Lyapunov-type inequality for a Riemann-Liouville fractional differential boundary value problem

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Abstract

The aim of this paper is to present a Lyapunov-type inequality for a Riemann-Liouville fractional differential equation of order $2 < \alpha \leq 3$ subject to mixed boundary conditions.

Keywords: Lyapunov’s inequality, Riemann-Liouville derivative, Caputo fractional derivative, mixed boundary conditions.

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1. Introduction

In this paper, we present a Lyapunov’s inequality for the following boundary value problem:

\[
\begin{cases}
(\alpha D^\alpha u)(t) + q(t)u(t) = 0, & a < t < b, \quad 2 < \alpha \leq 3, \\
u(a) = u'(a) = u'(b) = 0,
\end{cases}
\]

where $a$ and $b$ are consecutive zeros of the solution $u$. As $u = 0$ is a trivial solution, only non-negative solutions are taken in consideration. We prove that problem (1.1) has a non-trivial solution for $\alpha \in (2, 3]$ provided that the real and continuous function $q$ satisfies

\[
\int_a^b |q(t)| \, dt > \frac{\Gamma(\alpha)}{(b-a)^{(\alpha-1)}} \left( \frac{\alpha-1}{\alpha-2} \right)^{\alpha-2}.
\]

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Before we prove this result, let us dwell upon some references.

For the problem
\[
\begin{aligned}
&\frac{d^2}{dt^2} u(t) + q(t) u(t) = 0, \quad a < t < b \\
u(a) = u(b) = 0,
\end{aligned}
\]
where \(a\) and \(b\) are consecutive zeros of \(u\) and the function \(q \in C([a, b]; \mathbb{R})\). Lyapunov [7] proved a necessary condition of existence of non-trivial solutions is that
\[
\int_a^b |q(t)| dt > \frac{4}{b-a}.
\]

After this result, similar type inequalities have been obtained for other kind of differential equations and boundary conditions see [3], [8].

Concerning differential equation with fractional derivative’s in [2], Ferreira derived Lyapunov’s inequality for the problem
\[
\begin{aligned}
&\left(\frac{d^\alpha}{dt^\alpha} u(t) \right) + q(t) u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2, \\
u(a) = u(b) = 0,
\end{aligned}
\]
where \(q \in C([a, b]; \mathbb{R})\), \(a\) and \(b\) are consecutive zeros of \(u\), and \(\frac{d^\alpha}{dt^\alpha}\) is the Riemann-Liouville fractional derivative of order \(\alpha > 0\) defined for an absolute continuous function on \([a, b]\) by
\[
\left(\frac{d^\alpha}{dt^\alpha} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{\alpha-1} f(s) \, ds
\]
where \(n \in \mathbb{N}, n < \alpha \leq n+1\) (For more details of fractional derivatives see [6]). His inequality reads
\[
\int_a^b |q(t)| dt > \frac{\Gamma(\alpha)}{\Gamma(1-\alpha)} \left(\frac{4}{b-a}\right)^{\alpha-1} = \Gamma(\alpha) \left(\frac{2^{2(\alpha-1)}}{(b-a)^{\alpha-1}}\right),
\]
which in the particular case \(\alpha = 2\) corresponds to Lyapunov’s classical inequality (1).

Then, Ferreira [3] and Jleli and Samet [5] dealt with fractional differential boundary value problems with Caputo’s derivative which is defined for a function \(f \in AC^n[a, t]\) by
\[
\left(\frac{C^\alpha}{d^\alpha} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{\alpha-1} f^{(n)}(s) \, ds.
\]

For the boundary value problem
\[
\begin{aligned}
&\left(\frac{C^\alpha}{d^\alpha} u(t) \right) + q(t) u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2, \\
u(a) = u(b) = 0,
\end{aligned}
\]
where \(q \in C([a, b]; \mathbb{R})\) and \(a\) and \(b\) are consecutive zeros of \(u\), Ferreira [2] proved that if (1.6) has a nontrivial solution, then the following necessary condition is satisfied
\[
\int_a^b |q(t)| dt > \frac{\Gamma(\alpha) a^{\alpha}}{[(\alpha - 1)(b-a)]^{\alpha-1}}.
\]

In [5], Jleli and Samet considered the equation (1.6) subject to either
\[
\begin{aligned}
&u'(a) = 0, \quad u(b) = 0, \quad (1.8) \\
or \quad u(a) = 0, \quad u'(b) = 0, \quad (1.9)
\end{aligned}
\]
They showed that the associated non trivial solution exists if
\[
\int_a^b (b-s)^{\alpha-2} |q(t)| dt \geq \Gamma(\alpha)
\]
is satisfied. However, in the case of (1.9), the corresponding nontrivial solution exists if:

$$\int_a^b (b - s)^{\alpha - 2} |q(t)| \, dt \geq \frac{\Gamma(\alpha)}{\max\{\alpha - 1, 2 - \alpha\} (b - a)}.$$  

It was shown in [4] that a non trivial solution corresponding to equation (1.6) where $q \in C([a, b]; \mathbb{R})$, $a$ and $b$ are consecutive zeros of $u$, subject to the boundary conditions

$$u(a) - u'(a) = u(b) + u'(b) = 0,$$

exists if the following necessary condition

$$\int_a^b (b - s)^{\alpha - 2} (b - s + \alpha - 1) |q(s)| \, ds \geq \frac{(b - a + 2) \Gamma(\alpha)}{\max\{b - a + 1, \frac{2 - \alpha}{\alpha - 1} (b - a) - 1\}}$$

is satisfied.

2. Main results

2.1. A Lyapunov-type inequality for problem (1.1). The strategy in getting Lyapunov-type inequality for (1.1) is to re-write the considered problem in its equivalent integral form.

As in [2], the solution can be written in the integral form

$$u(t) = \int_a^t G(t, s) q(s) u(s) \, ds + \int_t^b G(t, s) q(s) u(s) \, ds,$$

where the Green function $G(x, t)$ is defined by

$$\Gamma(\alpha) G(t, s) = \begin{cases} 
\frac{(t - a)^{\alpha - 1}}{(b - a)^{\alpha - 2}} (b - s)^{\alpha - 2} - (t - s)^{\alpha - 1}, & a \leq s \leq t, \\
\frac{(t - a)^{\alpha - 1}}{(b - a)^{\alpha - 2}} (b - s)^{\alpha - 2}, & t \leq s \leq b.
\end{cases}$$

(2.1)

$$= \begin{cases} 
g_1(t, s), & a \leq s \leq t \leq b, \\
g_2(t, s), & a \leq t \leq s \leq b,
\end{cases}$$

(2.2)

which in the particular case $a = 0, b = 1$ corresponds to that of M. El-Shahed [1].

2.1. Theorem. The Green function $G$ satisfies:

1. $G(t, s) \geq 0$ for all $a \leq t, s \leq b$.
2. $\max_{t \in [a, b]} G(t, s) = G(b, s), \quad s \in [a, b]$.
3. $G(b, s)$ has a unique maximum given by:

$$\max_{s \in [a, b]} G(b, s) = \frac{1}{\Gamma(\alpha)} (b - a)^{\alpha - 1} \left(\frac{\alpha - 2}{\alpha - 1}\right)^{\alpha - 2}.$$

Proof. For the proof of Theorem 2.1, we start with the function $g_1(t, s)$. The function $g_1$ is non-decreasing. Indeed, to show this fact, we need to make the following observation of Ferreira in [2]:

$$\left(a + \frac{(s - a)(b - a)}{t - a}\right) \geq s \text{ is equivalent to } s \geq a;$$
this allows us to write
\[(t-s)^{\alpha-1} = (t-a + a - s)^{\alpha-1} = [(t-a)(1 + \frac{a-s}{t-a})]^{\alpha-1} = [(b-a)(1 + \frac{a-s}{t-a})]^{\alpha-1}(t-a)^{\alpha-1} = [b - (a + \frac{(s-a)(b-a)}{t-a})]^{\alpha-1}(t-a)^{\alpha-1},\]
which is used to show that \(g_1\) is positive and non-decreasing.

Indeed,
For \(a \leq s \leq t \leq b,\)
\[
g_1(t, s) := \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (t-s)^{\alpha-1}
\]
\[
= \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - [b - (a + \frac{(s-a)(b-a)}{t-a})]^{\alpha-1}(t-a)^{\alpha-1}
\]
\[
\geq \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (t-s)^{\alpha-2}(t-a)^{\alpha-1}(b-a)^{\alpha-2}
\]
\[
\geq 0.
\]
On the other hand
\[
\frac{\partial g_1}{\partial t}(t, s) = (\alpha-1) \frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (t-a)(t-s)^{\alpha-2}
\]
\[
= (\alpha-1) \frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (t-a)[b - (a + \frac{(s-a)(b-a)}{t-a})]^{\alpha-2}
\]
\[
\geq (\alpha-1) \frac{(t-a)^{\alpha-2}}{(b-a)^{\alpha-2}}(b-s)^{\alpha-2} - (b-s)^{\alpha-2}(t-a)^{\alpha-2}(b-a)^{\alpha-2}
\]
\[
\geq 0.
\]
Consequently,
\[
\max_{t, s \in [a, b]} g_1(t, s) = \max_{s \in [a, b]} g_1(b, s).
\]
In view of (2.1) – (2.2), \(g_1(b, s)\) is defined by: \(g_1(b, s) = (b-s)^{\alpha-2}(s-a)\). Its derivative with respect to \(s\) takes the form
\[
\frac{\partial g_1}{\partial s} = (b-s)^{\alpha-3}[(1-s)(1-a) + a(\alpha-2) + b].
\]
\[
\frac{\partial g_1}{\partial s} = 0 \iff s = s_* = a(\alpha-2) + b
\]
\[
\frac{\alpha-1}{a-1}.
\]
Hence
\[
\max_{s \in [a, b]} g_1(b, s) = (b-a)^{\alpha-1}(\frac{\alpha-2}{\alpha-1}).
\]
The function \(g_2\) is clearly positive and non-decreasing in \(t\), so
\[
\max_{t, s \in [a, b]} g_2(t, s) = \max_{s \in [a, b]} g_2(b, s) = g_2(s, s) = (\frac{s-a}{b-a})^\alpha = F(s).
\]
The function $F$ is increasing for
\[ s \leq s^* = \frac{(\alpha - 2)a + (\alpha - 1)b}{2\alpha - 3}; \]
and is decreasing for
\[ s \geq s^* = \frac{(\alpha - 2)a + (\alpha - 1)b}{2\alpha - 3}. \]
So
\[ \max F(s) = \max g_2(s, s) = g_2(s^*, s^*), \]
where
\[ g_2(s^*, s^*) = (b - a)^{\alpha - 1} \left( \frac{\alpha - 2}{\alpha - 1} \right)^{\alpha - 2}. \]

Now we need to compare $g_1(b, s_*)$ and $g_2(s^*, s^*)$.

Since $2 \leq \alpha \leq 3$ then $(2\alpha - 3)^{\alpha - 3} \geq (\alpha - 1)^{2\alpha - 3}$ and therefore
\[ (b - a)^{\alpha - 1} \left( \frac{\alpha - 2}{\alpha - 1} \right)^{\alpha - 2} \geq (b - a)^{\alpha - 1} \left( \frac{\alpha - 1}{\alpha - 2} \right)^{\alpha - 2}. \]

Consequently
\[ \max_{s \in [a, b]} G(b, s) = \frac{1}{\Gamma(\alpha)} (b - a)^{\alpha - 1} \left( \frac{\alpha - 2}{\alpha - 1} \right)^{\alpha - 1}. \]

We are now ready to prove the Lyapunov’s type inequality for problem (1.1).

2.2. Theorem. Let $u$ be a solution satisfying the following boundary value problem
\[ \begin{align*}
(aD^\alpha u)(t) + q(t)u(t) &= 0, \quad a < t < b, \ 2 < \alpha \leq 3, \\
u(a) = u'(a) = u'(b) &= 0,
\end{align*} \tag{2.3} \]
where $a$ and $b$ two consecutive zeros of $u$. Then (2.3) has a non-trivial solution provided that the real and continuous function $q$ satisfies the condition
\[ \int_a^b |q(t)| \, dt > \frac{\Gamma(\alpha)}{(b-a)^{\alpha - 1}} \left( \frac{\alpha - 1}{\alpha - 2} \right)^{\alpha - 1}. \tag{2.4} \]

Proof. For the proof of Theorem 2.2, we equip the Banach space $C[a, b]$ with the Chebyshev norm $||u|| = \max_{t \in [a, b]} |u(t)|$.

As
\[ u(t) := \int_a^b G(t, s)q(s)u(s) \, ds, \]
we have
\[ ||u|| \leq \int_a^b \max_{t, s \in [a, b]} |G(t, s)||q(s)||u|| \, ds. \]

Then since $u$ is a non-trivial solution, in view of Theorem 2.1, we get
\[ 1 \leq \int_a^b \frac{1}{\Gamma(\alpha)} (b - a)^{\alpha - 1} \left( \frac{\alpha - 2}{\alpha - 1} \right)^{\alpha - 1} |q(s)| \, ds. \]

Using the properties of $G$, the inequality (2.4) is obtained. □

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References


