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Solving linear Fredholm-Stieltjes integral equations of the second kind by using the generalized Simpson's rule

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Abstract: *In this paper, the generalized Simpson's rule (GSR) is applied to solve linear Fredholm-Stieltjes integral equations of the second kind (LFSIESK). A numerical example is presented to illustrate the method by using Maple. In some cases depending on the number of subintervals "n" , the results are calculated and compared. The graph of these results is plotted. An algorithm of this application is given by using Maple.*

Keywords: *Approximate Solutions, Linear Fredholm-Stieltjes Integral Equations, Simpson's Rule.*

İkinci tür lineer Fredholm-Stieltjes integral denklemlerinin genelleştirilmiş Simpson kuralı ile çözümü

Öz: *Bu çalışmada, ikinci tür lineer Fredholm-Stieltjes integral denklemlerinin çözümü için genelleştirilmiş Simpson kuralı uygulanmıştır. Metodu göstermek için Maple programı kullanılarak sayısal bir örnek sunulmuştur. "n"nin alt aralıklarına göre bazı durumlarda sonuçlar hesaplanmış ve karşılaştırılmıştır. Bu sonuçların grafiği çizilmiştir.Maple kullanılarak oluşturulmuş bu uygulamanın algoritması verilmiştir.*

Anahtar Kelimeler: *Yaklaşık Çözümler, Lineer Fredholm-Stieltjes İntegral Denklemleri, Simpson Kuralı.*

INTRODUCTION

The theory of integral equation with its applications plays an important role in applied mathematics. Integral equations are used as mathematical models for many and varied physical situations and they also occur as reformulations of other mathematical problems [7]. For many integral equations, it is necessary to use approximation methods. As an example, most of the geophysical problems connected with electromagnetic and seismic wave propagation can only be solved approximately. Among the integral equations, linear Fredholm integral equations of second kind is one of the most popular types of integral equations [7] [13]. Many approximation methods can be used to solve linear Fredholm integral equations of second kind. However, only a few of them are useful to solve LFSIESK. The generalized Simpson's rule is one of the most suitable method with its pretty close result to solve LFSIESK.

Solvıng LFSIESK by Usıng the GSR

Given LFSIESK

$$u(x) = \lambda \int_a^b K(x,y)\,u(y)\,dg(y) + f(x), \qquad x \in [a,b] \tag{2.1}$$

where $K(x,y) \in C[a,b]$, $g(y)$ is the continuous function on the closed interval $[a,b]$ which can be written as a difference of two strictly increasing functions $\varphi(y), \psi(y)$ on the closed interval $[a,b]$, $f(x)$ given function and $u(x)$ is the unknown function to be determined. The parameter λ is a known quantity. Now, instead of $g(y)$ in (2.1), if the difference $\varphi(y) - \psi(y)$ is substituted, then it becomes of the form

$$u(x) = \lambda \int_a^b K(x,y)\,u(y)\,dg(y) + f(x) = \underbrace{\lambda \int_a^b K(x,y)\,u(y)\,d\varphi(y)}_{I} - \underbrace{\lambda \int_a^b K(x,y)\,u(y)\,d\psi(y)}_{II} + f(x) \tag{2.2}$$

The integrals in (2.2) can be calculated separately as follows by using the generalized Simpson`s rule which is called here shortly as GSR [1],

$$I = \lambda \int_a^b K(x,y)\,u(y)\,d\varphi(y) \approx \frac{\lambda}{6}\sum_{i=0}^{n-1}\left[K(x,x_{2i})u(x_{2i}) + 4K(x,x^*_{2i+1})u(x^*_{2i+1}) + K(x,x_{2i+2})u(x_{2i+2})\right]\left[\varphi(x_{2i+2}) - \varphi(x_{2i})\right] =$$

$$= \frac{\lambda}{6}\Big\{\left[K(x,x_0)u(x_0) + 4K(x,x^*_1)u(x^*_1) + K(x,x_2)u(x_2)\right]\left[\varphi(x_2) - \varphi(x_0)\right] + \ldots +$$

$$+ \left[K(x,x_{2n-2})u(x_{2n-2}) + 4K(x,x^*_{2n-1})u(x^*_{2n-1}) + K(x,x_{2n})u(x_{2n})\right]\left[\varphi(x_{2n}) - \varphi(x_{2n-2})\right]\Big\}$$

where $x^*_{2i-1} = \varphi^{-1}\left[\dfrac{\varphi(x_{2i}) + \varphi(x_{2i-2})}{2}\right]$ and

$$II = \lambda \int_a^b K(x,y)\,u(y)\,d\psi(y) \approx \frac{\lambda}{6}\sum_{i=0}^{n-1}\left[K(x,x_{2i})u(x_{2i}) + 4K(x,x^{**}_{2i+1})u(x^{**}_{2i+1}) + K(x,x_{2i+2})u(x_{2i+2})\right]\left[\psi(x_{2i+2}) - \psi(x_{2i})\right] =$$

$$= \frac{\lambda}{6}\Big\{\left[K(x,x_0)u(x_0) + 4K(x,x^{**}_1)u(x^{**}_1) + K(x,x_2)u(x_2)\right]\left[\psi(x_2) - \psi(x_0)\right] + \ldots +$$

$$+ \left[K(x,x_{2n-2})u(x_{2n-2}) + 4K(x,x^{**}_{2n-1})u(x^{**}_{2n-1}) + K(x,x_{2n})u(x_{2n})\right]\left[\psi(x_{2n}) - \psi(x_{2n-2})\right]\Big\}$$

where $x^{**}_{2i-1} = \psi^{-1}\left[\dfrac{\psi(x_{2i})+\psi(x_{2i-2})}{2}\right]$

Therefore, the integral equation (2.2) becomes

$$\begin{aligned}\overline{u}(x) = \frac{\lambda}{6}\sum_{i=0}^{n-1}\Big\{&\left[K(x,x_{2i})u(x_{2i})+4K(x,x^{*}_{2i+1})u(x^{*}_{2i+1})+K(x,x_{2i+2})u(x_{2i+2})\right]\left[\varphi(x_{2i+2})-\varphi(x_{2i})\right]-\\ &-\left[K(x,x_{2i})u(x_{2i})+4K(x,x^{**}_{2i+1})u(x^{**}_{2i+1})+K(x,x_{2i+2})u(x_{2i+2})\right]\left[\psi(x_{2i+2})-\psi(x_{2i})\right]\Big\}+f(x)\end{aligned} \tag{2.3}$$

where $\overline{u}(x)$ is the approximate solution of (2.2).

Now, in (2.3), if we use the following substitution:

$$\left.\begin{aligned} A_i(x) &= \frac{1}{6}K(x,x_{2i-2})\left[\varphi(x_{2i})-\varphi(x_{2i-2})\right]\\ B_i(x) &= \frac{4}{6}K(x,x^{*}_{2i-1})\left[\varphi(x_{2i})-\varphi(x_{2i-2})\right]\\ C_i(x) &= \frac{1}{6}K(x,x_{2i})\left[\varphi(x_{2i})-\varphi(x_{2i-2})\right]\\ D_i(x) &= \frac{1}{6}K(x,x_{2i-2})\left[\psi(x_{2i})-\psi(x_{2i-2})\right]\\ E_i(x) &= \frac{4}{6}K(x,x^{**}_{2i-1})\left[\psi(x_{2i})-\psi(x_{2i-2})\right]\\ F_i(x) &= \frac{1}{6}K(x,x_{2i})\left[\psi(x_{2i})-\psi(x_{2i-2})\right]\end{aligned}\right\}\text{ for } i=1,2,3,\ldots,n, \tag{2.4}$$

then the equation (2.3) can be written as

$$\overline{u}(x) = \sum_{i=1}^{n}\left[A_i(x)u(x_{2i-2})+B_i(x)u(x^{*}_{2i-1})+C_i(x)u(x_{2i})-D_i(x)u(x_{2i-2})-E_i(x)u(x^{**}_{2i-1})-F_i(x)u(x_{2i})\right]+f(x) \tag{2.5}$$

From (2.5), we have

$$\overline{u}(x) = \lambda\left\{\sum_{i=1}^{n}\left(A_i(x)-D_i(x)\right)u(x_{2i-2})+\sum_{i=1}^{n}B_i(x)u(x^{*}_{2i-1})-\sum_{i=1}^{n}E_i(x)u(x^{**}_{2i-1})+\sum_{i=1}^{n}\left(C_i(x)-F_i(x)\right)u(x_{2i})\right\}+f(x) \tag{2.6}$$

However, some of the terms of the equation (2.6) can be written as

$$\lambda\sum_{i=1}^{n}\left(A_i(x)-D_i(x)\right)u(x_{2i-2}) = \lambda\left(A_1(x)-D_1(x)\right)u(x_0)+\lambda\sum_{i=1}^{n-1}\left(A_{i+1}(x)-D_{i+1}(x)\right)u(x_{2i}) \tag{2.7}$$

$$\lambda\sum_{i=1}^{n}\left(C_i(x)-F_i(x)\right)u(x_{2i}) = \lambda\sum_{i=1}^{n-1}\left(C_i(x)-F_i(x)\right)u(x_{2i})+\left(C_n(x)-F_n(x)\right)u(x_{2n}) \tag{2.8}$$

Taking into account (2.7) and (2.8), from (6), we get

$$\bar{u}(x) = \lambda\left(A_1(x) - D_1(x)\right)u(x_0) + \lambda\sum_{i=1}^{n-1}\left(A_{i+1}(x) - D_{i+1}(x) + C_i(x) - F_i(x)\right)u(x_{2i}) + \\ + \lambda\left(C_n(x) - F_n(x)\right)u(x_{2n}) + \lambda\sum_{i=1}^{n} B_i(x)u(x^*_{2i-1}) - \lambda\sum_{i=1}^{n} E_i(x)u(x^{**}_{2i-1}) + f(x) \tag{2.9}$$

In (2.9), if

$$P(x) = A_1(x) - D_1(x), R_i(x) = A_{i+1}(x) - D_{i+1}(x) + C_i(x) - F_i(x), S(x) = C_n(x) - F_n(x) \tag{2.10}$$

then (2.9) becomes

$$\bar{u}(x) = \lambda P(x)u(x_0) + \lambda\sum_{i=1}^{n-1} R_i(x)u(x_{2i}) + \lambda S(x)u(x_{2n}) + \lambda\sum_{i=1}^{n} B_i(x)u(x^*_{2i-1}) - \lambda\sum_{i=1}^{n} E_i(x)u(x^{**}_{2i-1}) + f(x) \tag{2.11}$$

Substituting x_{2j}, x^*_{2k-1} and x^{**}_{2k-1} for $j = 0,1,...,n$ and $k = 1,2,...,n$ into (2.11), we get the following system of linear equations

$$\bar{u}(x_{2j}) = \lambda P(x_{2j})u(x_0) + \lambda\sum_{i=1}^{n-1} R_i(x_{2j})u(x_{2i}) + \lambda S(x_{2j})u(x_{2n}) + \lambda\sum_{i=1}^{n} B_i(x_{2j})u(x^*_{2i-1}) - \\ - \lambda\sum_{i=1}^{n} E_i(x_{2j})u(x^{**}_{2i-1}) + f(x_{2j})$$

$$\bar{u}(x^*_{2k-1}) = \lambda P(x^*_{2k-1})u(x_0) + \lambda\sum_{i=1}^{n-1} R_i(x^*_{2k-1})u(x_{2i}) + \lambda S(x^*_{2k-1})u(x_{2n}) + \lambda\sum_{i=1}^{n} B_i(x^*_{2k-1})u(x^*_{2i-1}) - \\ - \lambda\sum_{i=1}^{n} E_i(x^*_{2k-1})u(x^{**}_{2i-1}) + f(x^*_{2k-1})$$

$$\bar{u}(x^{**}_{2k-1}) = \lambda P(x^{**}_{2k-1})u(x_0) + \lambda\sum_{i=1}^{n-1} R_i(x^{**}_{2k-1})u(x_{2i}) + \lambda S(x^{**}_{2k-1})u(x_{2n}) + \lambda\sum_{i=1}^{n} B_i(x^{**}_{2k-1})u(x^*_{2i-1}) - \\ - \lambda\sum_{i=1}^{n} E_i(x^{**}_{2k-1})u(x^{**}_{2i-1}) + f(x^{**}_{2k-1}) \tag{2.12}$$

From (2.12), we get the following system of linear equations,

$$\bar{u}(x_0) = \lambda P(x_0)u(x_0) + \lambda\sum_{i=1}^{n-1} R_i(x_0)u(x_{2i}) + \lambda S(x_0)u(x_{2n}) + \lambda\sum_{i=1}^{n} B_i(x_0)u(x^*_{2i-1}) - \\ - \lambda\sum_{i=1}^{n} E_i(x_0)u(x^{**}_{2i-1}) + f(x_0)$$

$$\vdots$$

$\vdots$

$$\bar{u}(x_{2n}) = \lambda P(x_{2n})u(x_0) + \lambda \sum_{i=1}^{n-1} R_i(x_{2n})u(x_{2i}) + \lambda S(x_{2n})u(x_{2n}) + \lambda \sum_{i=1}^{n} B_i(x_{2n})u(x_{2i-1}^{*}) -$$

$$-\lambda \sum_{i=1}^{n} E_i(x_{2n})u(x_{2i-1}^{**}) + f(x_{2n})$$

$$\bar{u}(x_1^{*}) = \lambda P(x_1^{*})u(x_0) + \lambda \sum_{i=1}^{n-1} R_i(x_1^{*})u(x_{2i}) + \lambda S(x_1^{*})u(x_{2n}) + \lambda \sum_{i=1}^{n} B_i(x_1^{*})u(x_{2i-1}^{*}) -$$

$$-\lambda \sum_{i=1}^{n} E_i(x_1^{*})u(x_{2i-1}^{**}) + f(x_1^{*})$$

$\vdots$

$$\bar{u}(x_{2n-1}^{*}) = \lambda P(x_{2n-1}^{*})u(x_0) + \lambda \sum_{i=1}^{n-1} R_i(x_{2n-1}^{*})u(x_{2i}) + \lambda S(x_{2n-1}^{*})u(x_{2n}) + \lambda \sum_{i=1}^{n} B_i(x_{2n-1}^{*})u(x_{2i-1}^{*}) -$$

$$-\lambda \sum_{i=1}^{n} E_i(x_{2n-1}^{*})u(x_{2i-1}^{**}) + f(x_{2n-1}^{*}) \tag{2.13}$$

If the system of linear equations (2.13) is converted into matrix form, then

$$\left(I - \lambda \underbrace{\begin{pmatrix} P(x_{2j}) & R_1(x_{2j}) & \cdots & R_{n-1}(x_{2j}) & S(x_{2j}) & B_i(x_{2j}) & \cdots & -E_i(x_{2j}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P(x_{2k-1}^{**}) & R_1(x_{2k-1}^{**}) & \cdots & R_{n-1}(x_{2k-1}^{**}) & S(x_{2k-1}^{**}) & B_i(x_{2k-1}^{**}) & \cdots & -E_i(x_{2k-1}^{**}) \end{pmatrix}}_{A} \right) \underbrace{\begin{pmatrix} u(x_{2j}) \\ \vdots \\ u(x_{2k-1}^{**}) \end{pmatrix}}_{U} = \underbrace{\begin{pmatrix} f(x_{2j}) \\ \vdots \\ f(x_{2k-1}^{**}) \end{pmatrix}}_{F} \tag{2.14}$$

for $j = 0,1,...,n, i = 1,2,...,n, k = 1,2,...,n$

Now, the system of linear equation (2.14) $(I - \lambda A) \cdot U = F$ has a unique solution $U = (I - \lambda A)^{-1} \cdot F$ if and only if $\det(I - \lambda A) \neq 0$.

Now, let $\varphi(x) \in C^{\alpha}[a,b],\ \psi(x) \in C^{\beta}[a,b]$ where $0 < \alpha \leq 1,\ 0 < \beta \leq 1$.

Then in the equations

$$A_i(x) = \frac{1}{6}K(x, x_{2i-2})[\varphi(x_{2i}) - \varphi(x_{2i-2})],\ B_i(x) = \frac{4}{6}K(x, x_{2i-1}^{*})[\varphi(x_{2i}) - \varphi(x_{2i-2})],$$

$$C_i(x) = \frac{1}{6}K(x, x_{2i})[\varphi(x_{2i}) - \varphi(x_{2i-2})],\ D_i(x) = \frac{1}{6}K(x, x_{2i-2})[\psi(x_{2i}) - \psi(x_{2i-2})],$$

$$E_i(x) = \frac{4}{6}K(x, x_{2i-1}^{**})[\psi(x_{2i}) - \psi(x_{2i-2})],\ F_i(x) = \frac{1}{6}K(x, x_{2i})[\psi(x_{2i}) - \psi(x_{2i-2})]$$ for $i = 1,2,...,n$

the terms approaches 0 as $x_{2i} - x_{2i-2}$ approaches 0, at least as fast as $|x_{2i} - x_{2i-2}|^{\alpha,\beta}$ approaches 0.

So, $A_i(x) \to 0$, $B_i(x) \to 0$, $C_i(x) \to 0$, $D_i(x) \to 0$, $E_i(x) \to 0$, $F_i(x) \to 0$ for all $i = 1,2,3,...,n$. [18], [19].

So, the coefficient matrix will be of the form $A \approx \begin{pmatrix} \approx 0 & \cdots & \approx 0 \\ \vdots & \ddots & \vdots \\ \approx 0 & \cdots & \approx 0 \end{pmatrix}$ and we can conclude that $\det(I-A) \neq 0$

.

Therefore, the system (2.14) has a unique solution, namely $U = (I-A)^{-1} \cdot F$.

Thus, if the solution of the system of linear equations (2.14) is substituted back into the (2.12), then the general solution is defined as

$$\bar{u}(x) = \lambda P(x).U_0 + \lambda \sum_{i=1}^{n-1} R_i(x).U_i + \lambda S(x).U_n + \lambda \sum_{i=1}^{n} B_i(x) U_{n+i} - \lambda \sum_{i=1}^{n} E_i(x) U_{2n+i} + f(x) \qquad (2.15)$$

Numerical Example

Let us consider the following LFSIESK

$$u(x) = \int_0^1 \left(1 + x^2 s\right) u(s)\, d\left(\ln\left(1+\sqrt{s}\right)\right) - \frac{x^2}{4} - \frac{x}{6} + x\sqrt{x}\,. \qquad (3.1)$$

Here $K(x,s) = 1 + x^2 s$, $\varphi(x) = \ln\left(1+\sqrt{x}\right)$, $\psi(x) = 0, \lambda = 1$ and $f(x) = -\frac{x^2}{4} - \frac{x}{6} + x\sqrt{x}$.

Let us take $n = 4$, then $h = \frac{b-a}{2n} = \frac{1-0}{8} = 0.125$ and $x_{2i} = a + 2ih = 0.125 \cdot 2i$ for $i = 1,2,3,4$.

If it is calculated, then it can be obtained as $x_0 = 0,\ x_2 = 0.25,\ x_4 = 0.5,\ x_6 = 0.75,\ x_8 = 1.0$.

Then, if the GSR is used to integrate (3.1),

$$I = \int_0^1 \left(1 + x^2 s\right) u(s)\, d\left(\ln\left(1+\sqrt{s}\right)\right) \approx$$
$$\approx \frac{1}{6} \sum_{i=0}^{3} \left[K(x, x_{2i}) u(x_{2i}) + 4K(x, x^*_{2i+1}) u(x^*_{2i+1}) + K(x, x_{2i+2}) u(x_{2i+2}) \right] \left[\varphi(x_{2i+2}) - \varphi(x_{2i})\right] \qquad (3.2)$$

where $x^*_{2i-1} = \varphi^{-1}\left[\frac{\varphi(x_{2i}) + \varphi(x_{2i-2})}{2}\right]$,

if calculated, then it can be obtained as $x^*_1 = 0.0505,\ x^*_3 = 0.3602,\ x^*_5 = 0.6159,\ x^*_7 = 0.8683$.

So the equation (3.2) becomes

$$\bar{u}(x) = \frac{1}{6} \sum_{i=0}^{3} \left[K(x, x_{2i}) u(x_{2i}) + 4K(x, x^*_{2i+1}) u(x^*_{2i+1}) + K(x, x_{2i+2}) u(x_{2i+2}) \right] \left[\varphi(x_{2i+2}) - \varphi(x_{2i})\right] + f(x) \qquad (3.3)$$

Here if

$$A_i(x) = \frac{1}{6} K(x, x_{2i-2}) \left[\varphi(x_{2i}) - \varphi(x_{2i-2})\right],\ B_i(x) = \frac{4}{6} K(x, x^*_{2i-1}) \left[\varphi(x_{2i}) - \varphi(x_{2i-2})\right],$$

$C_i(x) = \frac{1}{6} K(x, x_{2i}) \left[\varphi(x_{2i}) - \varphi(x_{2i-2})\right]$ for $i = 1,2,3,4$, then (3.3) becomes

$$\bar{u}(x) = \sum_{i=1}^{4}\left[A_i(x)u(x_{2i-2}) + B_i(x)u(x_{2i-1}^{*}) + C_i(x)u(x_{2i})\right] + f(x). \tag{3.4}$$

Then if the values $x_0, x_1^*, x_2, x_3^*, x_4, x_5^*, x_6, x_7^*, x_8$ are substituted into the equation (3.4), then the following system is obtained and solution is found by using Maple as follows

$$\begin{pmatrix} 0.9324 & -0.2703 & -0.0891 & -0.0862 & -0.0364 & -0.0593 & -0.0264 & -0.0462 & -0.0116 \\ -0.0676 & 0.7297 & -0.0892 & -0.0863 & -0.0364 & -0.0594 & -0.0264 & -0.0463 & -0.0116 \\ -0.0676 & -0.2712 & 0.9095 & -0.0882 & -0.0375 & -0.0616 & -0.0276 & -0.0487 & -0.0123 \\ -0.0676 & -0.2721 & -0.0920 & 0.9097 & -0.0388 & -0.0641 & -0.0290 & -0.0514 & -0.0131 \\ -0.0676 & -0.2724 & -0.0947 & -0.0940 & 0.9591 & -0.0685 & -0.0313 & -0.0563 & -0.0145 \\ -0.0676 & -0.2755 & -0.0976 & -0.0980 & -0.0433 & 0.9268 & -0.0339 & -0.0615 & -0.0159 \\ -0.0676 & -0.2780 & -0.1017 & -0.1037 & -0.0466 & -0.0799 & 0.9625 & -0.0688 & -0.0181 \\ -0.0676 & -0.2806 & -0.1059 & -0.1096 & -0.0501 & -0.0869 & -0.0413 & 0.9235 & -0.0203 \\ -0.0676 & -0.2840 & -0.1114 & -0.1173 & -0.0546 & -0.0959 & -0.0462 & -0.0864 & 0.9769 \end{pmatrix} \cdot \begin{pmatrix} u(x_0) \\ u(x_1^*) \\ u(x_2) \\ u(x_3^*) \\ u(x_4) \\ u(x_5^*) \\ u(x_6) \\ u(x_7^*) \\ u(x_8) \end{pmatrix} = \begin{pmatrix} 0.0000 \\ 0.0023 \\ 0.0677 \\ 0.1237 \\ 0.2077 \\ 0.2859 \\ 0.3839 \\ 0.4759 \\ 0.5833 \end{pmatrix} \Rightarrow \begin{pmatrix} u(x_0) \\ u(x_1^*) \\ u(x_2) \\ u(x_3^*) \\ u(x_4) \\ u(x_5^*) \\ u(x_6) \\ u(x_7^*) \\ u(x_8) \end{pmatrix} = \begin{pmatrix} 0.3068 \\ 0.3094 \\ 0.3821 \\ 0.4463 \\ 0.5449 \\ 0.6388 \\ 0.7591 \\ 1.0118 \end{pmatrix} \tag{3.5}$$

Then, this solution is substituted back into (3.4) and simplified by Maple to get

$$\bar{u}(x) = 0.306805528886602 - 0.128312149449382 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x} \tag{3.6}$$

which is pretty close to the exact solution $u(x) = 0.3058111302 - 0.1289085929 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$

As the number of subintervals "n" increased, the accuracy in the approximate solution increases and the error decreases. The following Table 1 shows how the approximate solution approaches the exact solution as the number of subintervals "n" increases.

Table 1. Comparison determinant of the coefficient matrix in (16) and the approximate solution, as n increases

n	$\det(A)$	$\bar{u}(x)$	$u(x)$
4	0.2619621	$0.3068055 - 0.1283121 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$	$0.3058111 - 0.1289085 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$
16	0.2621706	$0.3058447 - 0.1288960 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$	$0.3058111 - 0.1289085 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$
64	0.2621742	$0.3058122 - 0.1289082 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$	$0.3058111 - 0.1289085 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$
256	0.2621743	$0.3058111 - 0.1289085 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$	$0.3058111 - 0.1289085 \cdot x^2 - \frac{1}{6} \cdot x + x\sqrt{x}$

In the following Figure 3-1, we have Maple plot the solutions of the Table 1. It can be observed that as the number of subintervals "n" increases, the graph of the solutions are accumulating around the exact solution which is close enough to the solution of $n = 256$.

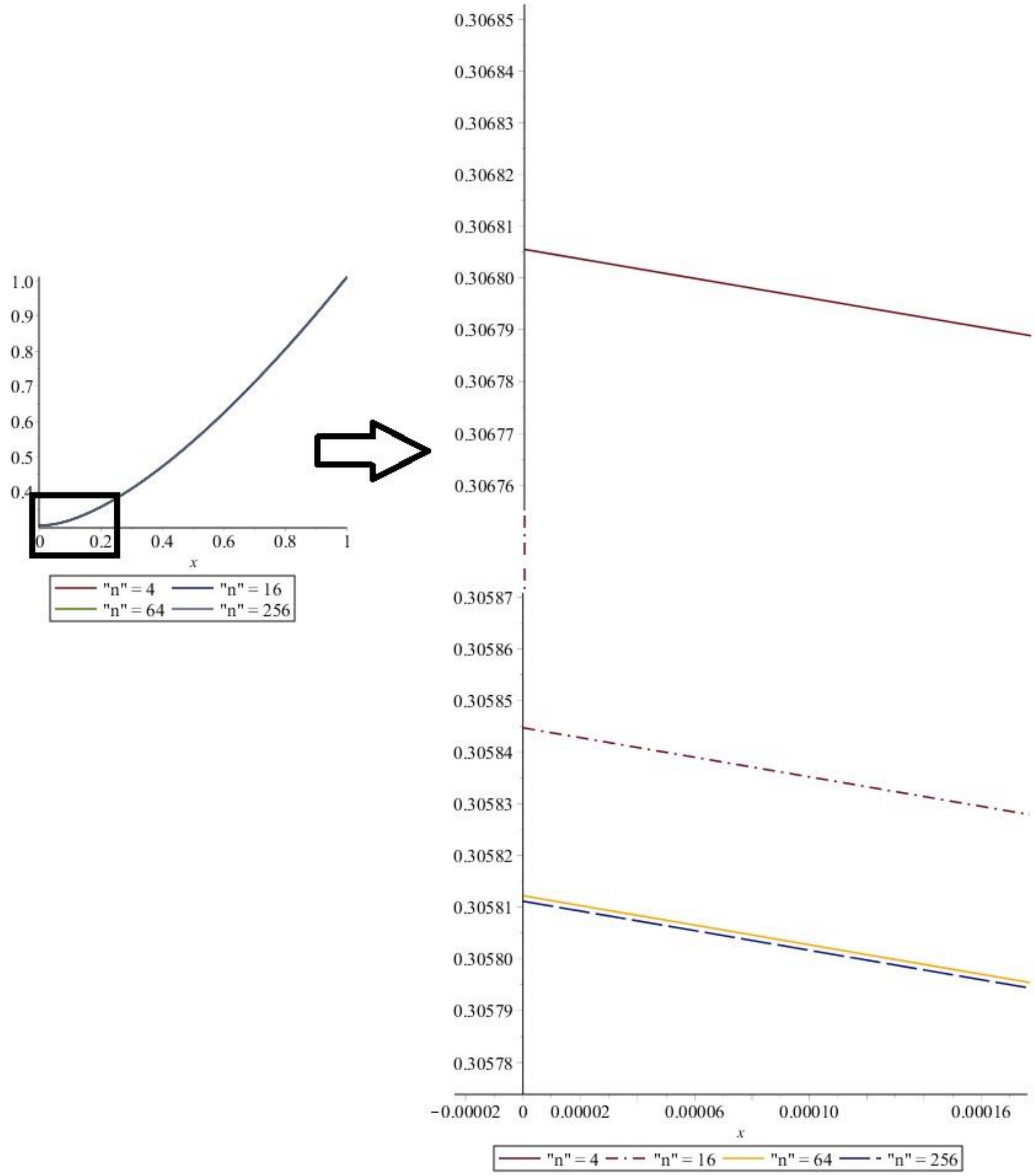

Figure 1: Comparison graphs of the approximate solutions as n increases

Algorıthm for Solvıng LFSIESK By Usıng The GSR In Maple

```
restart
with(LinearAlgebra) :
with(plots) :
```

Inputs

```
K := (x, s) → 1 + x^2·s :
f := x → -x^2/4 - x/6 + x·sqrt(x) :
p := x → ln(1 + sqrt(x)) :
t := x → 0 :
n := 4 :
a := 0.0 :
b := 1.0 :
lambda := 1 :
```

U(x)

```
Simpson_General_Solver := proc(K, p, t, f, a, b, lambda := 1, n := 100) :: function;
    local invp, h := (b - a)/(2·n), y, x :: list, Id, Temp, i, j, C, F, Pr, u :: list, U;
    invp := x → (solve(x = p(y), y));
    for i from 0 by 2 to 2·n do
        x_i := a + i·h;
    end do;
    for i from 1 to n do
        x_{2·i-1} := invp((p(x_{2·i}) + p(x_{2·i-2}))/2);
    end do;
    g := (y, i) → piecewise(i = 1, 1/6·K(y, x_{i-1})·(p(x_{i+1}) - p(x_{i-1})), i :: even, 4/6
    ·K(y, x_{i-1})·(p(x_i) - p(x_{i-2})), i = 2·n + 1, 1/6·K(y, x_{i-1})·(p(x_{i-1}) - p(x_{i-3})),
    1/6·K(y, x_{i-1})·(p(x_{i-1}) - p(x_{i-3})) + 1/6·K(y, x_{i-1})·(p(x_{i+1}) - p(x_{i-1})));
    A := Matrix(2·n + 1);
    for j from 1 to 2·n + 1 do for i from 1 to 2·n + 1 do A(j, i) := g(x[j - 1], i); end do end do;
    C := IdentityMatrix(2·n + 1) - A;
    F := Matrix( 2·n + 1, 1);
    for i from 1 to 2·n + 1
        do F(i, 1) := f(x_{i-1});
    end do;
    Pr := Multiply(MatrixInverse(C), F);
    for i from 1 to 2·n + 1 do
        u_i := Pr(i, 1);
    end do;
    U := y → add(lambda·g(y, i)·u[i], i = 1 ..2·n + 1) + f(y);
    U
end proc:
```

CONCLUSION

This paper deals with the operative algorithms for solving LFSIESK. Indeed, it enables the algorithms that implement the approximation method of the generalized Simpson's rule and its modification using Maple. A numerical example is given and the approximate results are compared with respect to the number of subintervals "n". Also the graph of the example is plotted with respect to increasing "n". Eventually, this shows that the algorithm yield acceptable results.

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