_2\).
8. \(X\) is \(S\Delta T_{3\frac{1}{2}}\) iff \(X\) is a subspace of a strongly compact \(\Delta T_2\).
9. \(X\) is \(STST_{3\frac{1}{2}}\) iff \(X\) is a subspace of a strongly compact \(ST_2\).
10. \(X\) is \(SLT_{3\frac{1}{2}}\) iff \(X\) is a subspace of a strongly compact \(T_2\).

**Remark 6.1.** For the category \(\text{Top}\) of topological spaces, all six of the properties defined in Definition 6.1 are equivalent and reduce to the usual \(T_{3\frac{1}{2}}\) = Tychonoff, i.e, completely regular \(T_1\) spaces [27], Remark 5.2, and Remark 6.2.

**Lemma 6.1.** (cf. [21]) All objects in \(CHY\) are (strongly) compact.

**Theorem 6.1.** Let \((A, K)\) be a Cauchy space. Then the followings are equivalent:

1. \((A, K)\) is \(\Delta T_{3\frac{1}{2}}\),
2. \((A, K)\) is \(ST_{3\frac{1}{2}}\),
3. \((A, K)\) is \(C\Delta T_{3\frac{1}{2}}\),
4. \((A, K)\) is \(CST_{3\frac{1}{2}}\),
5. \((A, K)\) is \(\Delta T_{3\frac{1}{2}}\),
6. \((A, K)\) is \(STST_{3\frac{1}{2}}\),
7. for each distinct pair \(x\) and \(y\) in \(A\), we have \([x] \cap [y] \notin K\).

**Proof.** It follows from Corollary 3.1, Theorem 5.1, Theorem 5.2, Definition 6.1 and Lemma 6.1.

**Example 6.1.** Let \(X = \{a, b\}\), \(\delta = \{(X, X), \{(a), \{a\}\}, \{(b), \{b\}\}, \{(a), \{a\}\}, \{(a), \{b\}\}, \{(a), \{b\}\}, \{(b), \{a\}\}, \{(b), \{b\}\}, \{(a), \{b\}\}, \{(b), \{a\}\}\) and \(\delta_1 = \{(X, X), \{(a), \{a\}\}, \{(b), \{b\}\}, \{(a), \{a\}\}, (X, \{b\}), (\{a\}, X), (X, \{b\}), (\{b\}, X), (\{a\}, \{b\}), (\{b\}, \{a\})\}\). Then \((X, \delta)\) is \(C\Delta T_{3\frac{1}{2}}\), but \((X, \delta_1)\) is not \(C\Delta T_{3\frac{1}{2}}\), since \((\{a\}, \{b\}) \in \delta\) with \(a \neq b\).
Theorem 6.2. Let \((A, K)\) be a Cauchy space. Then the followings are equivalent:

1. \((A, K)\) is \(T_1\).
2. \((A, K)\) is \(ST_{3\frac{1}{2}}\).
3. \(A\) is a point or the empty set.

Proof. It follows from Theorem 5.1 and Definition 6.1.

Theorem 6.3. Let \((A, K)\) be a Cauchy space. Then the followings are equivalent:

1. \((A, K)\) is \(LT_{3\frac{1}{2}}\).
2. \((A, K)\) is \(SLT_{3\frac{1}{2}}\).
3. For each pair of distinct points \(x\) and \(y\) in \(A\), we have \([x] \cap [y] \in K\) (equivalently, for each finite subset \(F\) of \(A\), we have \([F] \in K\).

Proof. It follows from Theorem 3.1, Definition 6.1 and Lemma 6.1.

We can infer the following results.

Remark 6.2. Let \((A, K)\) be in \(\text{CHY}\). The followings are equivalent:

1. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1, \((A, K)\) is \(T_1\) iff it is \(T_0\) iff it is \(T_0\) iff \((A, K)\) is \(ST_3\) iff it is \(T_3\) iff it is \(KT_3\) iff \((A, K)\) is \(ST_3\) iff it is \(T_1\) iff it is \(ST_2\) or \(\Delta T_2\) iff \((A, K)\) is \(ST_3\) or \(\Delta T_3\) iff \((A, K)\) is \(NT_2\) iff \((A, K)\) is \(\Delta T_{3\frac{1}{2}}\) iff \((A, K)\) is \(ST_{3\frac{1}{2}}\) iff \((A, K)\) is \(C\Delta T_{3\frac{1}{2}}\) iff \((A, K)\) is \(CS\Delta T_{3\frac{1}{2}}\) iff \((A, K)\) is \(S\Delta T_{3\frac{1}{2}}\) iff \((A, K)\) is \(SST_{3\frac{1}{2}}\).

2. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1, \((A, K)\) is \(T_2\) iff \((A, K)\) is \(ST_3\) iff \((A, K)\) is \(T_3\) iff \((A, K)\) is \(KT_3\) iff \((A, K)\) is \(ST_3\) iff \((A, K)\) is \(T_2\) or \(\Delta T_2\) iff \((A, K)\) is \(ST_3\) or \(\Delta T_3\) iff \((A, K)\) is \(NT_2\) iff \((A, K)\) is \(\Delta T_{3\frac{1}{2}}\) iff \((A, K)\) is \(ST_{3\frac{1}{2}}\) iff \((A, K)\) is \(C\Delta T_{3\frac{1}{2}}\) iff \((A, K)\) is \(CS\Delta T_{3\frac{1}{2}}\) iff \((A, K)\) is \(S\Delta T_{3\frac{1}{2}}\) iff \((A, K)\) is \(SST_{3\frac{1}{2}}\).

3. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1, \((A, K)\) is \(ST_3\) or \(T_3\) or \(KT_3\) or \(ST_3\) or \(T_2\) or \(\Delta T_2\) or \(ST_3\) or \(\Delta T_3\) or \(NT_2\) or \(\Delta T_{3\frac{1}{2}}\) or \(ST_{3\frac{1}{2}}\) or \(C\Delta T_{3\frac{1}{2}}\) or \(CS\Delta T_{3\frac{1}{2}}\) or \(S\Delta T_{3\frac{1}{2}}\) or \(SST_{3\frac{1}{2}}\), then \((A, K)\) is \(T_0\). But the converse of implication is not true, in general. For example, let \(A = \{x, y\}\) and \(K = \{[x], [y], [[x, y]], [\emptyset]\}\). Then \((A, K)\) is \(T_0\) but it is not \(ST_3\) or \(T_2\) or \(KT_3\) or \(ST_4\) or \(T_4\) or \(ST_2\) or \(\Delta T_2\) or \(ST_3\) or \(\Delta T_3\) or \(NT_2\) or \(\Delta T_{3\frac{1}{2}}\) or \(ST_{3\frac{1}{2}}\) or \(C\Delta T_{3\frac{1}{2}}\) or \(CS\Delta T_{3\frac{1}{2}}\) or \(S\Delta T_{3\frac{1}{2}}\) or \(SST_{3\frac{1}{2}}\).

4. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1, \((A, K)\) is \(ST_3\) or \(T_3\) or \(KT_3\) or \(ST_4\) or \(T_4\) or \(ST_2\) or \(\Delta T_2\) or \(ST_3\) or \(\Delta T_3\) or \(NT_2\) or \(\Delta T_{3\frac{1}{2}}\) or \(ST_{3\frac{1}{2}}\) or \(C\Delta T_{3\frac{1}{2}}\) or \(CS\Delta T_{3\frac{1}{2}}\) or \(S\Delta T_{3\frac{1}{2}}\) or \(SST_{3\frac{1}{2}}\), then \((A, K)\) is \(PreT_2\). But the converse of implication is not true, in general. For example, let \(A = \{x, y\}\) and \(K = \{[x], [y], [[x, y]], [\emptyset]\}\). Then \((A, K)\) is \(PreT_2\) but it is not \(ST_3\) or \(T_3\) or \(KT_3\) or \(ST_4\) or \(T_4\) or \(ST_2\) or \(\Delta T_2\) or \(ST_3\) or \(\Delta T_3\) or \(NT_2\) or \(\Delta T_{3\frac{1}{2}}\) or \(ST_{3\frac{1}{2}}\) or \(C\Delta T_{3\frac{1}{2}}\) or \(CS\Delta T_{3\frac{1}{2}}\) or \(S\Delta T_{3\frac{1}{2}}\) or \(SST_{3\frac{1}{2}}\).

5. By Theorems 3.1, 4.1 and 6.2, Corollary 3.1, the followings are equivalent:

(a) \((A, K)\) is \(PreT_3^2 (T^2_2)\), \(LT_3\), \(SLT_3\), and \(ST_3\) or \(T_3\) or \(KT_3\) or \(ST_4\) or \(T_4\) or \(ST_2\) or \(\Delta T_2\) or \(ST_3\) or \(\Delta T_3\) or \(NT_2\) or \(\Delta T_{3\frac{1}{2}}\) or \(ST_{3\frac{1}{2}}\) or \(C\Delta T_{3\frac{1}{2}}\) or \(CS\Delta T_{3\frac{1}{2}}\) or \(S\Delta T_{3\frac{1}{2}}\) or \(SST_{3\frac{1}{2}}\).

(b) \(A\) is a point or the empty set.

6. By Theorems 3.1, 4.1 and 6.1, Corollary 3.1 and Definition 3.2, \((A, K)\) is \(ST_3\) or \(T_3\) or \(KT_3\) or \(ST_4\) or \(T_4\) or \(ST_2\) or \(\Delta T_2\) or \(ST_3\) or \(\Delta T_3\) or \(NT_2\) or \(\Delta T_{3\frac{1}{2}}\) or \(ST_{3\frac{1}{2}}\) or \(C\Delta T_{3\frac{1}{2}}\) or \(CS\Delta T_{3\frac{1}{2}}\) or \(S\Delta T_{3\frac{1}{2}}\) or \(SST_{3\frac{1}{2}}\) iff \((A, K)\) is \(T_2\) (we will refer to it as the usual one).

References


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