ON A TYPE OF $\alpha$-COSYMPLECTIC MANIFOLDS

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Abstract. The object of this paper is to study $\alpha$-cosymplectic manifolds admitting a $W_2$-curvature tensor.

1. Introduction

A $(2m+1)$-dimensional differentiable manifold $M$ of class $C^\infty$ is said to have an almost contact structure if the structural group of its tangent bundle reduces to $U(m) \times 1$ ([3], [14]), equivalently an almost contact structure is given by a triple $(\varphi, \zeta, \eta)$ satisfying certain conditions. Many different types of almost contact structures are defined in the literature. In [12], Pokhariyal and Mishara have introduced new tensor fields which is called $W_2$ and $E$-tensor fields in a Riemmanian manifold and studied their properties. Then, Pokhariyal [13] has studied some properties of this tensor fields in Sasakian manifold. Recently, Matsumoto et al. [9] have studied $P$-Sasakian manifolds admitting $W_2$ and $E$-tensor fields and De and Sarkar [5] have studied Sasakian manifolds admitting tensor field. The curvature tensor $W_2$ is defined by

$$W_2(X,Y,U,V) = R(X,Y,U,V) + \frac{1}{n-1} [g(X,U)S(Y,V) - g(Y,U)S(X,V)],$$

where $S$ is a Ricci tensor of type $(0,2)$ [12]. In [10], Yildiz and De have studied geometric and relativistic properties of Kenmotsu manifolds admitting $W_2$-curvature tensor.

In the present paper, we have studied the some curvature conditions on $\alpha$-cosymplectic manifolds. We also have classified $\alpha$-cosymplectic manifolds which satisfy the conditions $P.W_2 = 0$, $\tilde{Z}.W_2 = 0$, $C.W_2 = 0$ and $\tilde{C}.W_2 = 0$ where $P$ is the projective curvature tensor, $\tilde{Z}$ is the concircular curvature tensor, $\tilde{C}$ is the quasi-conformal curvature tensor and $C$ is the conformal curvature tensor.

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2. Preliminaries

Let \((M^n, \varphi, \xi, \eta, g)\) be an \(n\)-dimensional (where \(n = 2m + 1\)) almost contact metric manifold, where \(\varphi\) is a \((1, 1)\)-tensor field, \(\xi\) is the structure vector field, \(\eta\) is a 1-form and \(g\) is the Riemannian metric. It is well known that the \((\varphi, \xi, \eta, g)\) structure satisfies the conditions [4].

\[
\varphi \xi = 0, \quad \eta(\varphi \xi) = 0, \quad \eta(\xi) = 1,
\]

\[
\varphi^2 X = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(\xi) = 1,
\]

\[
g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

for any vector fields \(X\) and \(Y\) on \(M^n\).

If moreover

\[
\nabla_X \xi = -\alpha \varphi^2 X,
\]

\[
(\nabla_Y \eta)Y = \alpha [g(X, Y) - \eta(X)\eta(Y)],
\]

where \(\nabla\) denotes the Riemannian connection of hold and \(\alpha\) is a real number, then \((M^n, \varphi, \xi, \eta, g)\) is called a \(\alpha\)-cosymplectic manifold [5]. (See also: [1])

In this case, it is well known that [10]

\[
R(X, Y)\xi = \alpha^2 [\eta(X)Y - \eta(Y)X],
\]

\[
S(X, \xi) = -\alpha^2 (n - 1)\eta(X),
\]

where \(S\) denotes the Ricci tensor. From (7), it easily follows that

\[
R(X, \xi)Y = \alpha^2 [g(X, Y)\xi - \eta(Y)X]
\]

\[
R(X, \xi)\xi = \alpha^2 [\eta(X)\xi - X].
\]

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [15].

According to them a quasi-conformal curvature tensor \(\tilde{C}\) is defined by

\[
\tilde{C}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
\]

\[\quad - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y, Z)X - g(X, Z)Y],\]

where \(a\) and \(b\) are constants and \(R, S, Q\) and \(\eta\) are the Riemannian curvature tensor type of \((1, 3)\), the Ricci tensor of type \((0, 2)\), the Ricci operator defined by \(g(QX, Y) = S(X, Y)\) and scalar curvature of the manifold respectively. If \(a = 1\) and \(b = -\frac{1}{n-2}\) then takes the form

\[
\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
\]

\[\quad + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z,
\]

where \(C\) is the conformal curvature tensor [7].
We next define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ of $\chi(M)$ by

$$R(X,Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X,Y]} W,$$

$$(X \wedge_A Y)W = A(Y,W)X - A(X,W)Y,$$

respectively, where $X,Y,W \in \chi(M)$ and $A$ is the symmetric $(0,2)$-tensor. On the other hand, the projective curvature tensor $P$ and the concircular curvature tensor $\tilde{Z}$ in a Riemannian manifold $(M^n, g)$ are defined by

$$P(X,Y)W = R(X,Y)W - \frac{1}{n-1} (X \wedge_S Y)W,$$  \hfill (13)

$$\tilde{Z}(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)} (X \wedge_g Y)W,$$  \hfill (14)

respectively \cite{16}.

An $\alpha$-cosymplectic manifold is said to be an $\eta$-Einstein manifold if Ricci tensor $S$ satisfies condition

$$S(X,Y) = \lambda_1 g(X,Y) + \lambda_2 \eta(X)\eta(Y),$$  \hfill (15)

where $\lambda_1, \lambda_2$ are certain scalars.

A Riemannian or a semi-Riemannian manifold is said to semi-symmetric if $R(X,Y).R = 0$, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors $X$ and $Y$ \cite{16}.

In an $\alpha$-cosymplectic manifold, using \cite{8} and \cite{9}, equations \cite{11}, \cite{12}, \cite{13} and \cite{14} reduce to

$$P(\xi, X)Y = -\alpha^2 g(X,Y)\xi - \frac{1}{n-1} S(X,Y)\xi,$$  \hfill (16)

$$\tilde{Z}(\xi, X)Y = (\alpha^2 + \frac{r}{n(n-1)})[-g(X,Y)\xi + \eta(Y)X]$$

$$C(\xi, Y)W = \frac{\alpha^2(n-1) + r}{(n-1)(n-2)} [g(Y,W)\xi - \eta(W)Y] - \frac{1}{n-2} [S(Y,W)\xi - \eta(W)QY],$$

$$\tilde{C}(\xi, Y)W = K[\eta(W)Y - g(Y,W)\xi] - b[S(Y,W)\xi - \eta(W)QY],$$  \hfill (19)

respectively, where $K = a\alpha^2 + b\alpha^2(n-1) + \frac{r}{n} (\frac{a}{n-1} + 2b)$.

A $\alpha$-cosymplectic manifold $M^n$ is said to be Einstein if its Ricci tensor $S$ is of the form

$$S(X,Y) = \lambda_1 g(X,Y),$$  \hfill (20)

for any vector fields $X, Y$ and $\lambda_1$ is a certain scalar.

**Theorem 1.** A cosymplectic manifold is locally the Riemannian product of an almost Kaehler manifold with the real line \cite{11}. 

3. $\alpha$-cosymplectic manifolds satisfying $W_2 = 0$

In this section we consider a $\alpha$-cosymplectic manifold satisfying $W_2 = 0$.

**Theorem 2.** Let $M$ be an $n$-dimensional ($n > 3$) $\alpha$-cosymplectic manifold satisfying $W_2 = 0$. Then $M$ is an Einstein manifold and $M$ is locally isometric to the hyperbolic space $H^n(-\alpha^2)$.

**Proof.** If $M$ be an $n$-dimensional $\alpha$-cosymplectic manifold satisfying $W_2 = 0$, then we have from (1)

$$R(X, Y, U, V) = \frac{1}{n-1}[g(Y, U)S(X, V) - g(X, U)S(Y, V)].$$

(21)

Using $X = U = \xi$ in (21), we get

$$R(\xi, Y, \xi, V) = \frac{1}{n-1}[g(Y, \xi)S(\xi, V) - g(\xi, \xi)S(Y, V)].$$

From (2), (8) and (10), we get

$$S(Y, V) = -\alpha^2(n - 1)g(Y, V).$$

(22)

Thus $M$ is an Einstein manifold. Now using (22) in (21), we get

$$R(X, Y, U, V) = -\alpha^2 g(Y, U)g(X, V) + \alpha^2 g(X, U)g(Y, V).$$

Hence $M$ is of constant curvature $-\alpha^2$. Then $M$ is locally isometric to the hyperbolic space $H^n(-\alpha^2)$. \hfill \square

4. $W_2$-semisymmetric $\alpha$-cosymplectic manifolds

**Definition 3.** An $n$-dimensional $\alpha$-cosymplectic manifolds is called $W_2$-semisymmetric if it satisfies

$$R(X, Y).W_2 = 0,$$

(23)

where $R(X, Y)$ is to be considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors $X, Y$.

**Proposition 4.** Let $M$ be an $n$-dimensional $\alpha$-cosymplectic manifold. Then the $W_2$-curvature tensor on $M$ satisfies the condition

$$W_2(X, Y, U, \xi) = 0.$$

(24)

**Proof.** The proof is clear from (1) and (7). \hfill \square

**Theorem 5.** A $W_2$-semisymmetric $\alpha$-cosymplectic manifold is a locally the Riemannian product of an almost Kaehler manifold with the real line or a locally isometric to the hyperbolic space $H^n(-\alpha^2)$. 


Proof. From (23) we have

(25)

If we multiply this equation by \(\xi\), we have

\[g(R(X,Y)W_2(Z,U)V,\xi) - g(W_2(R(X,Y)Z,U)V,\xi) - g(W_2(Z,R(X,Y)U)V,\xi) - g(W_2(Z,U)R(X,Y)V,\xi) = 0.\]  
(26)

Putting \(X = \xi\) in (26) we obtain

\[g(R(\xi,Y)W_2(Z,U)V,\xi) - g(W_2(R(\xi,Y)Z,U)V,\xi) - g(W_2(Z,R(\xi,Y)U)V,\xi) - g(W_2(Z,U)R(\xi,Y)V,\xi) = 0.\]  
(27)

Using (7), (9) and (10) in (27), we get

\[-\alpha^2 g(Y,W_2(Z,U)V) + \alpha^2 g(Z,U)V,\xi) - \alpha^2 g(Y,W_2(Z,U)V)\xi) - \alpha^2 g(Y,W_2(Z,U)V)\xi) - \alpha^2 g(Y,W_2(Z,U)V)\xi) = 0.\]  
(28)

Using (24) in (28), we obtain

\[\alpha^2 W_2(Z,U,V,Y) = 0.\]

Then \(\alpha = 0\) or \(W_2 = 0\). The proof is completed from Theorem 1 and Theorem 2. \(\square\)

5. \(\alpha\)-cosymplectic manifolds satisfying \(P(X,Y), W_2 = 0\)

In this section we consider a \(\alpha\)-cosymplectic manifold \(M^\alpha\) satisfying the condition

\[P(X,Y), W_2 = 0.\]

This equation implies

(29)

Taking the inner product with \(\xi\) and putting \(X = \xi\)

\[g(P(\xi,Y)W_2(Z,U)V,\xi) - g(W_2(P(\xi,Y)Z,U)V,\xi) - g(W_2(Z,P(\xi,Y)U)V,\xi) - g(W_2(Z,U)P(\xi,Y)V,\xi) = 0.\]  
(30)
Using (16) in (30), we have

\[- \alpha^2 g(Y, W_2(Z, U)V) - \frac{1}{n-1} S(Y, W_2(Z, U)V) \]
\[+ \alpha^2 g(Y, Z)g(W_2(\xi, U)V, \xi) + \frac{1}{n-1} S(Y, Z)g(W_2(\xi, U)V, \xi) \]
\[+ \alpha^2 g(Y, U)g(W_2(Z, \xi)V, \xi) + \frac{1}{n-1} S(Y, U)g(W_2(Z, \xi)V, \xi) \]
\[+ \alpha^2 g(Y, U)g(W_2(Z, U)\xi, \xi) + \frac{1}{n-1} S(Y, U)g(W_2(Z, U)\xi, \xi) = 0. \tag{31} \]

Using (24) in (31), we get

\[S(Y, W_2(Z, U)V) = 2(1-n)g(Y, W_2(Z, U)V). \tag{32} \]

So, \(M^n\) is an Einstein manifold. Now using (1) in (32), we get

\[\alpha^2 R(Z, U, V, Y) + \frac{\alpha^2}{n-1} [g(Z, V)S(U, Y) - g(U, V)S(Z, Y)] \]
\[+ \frac{1}{n-1} R(Z, U, V, QY) + \frac{1}{(n-1)^2} [g(Z, V)S(U, QY) - g(U, V)S(Z, QY)] = 0. \tag{33} \]

Again using \(Z = V = \xi\) in (33) and from (8), (10), we get

\[S(U, QY) = -2\alpha^2(n-1)S(U, Y) - \alpha^4(n-1)^2 g(U, Y). \tag{34} \]

Hence we have the following

**Theorem 6.** In an \(n\)-dimensional \((n > 3)\) \(\alpha\)-cosymplectic manifold \(M^n\) if the condition \(P(X, Y)W_2 = 0\) holds, then \(M^n\) is an Einstein manifold and the equation (34) is satisfied on \(M^n\).

**Lemma 7.** \([6]\) Let \(A\) be a symmetric \((0, 2)\)-tensor at point \(x\) of a semi-Riemannian manifold \((M^n, g), n > 3\), and let \(T = g^\wedge A\) be the Kulkarni-Nomizu product of \(g\) and \(A\). Then, the relation

\[T.T = k.Q(g, T), k \in R \]

is satisfied at \(x\) if and only if the condition

\[A^2 = k.A + \lambda g, \lambda \in R \]

holds at \(x\).

From Theorem 6 and Lemma 7, we get the following:

**Corollary 8.** Let \(M\) be an \(n\)-dimensional \((n > 3)\) \(\alpha\)-cosymplectic manifold satisfying the condition \(P(X, Y)W_2 = 0\), then \(T.T = k.Q(g, T)\), where \(T = g^\wedge S\) and \(k = -2\alpha^2(n-1)\).
In this section we consider a $\alpha$-cosymplectic manifold $M^n$ satisfying the condition
\[ \bar{Z}(X,Y).W_2 = 0. \]

This equation implies
\[
\begin{align*}
\bar{Z}(X,Y)W_2(Z,U)V - W_2(\bar{Z}(X,Y)Z, U)V \\
- W_2(Z, \bar{Z}(X,Y)U)V - W_2(Z, U)\bar{Z}(X,Y)V = 0.
\end{align*}
\] (35)

Now $X = \xi$ in (35), we have
\[
\begin{align*}
\bar{Z}(\xi,Y)W_2(Z,U)V - W_2(\bar{Z}(\xi,Y)Z, U)V \\
- W_2(Z, \bar{Z}(\xi,Y)U)V - W_2(Z, U)\bar{Z}(\xi,Y)V = 0.
\end{align*}
\] (36)

Using (17) in (36), we get
\[
\begin{align*}
\{\alpha^2 + \frac{r}{n(n-1)}\} & \{ -g(Y, W_2(Z,U)V)\xi + g(W_2(Z,U)V, \xi)Y \\
& + g(Y, Z)W_2(\xi,U)V - \eta(Z)W_2(Y,U)V \\
& + g(Y, U)W_2(Z,\xi)V - \eta(U)W_2(Z,Y)V \\
& + g(Y, U)W_2(Z,\xi)V - \eta(V)W_2(Z,U)V \}. \\
& = 0.
\end{align*}
\] (37)

Taking the inner product with $\xi$ and using (24) in (37), we have
\[
\{\alpha^2 + \frac{r}{n(n-1)}\} g(Y, W_2(Z,U)\xi) = 0.
\]

Again from (17) we have $\alpha^2 + \frac{r}{n(n-1)} \neq 0$. Hence we have
\[ W_2(Z, U, V, Y) = 0. \]

From the proof of Theorem 2 and Theorem 5 we can say:

**Theorem 9.** An $n$-dimensional ($n > 3$) $\alpha$-cosymplectic manifold $M$ satisfying the condition $\bar{Z}(\xi,Y).W_2 = 0$ is an Einstein manifold and locally isometric to the hyperbolic space $H^n(-\alpha^2)$.

In this section we consider a $\alpha$-cosymplectic manifold $M^n$ satisfying the condition
\[ C(X,Y).W_2 = 0. \]

**Theorem 10.** Let $M^n$ be an $n$-dimensional ($n > 3$) $\alpha$-cosymplectic manifold satisfying the condition $C(X,Y).W_2 = 0$. Then $M^n$ is an Einstein manifold.

**Proof.** This equation implies
\[
\begin{align*}
C(X,Y)W_2(Z,U)V - W_2(C(X,Y)Z, U)V \\
- W_2(Z, C(X,Y)U)V - W_2(Z, U)C(X,Y)V = 0.
\end{align*}
\] (38)
Putting $X = \xi$ in (38), we have
\[
C(\xi, Y)W_2(Z, U)V - W_2(C(\xi, Y)Z, U)V
- W_2(Z, C(\xi, Y)U)V - W_2(Z, U)C(\xi, Y)V = 0.
\] (39)

Using (18) in (39), we have
\[
A_g(Y, W_2(Z, U)V) - A_\eta(W_2(Z, U)V)Y - BS(Y, W_2(Z, U)V)\xi + B_\eta(W_2(Z, U)V)QY
- A_g(Y, Z)W_2(\xi, U)V + A_\eta(Z)W_2(Y, U)V + BS(Y, Z)W_2(\xi, U)V - B_\eta(Z)W_2(QY, U)V
- A_g(Y, U)W_2(Z, \xi)V + A_\eta(U)W_2(Z, Y)V + BS(Y, U)W_2(Z, \xi)V - B_\eta(U)W_2(Z, QY)V
- A_g(Y, V)W_2(Z, U)\xi + A_\eta(V)W_2(Z, Y)V + BS(Y, V)W_2(Z, U)\xi - B_\eta(V)W_2(Z, U)QY
\] (40)
respectively, where $A = \frac{\alpha^2(n-1)+r}{(n-1)(n-2)}$ and $B = \frac{1}{n-2}$. Taking the inner product with $\xi$ and using (24), we obtain
\[
A_g(Y, W_2(Z, U)V) - BS(Y, W_2(Z, U)V) = 0.
\] (41)
Thus $M$ is an Einstein manifold.

8. $\alpha$-cosymplectic manifolds satisfying $\tilde{C}(X, Y).W_2 = 0$

In this section we consider a $\alpha$-cosymplectic manifold $M^n$ satisfying the condition
\[
\tilde{C}(X, Y).W_2 = 0.
\]

**Theorem 11.** Let $M$ be an $n$-dimensional ($n > 3$) $\alpha$-cosymplectic manifold satisfying the condition $\tilde{C}(X, Y).W_2 = 0$. Then we get

1) if $b = 0$, then $M$ is an Einstein manifold and $M$ is locally isometric to the hyperbolic space $H^n(-\alpha^2)$.

2) if $b \neq 0$, then $M$ is an Einstein manifold.

**Proof.** This equation implies
\[
\tilde{C}(X, Y)W_2(Z, U)V - W_2(\tilde{C}(X, Y)Z, U)V
- W_2(Z, \tilde{C}(X, Y)U)V - W_2(Z, U)\tilde{C}(X, Y)V = 0.
\] (42)

Putting $X = \xi$ in (42), we have
\[
\tilde{C}(\xi, Y)W_2(Z, U)V - W_2(\tilde{C}(\xi, Y)Z, U)V
- W_2(Z, \tilde{C}(\xi, Y)U)V - W_2(Z, U)\tilde{C}(\xi, Y)V = 0.
\] (43)
Using (19) in (43), we have
\[
K\{g(W_2(Z, U)V, \xi)Y - g(Y, W_2(Z, U)V)\xi - \eta(Z)W_2(Y, U)V \\
+ g(Y, Z)W_2(\xi, U)V - \eta(U)W_2(Z, Y)V + g(Y, U)W_2(Z, \xi)V \\
- \eta(V)W_2(Z, U)Y + g(Y, V)W_2(Z, U)\xi \}
\]
\[
= b[S(Y, W_2(Z, U)V)\xi - g(W_2(Z, U)V, \xi)QY - S(Y, Z)W_2(\xi, U)V \\
+ \eta(Z)W_2(QY, U)V - S(Y, U)W_2(Z, \xi)V + \eta(U)W_2(Z, QY)V \\
- S(Y, Z)W_2(Z, U)\xi + \eta(V)W_2(Z, U)QY} = 0,
\]
where \( K = a\alpha^2 + b\alpha^2(n - 1) + \frac{\xi}{n}(\frac{\alpha}{n+1} + 2b) \). Taking the inner product with \( \xi \) and using (24) in (44), we have
\[
Kg(Y, W_2(Z, U)V) - bS(Y, W_2(Z, U)V) = 0. \tag{45}
\]

From this equation, if \( b = 0 \) then \( W_2 = 0 \) and if \( b \neq 0 \) then \( S(Y, W_2(Z, U)V) = \frac{k}{b}g(Y, W_2(Z, U)V) \). Hence, the proof is completed. \( \square \)

**Corollary 12.** Let \( M \) be an \( n \)-dimensional \((n > 3)\) \( \alpha \)-cosymplectic manifold satisfying the condition \( \tilde{C}(\xi, Y)W_2 = 0 \), then \( T.T = kQ(g, T) \), where \( T = g\wedge S \) and \( k = \frac{K}{b} - \alpha^2(n - 1) \).

**Proof.** If \( b \neq 0 \), then using \( Z = V = \xi \) in (45) and from (1) and (10), we have
\[
S(QY, U) = \frac{K}{b} - \alpha^2(n - 1))S(U, Y) + \alpha^2(n - 1)\frac{K}{b}g(U, Y).
\]

Hence, we have desired result from Lemma 7. \( \square \)

**References**


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