SOME INEQUALITIES FOR POSITIVE MULTILINEAR MAPPINGS

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Abstract. This paper devoted to obtaining some inequalities for positive multilinear mappings. More precisely, we present some Kantorovich and arithmetic-geometric mean inequalities for this kind of mappings. Our results improve earlier results by Kian and Dehghani.

1. Introduction

Let $M_n(\mathbb{C}) = M_n$ be the algebra of all $n \times n$ complex matrices and assume that $M$ and $m$ are scalars and $I$ denotes the identity matrix. We write $A \geq 0$ to mean that the matrix $A$ is positive semidefinite matrix and identify $A = B$ with $A - B \geq 0$. Likewise, we write $A > 0$ to refer that $A$ is a positive definite matrix. The operator norm is denoted by $\| \cdot \|$.

A linear map $\Phi : M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is called positive if $\Phi(A) \geq 0$, whenever $A \geq 0$. Also $\Phi$ is strictly positive if $\Phi(A) > 0$, whenever $A > 0$ and $\Phi$ is called unital if $\Phi(I) = I$. A real-valued continuous function $f$ defined on $[0, \infty)$ is called matrix monotone if $f(A) \geq f(B)$ for $A \geq B \geq 0$. It is well known that $f(t) = t^r$ ($0 \leq r \leq 1$) is a matrix monotone function, namely

$$A \geq B \implies A^p \geq B^p \quad \text{for} \quad 0 \leq p \leq 1.$$ 

Although,

$$A \geq B \implies A^p \geq B^p \quad \text{for} \quad 1 \leq p$$

is not true in general.

If $\Phi : M_n \to M_p$ is a unital positive linear mapping, then Kadison’s inequality states that $\Phi^2(A) \leq \Phi(A^2)$ for every Hermitian matrix $A$ and Choi’s inequality says that $\Phi^{-1}(A) \leq \Phi(A^{-1})$ for every strictly positive matrix $A$, see [4]. There have been a lot of works in which counterparts of these inequalities are presented. Especially see [8, 9].

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A mapping $\Phi : M^k_n := M_n \times \ldots \times M_n \to M_p$ is said to be multilinear if it is linear in each of its variable. A multilinear mapping $\Phi : M^k_n \to M_p$ is called positive if $\Phi(A_1, \ldots, A_k) \geq 0$ whenever $A_i \geq 0$ for $i = 1, \ldots, k$. It is called strictly positive if $A_i > 0$ for $i = 1, \ldots, k$ implies that $\Phi(A_1, \ldots, A_k) > 0$ and $\Phi$ is called unital if $\Phi(I, \ldots, I) = I$; see [3].

Recently, Dehghani et al. [5] obtained an extension of the Choi’s inequality and Kadison’s inequality for positive multilinear mappings:

**Lemma 1.** If $\Phi : M^k_n \to M_p$ is a unital positive multilinear mapping, then

$$\Phi^{-1}(A_1, \ldots, A_k) \leq \Phi(A_1^{-1}, \ldots, A_k^{-1})$$

(1)

and

$$\Phi^2(A_1, \ldots, A_k) \leq \Phi(A_1^2, \ldots, A_k^2)$$

(2)

for all strictly positive matrices $A_i \in M_n$ ($i = 1, \ldots, k$).

In the same paper, the authors presented a Pólya–Szegö type inequality for strictly positive multilinear mappings as follows: If $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$ are $k$–tuples of positive matrices with $0 < mI \leq A_i, B_i \leq MI$ ($i = 1, \ldots, k$) for some positive real numbers $m < M$, then

$$\Phi(A_1, \ldots, A_k) \Phi(B_1, \ldots, B_k) \leq \frac{M^k + m^k}{2M^k m^k} \Phi(A_1^2 B_1, \ldots, A_k^2 B_k)$$

(3)

where $A^\frac{1}{2} = (A^{-\frac{1}{2}} BA^{-\frac{1}{2}})^{\frac{1}{2}} A^\frac{1}{2}$ is called geometric mean of $A, B$. In [3], Kian and Dehghani presented a Kantorovich type inequality for positive multilinear mappings which is a counterpart of [1] as follows:

**Lemma 2.** If $A_i \in M_n$ ($i = 1, \ldots, k$) are positive matrices with $0 < mI \leq A_i \leq MI$ for some scalars $m < M$ and $\Phi : M^k_n \to M_p$ is a unital positive multilinear mapping, then

$$\Phi(A_1^{-1}, \ldots, A_k^{-1}) \leq \frac{(M^k + m^k)^2}{4M^k m^k} \Phi^{-1}(A_1, \ldots, A_k).$$

(4)

With the same assumptions of Lemma 2 Kian and Dehghani obtained

$$\Phi^p(A_1^{-1}, \ldots, A_k^{-1}) \leq \left(\frac{(M^k + m^k)^2}{4M^k m^k}\right)^p \Phi^{-p}(A_1, \ldots, A_k) \quad \text{for } p \geq 2.$$  

(5)

Notice that the inequality

$$\Phi(A_1, \ldots, A_k) + M^k m^k \Phi(A_1^{-1}, \ldots, A_k^{-1}) \leq (M^k + m^k) I$$

holds for every unital positive multilinear mappings. By taking $0 < m^2 I \leq A_i^2 \leq M^2 I$ in the inequality 4 and using inequality 2, we can write the following inequality which will be an important tool for getting our results

$$\Phi^2(A_1, \ldots, A_k) + M^{2k} m^{2k} \Phi^2(A_1^{-1}, \ldots, A_k^{-1}) \leq (M^{2k} + m^{2k}) I.$$  

(7)
In the paper [3], Kian and Dehgani proved that if \((A_1, ..., A_k)\) and \((B_1, ..., B_k)\) are \(k\)-tuples of positive matrices with \(0 < mI \leq A_i, B_i \leq MI \ (i = 1, ..., k)\) for some positive real numbers \(m < M\), then
\[
\Phi^2 \left( \frac{A_1 + B_1}{2}, ..., \frac{A_k + B_k}{2} \right) \leq \left( \frac{(M^k + m^k)^2}{4M^k m^k} \right)^2 \Phi^2 \left( A_1 B_1, ..., A_k B_k \right). \tag{8}
\]

In this paper, we will present some operator inequalities for positive unital multilinear mappings which are generalization of the inequality (8) and improvement of the inequality (5) for \(p \geq 4\). Our idea throughout the paper is similar to the study of Fu and He [10] and Zhang [6] for positive linear maps. Moreover, we will give a squared version of the inequality (3).

2. Main Results

Let’s give some well known lemmas before we give the main theorems of this paper.

Lemma 3. (i) [2, Theorem 1] Let \(A, B > 0\). Then the following norm inequality holds:
\[
\|AB\| \leq \frac{1}{4} \|A + B\|^2.
\]

(ii) [1, Theorem 3] Let \(A\) and \(B\) be positive operators. Then
\[
\|A^r + B^r\| \leq \|(A + B)^r\| \quad \text{for } 1 \leq r \leq \infty.
\]

Theorem 4. Let \(A_i \in M_n\) with \(0 < m \leq A_i \leq M \ (i = 1, ..., k)\). If \(\Phi : M_n^k \to M_1\) is a unital positive multilinear mapping, then for \(4 \leq p < \infty\)
\[
\Phi^p \left( A_1^{-1}, ..., A_k^{-1} \right) \leq \left( \frac{M^{2k} + m^{2k}}{4^{\frac{k}{2}} M^k m^k} \right)^p \Phi^{-p} \left( A_1, ..., A_k \right). \tag{9}
\]

Proof. The matrix inequality (9) is equivalent to
\[
\left\| \Phi^\frac{p}{2} \left( A_1^{-1}, ..., A_k^{-1} \right) \Phi^\frac{p}{2} \left( A_1, ..., A_k \right) \right\| \leq \frac{1}{4} \left( \frac{M^{2k} + m^{2k}}{M^k m^k} \right)^\frac{p}{2}.
\]
Compute
\[ \left\| \Phi^{\frac{p}{2}} \left( A_1^{-1}, \ldots, A_k^{-1} \right) M^{\frac{kp}{2}} m^{\frac{kp}{2}} \Phi^{\frac{p}{2}} (A_1, \ldots, A_k) \right\| \]
\[ \leq \frac{1}{4} \left\| \Phi^{\frac{p}{2}} (A_1, \ldots, A_k) + M^{\frac{kp}{2}} m^{\frac{kp}{2}} \Phi^{\frac{p}{2}} (A_1^{-1}, \ldots, A_k^{-1}) \right\|^2 \]
(by Lemma 3 (i))
\[ \leq \frac{1}{4} \left\| \Phi^{\frac{p}{2}} (A_1, \ldots, A_k) + M^{2k} m^{2k} \Phi^{2} (A_1^{-1}, \ldots, A_k^{-1}) \right\|^2 \]
(by Lemma 3 (ii))
\[ = \frac{1}{4} \left\| \Phi^{\frac{p}{2}} (A_1, \ldots, A_k) + M^{2k} m^{2k} \Phi^{2} (A_1^{-1}, \ldots, A_k^{-1}) \right\|^{\frac{p}{2}} \]
\[ \leq \frac{1}{4} \left\| \Phi^{\frac{p}{2}} (A_1, \ldots, A_k) \right\|^{\frac{p}{2}} \]
(by (7))
\[ \leq \frac{1}{4} \left( M^{2k} + m^{2k} \right)^{\frac{p}{2}}. \]

So
\[ \left\| \Phi^{\frac{p}{2}} (A_1^{-1}, \ldots, A_k^{-1}) \right\| \leq \frac{\left( M^{2k} + m^{2k} \right)^{\frac{p}{2}}}{4 M^{\frac{kp}{2}} m^{\frac{kp}{2}}}. \]

Thus inequality (9) holds. \qed

Remark 5. It is obvious that inequality (9) is tighter than inequality (8) for \( p \geq 4 \).

Now, let's give the generalization of the inequality (8).

Theorem 6. Let \( A_i, B_i \in M_n \) with \( 0 < m \leq A_i, B_i \leq M \) for some positive real numbers \( m < M \) \( (i = 1, \ldots, k) \). If \( \Phi : M_n^k \to M_1 \) is a unital positive multilinear mapping, then for \( 2 \leq p < \infty \)
\[ \Phi^p \left( \frac{A_1 + B_1}{2}, \ldots, \frac{A_k + B_k}{2} \right) \leq \left( \frac{(M^k + m^k)^2}{4^{\frac{p}{2}} M^k m^k} \right)^p \Phi^p (A_1 \sharp B_1, \ldots, A_k \sharp B_k). \quad (10) \]

Proof. The claimed inequality is equivalent to
\[ \left\| \Phi^{\frac{p}{2}} \left( \frac{A_1 + B_1}{2}, \ldots, \frac{A_k + B_k}{2} \right) M^{\frac{kp}{2}} m^{\frac{kp}{2}} \Phi^{\frac{p}{2}} (A_1 \sharp B_1, \ldots, A_k \sharp B_k) \right\| \leq \frac{1}{4} \left( M^k + m^k \right)^p. \]
By computation, we have
\[
\left\| \Phi^\frac{p}{2} \left( \frac{A_1+B_1}{2}, \ldots, \frac{A_k+B_k}{2} \right) M^\frac{kp}{2} m^\frac{kp}{2} \Phi^{-\frac{p}{2}} (A_1^\# B_1, \ldots, A_k^\# B_k) \right\| \\
\leq \frac{1}{4} \left\| \Phi^\frac{p}{2} \left( \frac{A_1+B_1}{2}, \ldots, \frac{A_k+B_k}{2} \right) + M^\frac{kp}{2} m^\frac{kp}{2} \Phi^{-\frac{p}{2}} (A_1^\# B_1, \ldots, A_k^\# B_k) \right\|^2
\]
(by Lemma 3 (i))
\[
\leq \frac{1}{4} \left\| \Phi \left( \frac{A_1+B_1}{2}, \ldots, \frac{A_k+B_k}{2} \right) + M^k m^k \Phi \left( (A_1^\# B_1)^{-1}, \ldots, (A_k^\# B_k)^{-1} \right) \right\|^2
\]
(by Lemma 3 (ii))
\[
= \frac{1}{4} \left\| \Phi \left( \frac{A_1+B_1}{2}, \ldots, \frac{A_k+B_k}{2} \right) + M^k m^k \Phi \left( (A_1^\# B_1)^{-1}, \ldots, (A_k^\# B_k)^{-1} \right) \right\|^p.
\]
By operator arithmetic-geometric mean inequality
\[
\leq \frac{1}{4} \left\| \phi \left( \frac{A_1+B_1}{2}, \frac{A_2+B_2}{2}, \ldots, \frac{A_k+B_k}{2} \right) + M^k m^k \phi \left( \frac{A_1^{-1}+B_1^{-1}}{2}, \ldots, \frac{A_k^{-1}+B_k^{-1}}{2} \right) \right\|^p
\]
\[
= \frac{1}{4} \frac{1}{2^k} \left\| \phi \left( A_1+B_1, \ldots, A_k+B_k \right) + M^k m^k \phi \left( A_1^{-1}+B_1^{-1}, \ldots, A_k^{-1}+B_k^{-1} \right) \right\|^p
\]
\[
\leq \frac{1}{4} \frac{1}{2^k} \left\| \phi \left( A_1, A_2, \ldots, A_k \right) + M^k m^k \phi \left( A_1^{-1}, A_2^{-1}, \ldots, A_k^{-1} \right) + \phi \left( B_1, A_2, \ldots, A_k \right) + \phi \left( B_1, A_2, \ldots, A_k \right) + \phi \left( B_1, B_2, \ldots, B_k \right) + M^k m^k \phi \left( B_1^{-1}, B_2^{-1}, \ldots, B_k^{-1} \right) \right\|^p
\]
\[
\leq \frac{1}{4} \frac{1}{2^k} 2^k (M^k + m^k)^p \quad (\text{by } [6])
\]
\[
\leq \frac{1}{4} (M^k + m^k)^p.
\]
So
\[
\left\| \Phi^\frac{p}{2} \left( \frac{A_1+B_1}{2}, \ldots, \frac{A_k+B_k}{2} \right) \Phi^{-\frac{p}{2}} (A_1^\# B_1, \ldots, A_k^\# B_k) \right\| \leq \frac{1}{4} \frac{(M^k + m^k)^p}{M^p m^p}.
\]
Thus (8) holds. \( \square \)

**Remark 7.** Inequality (8) is a special case of Theorem 6 by taking \( p = 2 \). Thus (10) is a generalization of (8).

Finally, let’s give squared version of (10). For our object, we need the following lemma (see [7, Theorem 6]).

**Lemma 8.** Let \( A, B \in M_n \) such that \( 0 < A \leq B \) and \( 0 < m \leq A \leq M \). Then
\[
A^2 \leq K (h) B^2,
\]
where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$.

**Theorem 9.** Let $A_i$ and $B_i$ be positive matrices with $0 < mI \leq A_i, B_i \leq MI$ $(i = 1, \ldots, k)$ for some positive real numbers $m < M$ and $\Phi$ be a strictly positive unital multilinear map. Then

$$
(\Phi(A_1, \ldots, A_k) \hat{\otimes} \Phi(B_1, \ldots, B_k))^2 \leq \left( \frac{(M^k + m^k)^2}{4M^k m^k} \right)^2 \Phi^2(A_1 \hat{\otimes} B_1, \ldots, A_k \hat{\otimes} B_k). \tag{11}
$$

**Proof.** We have

$$
\Phi(A_1, \ldots, A_k) \hat{\otimes} \Phi(B_1, \ldots, B_k) \leq \frac{M^k + m^k}{2M^k m^k} \Phi(A_1 \hat{\otimes} B_1, \ldots, A_k \hat{\otimes} B_k).
$$

Since $m^k \leq \Phi(A_1, \ldots, A_k) \hat{\otimes} \Phi(B_1, \ldots, B_k) \leq M^k$, by applying Lemma 8 we get the inequality (11). □

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