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Random evolution equations in Fréchet spaces

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Abstract

This paper deals with the existence of random mild solutions for some classes of first and second order evolution equations with random effects in Fréchet spaces. The technique used is a generalization of the classical Darbo fixed point theorem for Fréchet spaces associated with the concept of measure of noncompactness.

Keywords: Evolution equation; evolution system; densely defined operator; mild solution fixed point; Fréchet space.
2010 MSC: 34G20, 34G25

Dedicated to Juan J. Nieto for his 60th birthday.

1. Introduction

There has been a significant development in functional evolution equations in recent years; see the monographs [2, 3, 17, 23, 25] and the references therein. By means of a nonlinear alternative of Leray–Schauder type for contraction operators on Fréchet spaces [16], Baghli and Benchohra [4, 5] provided sufficient conditions for the existence of mild solutions of some classes of evolution equations, while in [6, 7, 8] the authors presented some global existence and stability results for functional evolution equations and inclusions in the space of continuous and bounded functions. In [1], an iterative method is used for the existence of mild solutions of evolution equations and inclusions. However in the previous papers some restrictions are supposed like, the compactness of the semigroup, the Lipschitz conditions on the nonlinear term or the boundedness of the obtained mild solutions.

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The nature of a dynamic system in engineering or natural sciences depends on the accuracy of the information we have concerning the parameters that describe that system. If the knowledge about a dynamic system is precise then a deterministic dynamical system arises. Unfortunately in most cases the available data for the description and evaluation of parameters of a dynamic system are inaccurate, imprecise or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system are of statistical nature, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random differential equations, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the monographs [9, 11, 20, 21, 24].

In this paper, we discuss the existence of random mild solutions for the evolution equation

$$u'(t,w) = A(t)u(t,w) + f(t,u(t,w),w); \quad \text{if } t \in \mathbb{R}_+ := [0,\infty),\ w \in \Omega, \tag{1.1}$$

with the initial condition

$$u(0,w) = u_0(w) \in E,\ w \in \Omega, \tag{1.2}$$

where (Ω, F, P) is a complete probability space, $u_0 : \Omega \to E$ is a given function, $f : \mathbb{R}_+ \times E \times \Omega \to E$ is a given function, $(E, \|\cdot\|)$ is a (real or complex) Banach space, and $\{A(t)\}_{t>0}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of bounded linear operators $\{U(t,s)\}_{(t,s)\in\mathbb{R}_+\times\mathbb{R}_+}$; for $(t,s) \in \Lambda := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \le s \le t < +\infty\}$.

Next, we discuss the existence of random mild solutions for the following second order evolution problem

$$\begin{cases} u''(t,w) - A(t)u(t,w) = g(t,u(t,w),w); \quad \text{if } t \in \mathbb{R}_+ := [0,\infty),\ w \in \Omega, \\ \\ u(0,w) = \underline{u}(w), \quad u'(0,w) = \bar{u}(w),\ w \in \Omega, \end{cases} \tag{1.3}$$

where $E, \{A(t)\}_{t>0}$ are as problem (1.1)-(1.2) and $\underline{u}, \bar{u} : \Omega \to E$ and $g : \mathbb{R}_+ \times E \times \Omega \to E$ are given functions.

This paper initiates the existence of random mild solutions for evolution equations in Fréchet spaces with an application of a generalization of the classical Darbo fixed point theorem, and the concept of measure of noncompactness.

2. Preliminaries

Let $I := [0,T];\ T > 0$. A measurable function $u : I \to E$ is Bochner integrable if and only if $\|u\|$ is Lebesgue integrable. For properties of the Bochner integral, see for instance, Yosida [26].
By $B(E)$ we denote the Banach space of all bounded linear operators from E into E, with the norm

$$\|N\|_{B(E)} = \sup_{\|u\|=1} \|N(u)\|.$$

As usual, $L^1(I,E)$ denotes the Banach space of measurable functions $u : I \to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1} = \int_0^T \|u(t)\|dt.$$

By $\mathcal{C} := C(I)$ we denote the Banach space of all continuous functions from I into E with the norm $\|\cdot\|_\infty$ defined by

$$\|u\|_\infty = \sup_{t\in I} \|u(t)\|.$$

Let β_E be the σ-algebra of Borel subsets of E. A mapping $v : \Omega \to E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 2.1. *A mapping $T : \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$, one has*

$$T^{-1}(B) = \{(w, v) \in \Omega \times E : T(w, v) \in B\} \subset \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E those defined in Ω and E respectively.

Definition 2.2. *A function $T : \Omega \times E \to E$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in E$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.*

Definition 2.3. *A function $f : I \times E \times \Omega \to E$ is called random Carathéodory if the following conditions are satisfied:*

(i) The map $(t, w) \to f(t, u, w)$ is jointly measurable for all $u \in E$, and

(ii) The map $u \to f(t, u, w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [18].

Definition 2.4. *[14] Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w, y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for $P-$almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.*

In what follows, for the family $\{A(t),\ t \geq 0\}$ of closed densely defined linear unbounded operators on the Banach space E we assume that it satisfies the following assumptions (see [3], p. 158).

(P_1) The domain $D(A(t))$ is independent of t and is dense in E,

(P_2) For $t \geq 0$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ exists for all λ with $Re\lambda \leq 0$, and there is a constant K independent of λ and t such that

$$\|R(t, A(t))\| \leq K(1 + |\lambda|)^{-1},\ \text{for } Re\lambda \leq 0,$$

(P_3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\|(A(t) - A(\theta))A^{-1}(\tau)\| \leq L|t - \tau|^{\alpha},\ \text{for } t, \theta, \tau \in I.$$

Lemma 2.5. *([3], p. 159) Under assumptions $(P_1) - (P_3)$, the Cauchy problem*

$$u'(t) - A(t)u(t) = 0,\ t \in I,\ u(0) = y_0,$$

has a unique evolution system $U(t, s)$, $(t, s) \in \Delta := \{(t, s) \in J \times J : 0 \leq s \leq t \leq T\}$ satisfying the following properties:

1. *$U(t,t) = I$ where I is the identity operator in E,*
2. *$U(t,s)\ U(s,\tau) = U(t,\tau)$ for $0 \leq \tau \leq s \leq t \leq T$,*
3. *$U(t,s) \in B(E)$ the space of bounded linear operators on E, where for every $(t,s) \in \Delta$ and for each $u \in E$, the mapping $(t,s) \to U(t,s)u$ is continuous.*

More details on evolution systems and their properties can be found in the books of Ahmed [3] and Pazy [23].

Let $X := C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions v from $\mathbb{R}_+$ into E, equipped with the family of seminorms

$$\|v\|_n = \sup_{t\in[0,n]} \|v(t)\|;\ n \in \mathbb{N},$$

and the distance

$$d(u,v) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|u-v\|_n}{1+\|u-v\|_n};\ u,v \in X.$$

We recall the following definition of the notion of a sequence of measures of noncompactness [12, 13].

Definition 2.6. *Let $\mathcal{M}_{\mathcal{X}}$ be the family of all nonempty and bounded subsets of a Fréchet space $\mathcal{X}$. A family of functions $\{\mu_n\}_n \in \mathbb{N}$ where $\mu_n : \mathcal{M}_X \to [0,\infty)$ is said to be a family of measures of noncompactness in the real Fréchet space $\mathcal{X}$ if it satisfies the following conditions for all $B, B_1, B_2 \in \mathcal{M}_{\mathcal{X}}$:*

(a) $\{\mu_n\}_{n\in\mathbb{N}}$ is full, that is: $\mu_n(B) = 0$ for $n \in \mathbb{N}$ if and only if B is precompact,

(b) $\mu_n(B_1) \leq \mu_n(B_2)$ for $B_1 \subset B_2$ and $n \in \mathbb{N}$,

(c) $\mu_n(ConvB) = \mu_n(B)$ for $n \in \mathbb{N}$,

(d) If $\{B_i\}_{i=1,\cdots}$ is a sequence of closed sets from $\mathcal{M}_{\mathcal{X}}$ such that $B_{i+1} \subset B_i;\ i = 1,\cdots$ and if $\lim_{i\to\infty} \mu_n(B_i) = 0$, for each $n \in \mathbb{N}$, then the intersection set $B_\infty := \cap_{i=1}^{\infty} B_i$ is nonempty.

Some Properties:

(e) We call the family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ to be homogeneous if $\mu_n(\lambda B) = |\lambda|\mu_n(B)$; for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$.

(f) If the family $\{\mu_n\}_{n\in\mathbb{N}}$ satisfies the condition $\mu_n(B_1 \cup B_2) \leq \mu_n(B_1) + \mu_n(B_2)$, for $n \in \mathbb{N}$, it is called subadditive.

(g) It is sublinear if both conditions (e) and (f) hold.

(h) We say that the family of measures $\{\mu_n\}_{n\in\mathbb{N}}$ has the maximum property if

$$\mu_n(B_1 \cup B_2) = \max\{\mu_n(B_1), \mu_n(B_2)\},$$

(i) The family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if if the conditions (a), (g) and (h) hold; (full sublinear and has maximum property).

Example 2.7. *For $B \in \mathcal{M}_X,\ x \in B,\ n \in \mathbb{N}$ and $\epsilon > 0$, let us denote by $\omega^n(x,\epsilon)$ for $n \in \mathbb{N}$; the modulus of continuity of the function x on the interval $[0,n]$; that is,*

$$\omega^n(x,\epsilon) = \sup\{|x(t) - x(s)| : t,s \in [0,n], |t-s| \leq \epsilon\}.$$

Further, let us put

$$\omega^n(B,\epsilon) = \sup\{\omega^n(x,\epsilon) : x \in B\},$$

$$\omega_0^n(B) = \lim_{\epsilon \to 0^+} \omega^n(B,\epsilon),$$

$$\bar{\alpha}^n(B) = \sup_{t\in[0,n]} \alpha(B(t)),$$

and

$$\beta_n(B) = \omega_0^n(B) + \bar{\alpha}^n(B).$$

The family of mappings $\{\beta_n\}_{n\in\mathcal{N}}$ where $\beta_n : \mathcal{M}_X \to [0,\infty)$, satisfies the conditions (a)-(d) from Definition 2.6.

Definition 2.8. *A nonempty subset $B \subset \mathcal{X}$ is said to be bounded if*

$$\sup_{v\in\mathcal{X}} \|v\|_n < \infty; \; for\ n \in \mathbb{N}.$$

Lemma 2.9. *[10] If Y is a bounded subset of Fréchet space $\mathcal{X}$, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\mu_n(Y) \le 2\mu_n(\{y_k\}_{k=1}^{\infty}) + \epsilon; \; for\ n \in \mathbb{N}.$$

Lemma 2.10. *[22] If $\{u_k\}_{k=1}^{\infty} \subset L^1(I)$ is uniformly integrable, then $\mu_n(\{u_k\}_{k=1}^{\infty})$ is measurable for $n \in \mathbb{N}$, and*

$$\mu_n\left(\left\{\int_0^t u_k(s)ds\right\}_{k=1}^{\infty}\right) \le 2\int_0^t \mu_n(\{u_k(s)\}_{k=1}^{\infty})ds,$$

for each $t \in [0,n]$.

Definition 2.11. *Let Ω be a nonempty subset of a Fréchet space $\mathcal{X}$, and let $A : \Omega \to \mathcal{X}$ be a continuous operator which transforms bounded subsets of onto bounded ones. One says that A satisfies the Darbo condition with constants $(k_n)_{n\in\mathbb{N}}$ with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$, if*

$$\mu_n(A(B)) \le k_n \mu_n(B)$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}$.
If $k_n < 1$; $n \in \mathbb{N}$ then A is called a contraction with respect to $\{\mu_n\}_{n\in\mathbb{N}}$.

In the sequel we will make use of the following generalization of the classical Darbo fixed point theorem for Fréchet spaces.

Theorem 2.12. *[12, 13] Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space $\mathcal{X}$ and let $V : \Omega \to \Omega$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$. Then V has at least one fixed point in the set Ω.*

3. First Order Random Evolution Equations

In this section, we present the main results for the global existence of random mild solutions for the problem (1.1)-(1.2).

Definition 3.1. *We say that a continuous function $u(\cdot,w) : \mathbb{R}_+ \times \Omega \to E$ is a mild solution of the problem (1.1)-(1.2), if u satisfies the following integral equation*

$$u(t,w) = U(t,0)u_0(w) + \int_0^t U(t,s)\ f(s,u(s,w),w)ds; \; for\ each\ t \in \mathbb{R}_+,\ and\ w \in \Omega.$$

Let us introduce the following hypotheses.

(H_1) There exists a constant $M \geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq M; \text{ for every } (t,s) \in \Lambda.$$

(H_2) The function f is random Carathéodory on $\mathbb{R}_+ \times E \times \Omega$.

(H_3) There exists a continuous function $p : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that for any $w \in \Omega$, we have

$$\|f(t,u,w)\| \leq p(t,w)(1 + \|u\|); \text{ for a.e. } t \in \mathbb{R}_+, \text{ and each } u \in E.$$

(H_4) For each bounded set $B \subset E$ and for any $w \in \Omega$, we have

$$\mu(f(t,B,w)) \leq p(t,w)\mu(B); \text{ for a.e. } t \in \mathbb{R}_+,$$

where μ is a measure of noncompactness on the Banach space E.

Set

$$p_n^*(w) = \operatorname*{ess\,sup}_{t\in[0,n]} p(t,w); \ for\ n \in \mathbb{N}.$$

Define on X the family of measure of noncompactness by

$$\mu_n(D) = \sup_{t\in[0,n]} e^{-4Mp_n^*(w)\tau t}\mu(D(t)),$$

where $\tau > 1$, and $D(t) = \{v(t) \in E : v \in D\}; \ t \in [0,n]$.

Theorem 3.2. *Assume that the hypotheses $(H_1) - (H_4)$ are satisfied, and $nMp_n^*(w) < 1$ for each $n \in \mathbb{N}$, and $w \in \Omega$. Then the problem (1.1)-(1.2) has at least one random mild solution in X.*

Proof. Consider the operator $N : \Omega \times X \to X$ defined by:

$$(N(w)u)(t) = U(t,0)u_0(w) + \int_0^t U(t,s)\ f(s,u(s,w),w)ds. \tag{3.1}$$

The function f is continuous on $\mathbb{R}_+$, then $N(w)$ defines a mapping $N : \Omega \times X \to X$. Thus u is a random solution for the problem (1.1)-(1.2) if and only if $u = (N(w))u$. We shall show that the operator N satisfies all conditions of Lemma 2.12. The proof will be given in several steps.

Step 1. *$N(w)$ is a random operator with stochastic domain on X.*
Since $f(t,u,w)$ is random Carathéodory, the map $w \to f(t,u,w)$ is measurable in view of Definition 2.2. Therefore, the map

$$w \mapsto U(t,0)u_0(w) + \int_0^t U(t,s)f(s,u(s,w),w)ds,$$

is measurable. As a result, N is a random operator on $\Omega \times X$ into X.

Let $W : \Omega \to \mathcal{P}(X)$ be the ball

$$W(w) := B(0,R_n(w)) = \{v \in X : \|v\|_n \leq R_n(w)\}; \ w \in \Omega, \ n \in \mathbb{N},$$

where $R_n : \Omega \to (0,\infty)$ is a function such that

$$R_n(w) \geq \frac{M\|u_0(w)\| + nMp_n^*(w)}{1 - nMp_n^*(w)}.$$

Since $W(w)$ bounded, closed, convex and solid for all $w \in \Omega$, then W is measurable by Lemma 17 of [14]. Let $w \in \Omega$ be fixed, then from (H_3), for any $u \in w(w)$, and each $t \in [0, n]$ we have

$$\begin{aligned}
\|(N(w)u)(t)\|_E &\leq \|U(t,0)u_0(w) + \int_0^t U(t,s)\ f(s,u(s,w),w)ds\|_E \\
&\leq M\|u_0(w)\| + M\left(\int_0^t p(s,w)(1+\|u(s,w)\|ds\right) \\
&\leq M\|u_0(w)\| + nMp_n^*(w) + nMp_n^*(w)R_n \\
&\leq R_n(w).
\end{aligned}$$

Therefore, N is a random operator with stochastic domain W and $N(w) : W(w) \to N(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in X.

Step 2. *$N(w) : B_{R_n} \to B_{R_n}$ is continuous.*
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k \to u$ in $B_{R_n}(w)$. Then, for each $t \in [0, n]$ and $w \in \Omega$, we have

$$\|(N(w)u_k)(t) - (N(w)u)(t)\|$$

$$\begin{aligned}
&\leq \int_0^t \|U(t,s)\|_{B(E)}\|f(s,u_k(s.w),w) - f(s,u(s,w),w)\|ds \\
&\leq M\int_0^t \|f(s,u_k(s,w),w) - f(s,u(s,w),w)\|ds.
\end{aligned}$$

Since $u_k \to u$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N(w)(u_k) - N(w)(u)\|_n \to 0 \quad \text{as } k \to \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \to N(w)$ is a continuous random operator with stochastic domain W, and $N(w)(W(w))$ is bounded.

Step 3. *For each bounded subset D of $W(w)$, $\mu_n(N(w)(D)) \leq \ell_n\mu_n(D)$.*
From Lemmas 2.9 and 2.10, for any $D \subset B_{R_n}$ and any $\epsilon > 0$, there exists a sequence $\{u_k\}_{k=0}^{\infty} \subset D$, such that for all $t \in [0, n]$ and $w \in \Omega$, we have

$$\begin{aligned}
\mu((N(w)D)(t)) &= \mu\left(\left\{U(t,0)u_0 + \int_0^t U(t,s)\ f(s,u(s,w),w)ds;\ u \in D\right\}\right) \\
&\leq 2\mu\left(\left\{\int_0^t U(t,s)f(s,u_k(s,w),w)ds\right\}_{k=1}^{\infty}\right) + \epsilon \\
&\leq 4\int_0^t \mu\left(\|U(t,s)\|_{B(E)}\{f(s,u_k(s,w),w)\}_{k=1}^{\infty}\right)ds + \epsilon \\
&\leq 4M\int_0^t \mu\left(\{f(s,u_k(s,w),w)\}_{k=1}^{\infty}\right)ds + \epsilon \\
&\leq 4M\int_0^t p_n(s)\mu\left(\{u_k(s,w)\}_{k=1}^{\infty}\right)ds + \epsilon \\
&\leq 4Mp_n^*(w)\int_0^t e^{4Mp_n^*(w)\tau s}e^{-4Mp_n^*(w)\tau s}\mu\left(\{u_k(s,w)\}_{k=1}^{\infty}\right)ds + \epsilon \\
&\leq \frac{(e^{4Mp_n^*\tau t} - 1)}{\tau}\mu_n(D) + \epsilon \\
&\leq \frac{e^{4Mp_n^*(w)\tau t}}{\tau}\mu_n(D) + \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\mu((N(w)D)(t)) \leq \frac{e^{4Mp_n^*(w)\tau t}}{\tau}\mu_n(D).$$

Thus

$$\mu_n(N(w)(D)) \leq \frac{1}{\tau}\mu_n(D).$$

As a consequence of steps 1 to 3 together with Theorem 2.12, we can conclude that N has at least one fixed point in $W(w)$ which is a random mild solution of problem (1.1)-(1.2). □

4. Second Order Random Evolution Equations

In this section, we present the main results for the global existence of random mild solutions for problem (1.3).

In what follows, let $\{A(t),\ t \geq 0\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ that is dense in E and independent of t. The existence of solutions to our problem is related to the existence of an evolution operator $U(t,s)$ for the homogeneous problem

$$u''(t) = A(t)u(t);\ t \in \mathbb{R}_+. \tag{4.1}$$

This concept of evolution operator has been developed by Kozak [19].

Definition 4.1. *A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t,s) : E \to E$; $(t,s) \in \{(t,s) : s \leq t\}$, is called an evolution operator of the equation (4.1) if the following conditions hold;*

(P_1) *For any $u \in E$, the map $(t,s) \to \mathcal{U}(t,s)u$ is continuously differentiable and:*

- (a) *for any $(t \in \mathbb{R}_+ : \mathcal{U}(t,t) = 0$;*
- (b) *for all $(t,s) \in \Delta$ and for any $u \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t,s)u|_{t=s} = u$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)u|_{t=s} = -u$.*

(P_2) *For all $(t,s) \in \Delta$ if $u \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)u \in D(A(t))$, the map $(t,s) \to \mathcal{U}(t,s)u$ is of class C^2, and*

- (a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)u = A(t)\mathcal{U}(t,s)u$;
- (b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)u = \mathcal{U}(t,s)A(s)u$;
- (c) $\frac{\partial^2}{\partial t \partial s}\mathcal{U}(t,s)u|_{t=s} = 0$.

(P_3) *For all $(t,s) \in \Delta$ if $u \in D(A(t))$, then the map $(t,s) \to A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)u$ is continuous, $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)u$ and $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)u$ exist and*

- (a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)u = A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)u$;
- (b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)u = A(t)\frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)u$.

Let $X := C(\mathbb{R}_+)$ be the Fréchet space of all continuous functions from $\mathbb{R}_+$ into E. Let us introduce the definition of the mild solution of the problem (1.3).

Definition 4.2. *We say that a function $u \in X$ is a random mild solution of the problem (1.3) if u satisfies the following integral equation*

$$u(t) = -\frac{\partial}{\partial s}U(t,0)\underline{u}(w) + U(t,0)\bar{u}(w) + \int_0^t U(t,s)\ g(s,u(s,w),w)ds;$$

$t \in \mathbb{R}_+,\ w \in \Omega$.

Let us introduce the following hypotheses.

(H_1') There exist constants $M_1,\ M_2 > 0$ such that for every $(t,s) \in \Lambda$, we have

$$\left\| \frac{\partial}{\partial s} U(t,s) \right\|_{B(E)} \leq M_1 \quad \text{and} \quad \|U(t,s)\|_{B(E)} \leq M_2.$$

(H_2') The function g is random Carathéodory on $\mathbb{R}_+ \times E \times \Omega$.

(H_3') There exists a continuous function $q : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that for any $w \in \Omega$, we have

$$\|g(t,u,w)\| \leq q(t,w)(1 + \|u\|);\ \text{for a.e. } t \in \mathbb{R}_+,\ \text{and each } u \in E.$$

(H_4') For each bounded and measurable set $B \subset E$ and for any $w \in \Omega$, we have

$$\mu(g(t,B,w)) \leq q(t,w)\mu(B);\ \text{for a.e. } t \in \mathbb{R}_+,$$

Set

$$q_n^*(w) = \operatorname*{ess\,sup}_{t\in[0,n]} q(t,w);\ \ for\ n \in \mathbb{N}.$$

Now we present (without proof) existence of random mild solution for problem (1.3).

Theorem 4.3. *Assume that the hypotheses $(H_1') - (H_4')$ are satisfied. If $nM_2q_n^*(w) < 1$ for each $n \in \mathbb{N}$, and $w \in \Omega$, then the problem (1.3) has at least one random mild solution in X.*

5. An Example

Let be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u : \Omega \to L^2([0,\pi],\mathbb{R})$, we consider the following functional random evolution problem of the form

$$\begin{cases} \frac{\partial z}{\partial t}(t,x,w) = a(t,x,w)\frac{\partial^2 z}{\partial x^2}(t,x) & \\ +Q(t,z(t,x,w)); & t \in \mathbb{R}_+,\ x \in [0,\pi],\ w \in \Omega, \\ z(t,0,w) = z(t,\pi,w) = 0; & t \in \mathbb{R}_+,\ w \in \Omega, \\ z(0,x,w) = \Phi(x,w); & x \in [0,\pi],\ w \in \Omega, \end{cases} \tag{5.1}$$

where $a(t,x,w) : \mathbb{R}_+ \times [0,\pi] \times \Omega \to \mathbb{R}$ is a continuous function and is uniformly Hölder continuous in t, $Q : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : [0,\pi] \times \Omega \to \mathbb{R}$ are continuous functions.

Consider $E = L^2([0,\pi],\mathbb{R})$ and define $A(t)$ by $A(t)y = a(t,x,w)y''$ with domain

$$D(A) = \{y \in E : y, y' \text{ are absolutely continuous}, y'' \in E,\ y(0) = y(\pi) = 0\}.$$

Then $A(t)$ generates an evolution system $U(t,s)$ (see [15]).

For $x \in [0,\pi]$, we have

$$u(t,w)(x) = z(t,x,w); \quad t \in \mathbb{R}_+,\ w \in \Omega,$$

$$f(t,u(t,w),x,w) = Q(t,z(t,x,w)); \quad t \in \mathbb{R}_+,\ w \in \Omega,$$

and

$$u_0(x,w) = \Phi(x,w); \quad x \in [0,\pi],\ w \in \Omega.$$

Thus, under the above definitions of f, u_0 and $A(\cdot)$, the system (5.1) can be represented by the functional evolution problem (1.1)-(1.2). Furthermore, more appropriate conditions on Q ensure the hypotheses $(H_1) - (H_5)$. Consequently, Theorem 3.2 implies that the evolution problem (5.1) has at least one random mild solution.

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