Multiple Soliton Solutions of Some Nonlinear Partial Differential Equations

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Abstract

In this paper, we implemented an improved tanh function method for multiple soliton solutions of new coupled Konno-Oono equation and extended (3+1)-dimensional KdV-type equation.

1. Introduction

Nonlinear partial differential equations (NPDEs) have an important place in applied mathematics and physics [1], [2]. Many analytical methods have been found in literature [3]-[11]. Besides these methods, there are many methods which reach to solution by using an auxiliary equation. Using these methods, partial differential equations are transformed into ordinary differential equations. These nonlinear partial differential equations are solved with the help of ordinary differential equations. These methods are given in [12]-[39].

We used the improved tanh function method to find the multiple soliton solutions of new coupled Konno-Oono equation and extended (3+1)-dimensional KdV-type equation in this study. This method is presented by Chen and Zhang [15].

2. Analysis of method

Let’s introduce the method briefly. Consider a general partial differential equation of two variables,
\[ \psi (v, v_t, v_x, v_{xx}, \ldots) = 0. \]
(2.1)
and transform equation (2.1) with
\[ v(x, t) = v(\varnothing), \quad \varnothing = k(x - wt) \]
where \( k, w \) are constants. With this conversion, we obtain a nonlinear ordinary differential equation for \( v(\varnothing) \)
\[ \psi' (v', v'', v''' \ldots) = 0. \]
(2.2)
We can express the solution of equation (2.2) as below,
\[ v(\varnothing) = \sum_{i=0}^{n} a_i \varnothing^i, \]
here \( n \) is a positive integer and is found as the result of balancing the highest order linear term and the highest order nonlinear term found in the equation.
If we write these solutions in equation (2.2), we obtain a system of algebraic equations for \( F(\varphi) \), \( F^2(\varphi) \), \( \ldots \), \( F^i(\varphi) \), after, if the coefficients of \( F(\varphi) \), \( F^2(\varphi) \), \( \ldots \), \( F^i(\varphi) \) are equal to zero, we can find the \( k, w, a_0, a_1, \ldots, a_n \) constants. The basic step of the method is to make full use of the Riccati equation satisfying the tanh function and to use \( F(\varphi) \), solutions. The Riccati equation required in this method is given below

\[
F'(\varphi) = A + BF(\varphi) + CF^2(\varphi)
\]

where, \( F'(\varphi) = \frac{dF(\varphi)}{d\varphi} \) and \( A, B \) and \( C \) are constants. The authors expressed the solutions [15].

**Example 2.1.** Example 1. We consider the new coupled Konno-Oono equation,

\[
\begin{align*}
\nu_t + 2\mu u_x &= 0, \\
u_{tt} - 2\mu v &= 0. 
\end{align*}
\]

Using the variable \( \nu(x, t) = \nu(\varphi) \) and \( u(x, t) = u(\varphi) \), \( \varphi = k(x - \mu t) \), the equation (2.3) turns into an ordinary differential equation,

\[
-k\omega \nu' + 2\mu u' = 0
\]

\[
-k^2 \omega u'' - 2\mu v = 0. 
\]

When balancing \( \nu' \) with \( \omega u' \) and \( u'' \) with \( \omega v \) then \( n_1 = 1 \) and \( n_2 = 2 \) gives. The solution is as follows:

\[
\begin{align*}
u &= a_0 + a_1 F(\varphi) \\
v &= b_0 + b_1 F(\varphi) + b_2 F^2(\varphi) 
\end{align*}
\]

(2.5) are substituted in equations (2.4), a system of algebraic equations for \( k, w, a_0, a_1, b_0, b_1, b_2 \) are obtained. The obtained systems of algebraic equations are as follows:

\[
\begin{align*}
2Aka_0a_1 - Akwb_1 &= 0, \\
2Bk^2a_0a_1 + 2Aka_1^2 - Bkwb_1 - 2Akwb_2 &= 0, \\
2Cka_0a_1 + 2Bka_1^2 - Ckwb_1 - 2Bkwb_2 &= 0, \\
2Ck^2a_1^2 - 2Ckwb_2 = 0 - ABk^2wa_1 - 2a_0b_1 &= 0, \\
-B^2k_2wa_1 - 2ACK^2wa_1 - 2a_1b_1 - 2a_0b_1 &= 0, \\
-3BCK^2wa_1 - 2a_1b_1 - 2a_0b_2 &= 0, \\
-2C^2k^2wa_1 - 2a_1b_2 &= 0.
\end{align*}
\]

Solving the above system with the help of Mathematica, the coefficients are found as two cases:

**Case 1:**

\[
a_0 = 0, B = 0, b_1 = 0, A \neq 0, b_2 = \frac{Ch_0}{Aa_1}, k = \frac{i h_0}{Aa_1}, w = \frac{a_1^2}{b_2}, a_1 \neq 0, b_2 \neq 0.
\]

**Case 2:**

\[
a_0 = 0, B = 0, b_1 = 0, A = 0, b_2 \neq 0, k = \frac{i h_2}{Ca_1}, w = \frac{a_1^2}{b_2}, b_0 = 0, C \neq 0.
\]

After these procedures, the solutions:

**Solution 1:**

\[
\begin{align*}
u_1(x, t) &= a_1 \left[ \text{Coth} \left( \frac{2ib_0}{a_1} x + 2ia_1 t \right) \pm \text{Cosech} \left( \frac{2ib_0}{a_1} x + 2ia_1 t \right) \right] \\
v_1(x, t) &= b_0 - b_0 \left[ \text{Coth} \left( \frac{2ib_0}{a_1} x + 2ia_1 t \right) \pm \text{Cosech} \left( \frac{2ib_0}{a_1} x + 2ia_1 t \right) \right]^2 \\
u_2(x, t) &= a_1 \left[ \text{Tanh} \left( \frac{2ib_0}{a_1} x + 2ia_1 t \right) \pm \text{iSech} \left( \frac{2ib_0}{a_1} x + 2ia_1 t \right) \right]
\end{align*}
\]
\[ v_2(x,t) = b_0 - b_0 \left[ \tanh \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \pm is\text{eh} \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \right]^2 \]

**Solution 2:**

\[ u_3(x,t) = a_1 \left[ \sec \left( \frac{2ib_0}{a_1}x - 2ia_1t \right) \pm \tan \left( \frac{2ib_0}{a_1}x - 2ia_1t \right) \right] \]

\[ v_3(x,t) = b_0 + b_0 \left[ \sec \left( \frac{2ib_0}{a_1}x - 2ia_1t \right) \pm \tan \left( \frac{2ib_0}{a_1}x - 2ia_1t \right) \right]^2 \]

\[ u_4(x,t) = a_1 \left[ \csc \left( \frac{2ib_0}{a_1}x - 2ia_1t \right) \pm \cot \left( \frac{2ib_0}{a_1}x - 2ia_1t \right) \right] \]

\[ v_4(x,t) = b_0 + b_0 \left[ \csc \left( \frac{2ib_0}{a_1}x - 2ia_1t \right) \pm \cot \left( \frac{2ib_0}{a_1}x - 2ia_1t \right) \right]^2 \]

\[ u_5(x,t) = a_1 \left[ \sec \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \pm \tan \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \right] \]

\[ v_5(x,t) = b_0 + b_0 \left[ \sec \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \pm \tan \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \right]^2 \]

\[ u_6(x,t) = a_1 \left[ \csc \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \pm \cot \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \right] \]

\[ v_6(x,t) = b_0 + b_0 \left[ \csc \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \pm \cot \left( \frac{2ib_0}{a_1}x + 2ia_1t \right) \right]^2 \]

**Solution 3:**

\[ u_7(x,t) = a_1 \left[ \tanh \left( \frac{ib_0}{a_1}x + ia_1t \right) \right] \]

\[ v_7(x,t) = b_0 - b_0 \left[ \tanh \left( \frac{ib_0}{a_1}x + ia_1t \right) \right]^2 \]

\[ u_8(x,t) = a_1 \left[ \coth \left( \frac{ib_0}{a_1}x + ia_1t \right) \right] \]

\[ v_8(x,t) = b_0 - b_0 \left[ \coth \left( \frac{ib_0}{a_1}x + ia_1t \right) \right]^2 \]

**Solution 4:**

\[ u_9(x,t) = a_1 \left[ \tan \left( \frac{ib_0}{a_1}x - ia_1t \right) \right] \]

\[ v_9(x,t) = b_0 + b_0 \left[ \tan \left( \frac{ib_0}{a_1}x - ia_1t \right) \right]^2 \]

**Solution 5:**

\[ u_{10}(x,t) = a_1 \left[ \cot \left( \frac{ib_0}{a_1}x + ia_1t \right) \right] \]
\[ v_{10}(x,t) = b_0 + b_0 \left[ \text{Cot} \left( -\frac{ib_0}{a_1}x + ia_1t \right) \right]^2 \]

**Solution 6:**

\[ u_{11}(x,t) = -\frac{a_1}{\left( \frac{du}{a_1}x - ia_1t \right) + c_0} \]

\[ v_{11}(x,t) = b_2 \left( -\frac{1}{\left( \frac{du}{a_1}x - ia_1t \right) + c_0} \right)^2 \]

**Example 2.2.** Now let’s get the extended (3+1)-dimensional KdV-type equation,

\[ u_t + 6u_x u_y + u_{xxx} + 6u_t u_z + 10u_x u_{ty} + 20u_z u_{tx} + 6u_t u_z + u_{xxx} = 0, \tag{2.6} \]

Using the wave variable \( u(x,y,z,t) = u(\varphi) \) and, \( \varphi = k(x + \alpha y + \beta z - wt) \), the equation (2.6) turns into an ordinary differential equation,

\[ -wu'' + 6ka(u')^2 + k^2 a^2 u'' + k^2 a^3 u(5) + 60k^2 a^2 u'' + 6ka u'' + 6k^2 a u'' = 0, \tag{2.7} \]

When balancing \( u^{(5)} \) with \( u u'' \) then \( n = 1 \) gives. The solution is as follows:

\[ u = a_0 + a_1 \mathcal{F}(\varphi) \tag{2.8} \]

If (2.8) is substituted in equation (2.7), a system of algebraic equations for \( k, w, \alpha, \beta, a_0, a_1 \) can be obtained. The obtained systems of algebraic equations are as follows

\[ \begin{align*}
-\text{A}a_0 + \text{A}^2 a_1 + 2A^2 C_2 a_1 + A^2 C^2 a_1 + C^2 a_1 + A B^2 C^2 a_1 + C^2 a_1 & = 0, \\
+16A^2 C^2 a_1 + 6A^2 C a_1 + 6A^2 C a_1 + 30A^2 C^2 a_1 + 60A^2 C^2 a_1 + 60A^2 C^2 a_1 & = 0, \\
-\text{B} a_1 + 8A^2 C^2 a_1 + 8A B C^2 a_1 + 8A B C^2 a_1 + B^2 a_1 + 52A B C^2 a_1 + 3A B^2 C^2 a_1 + 136A^2 B C^2 a_1 & = 0, \\
+12A B C a_1 + 12A B C a_1 + 60A B C a_1 + 300A^2 B C^2 a_1 + 180A^2 B C^2 a_1 & = 0, \\
-\text{C} a_1 + 7B^2 C^2 a_1 + 8A C^2 a_1 + 7B^2 C^2 a_1 + 8A C^2 a_1 + 8A C^2 a_1 + 31B^2 C^2 a_1 & = 0, \\
+292A B^2 C^2 a_1 + 136A^2 C^2 a_1 + 6B^2 C a_1 + 12A C a_1 + 6B^2 C a_1 + 12A C a_1 + 30B^2 C a_1 + 480A B^2 C a_1 & = 0, \\
+30B^2 C a_1 + 300A^2 C^2 a_1 + 180A B^2 C^2 a_1 + 180A^2 B C^2 a_1 & = 0,
\end{align*} \]

If the system is solved, the coefficients are found as

\[ B = 0, \quad a_1 = \frac{1}{2} \sqrt{\frac{C}{A}}, \quad a_1 \neq 0, \quad A \neq 0, \quad k = -\frac{1}{4A a_1}, \quad \alpha = -w, \quad \beta \neq 0. \]

with the help of the Mathematica program. After these operations, the solutions of equation (2.6) as follow:

**Solution 1:**

\[ u_1(x,t) = \frac{i}{2} \left[ \text{Coth} \left( ix - iwy + i\beta z - iwt \right) \pm \text{Cosech} \left( ix - iwy + i\beta z - iwt \right) \right] \]

\[ u_2(x,t) = \frac{i}{2} \left[ \text{Tanh} \left( ix - iwy + i\beta z - iwt \right) \pm i\text{Sech} \left( ix - iwy + i\beta z - iwt \right) \right] \]

**Solution 2:**

\[ u_3(x,t) = \frac{1}{2} \left[ \text{Sec} \left( -x + wy - \beta z + wt \right) + \text{Tan} \left( -x + wy - \beta z + wt \right) \right] \tag{2.9} \]

\[ u_4(x,t) = \frac{1}{2} \left[ \text{Cosec} \left( -x + wy - \beta z + wt \right) - \text{Cot} \left( -x + wy - \beta z + wt \right) \right] \]

\[ u_5(x,t) = \frac{1}{2} \left[ \text{Sec} \left( x + wy + \beta z - wt \right) - \text{Tan} \left( x + wy + \beta z - wt \right) \right] \tag{2.10} \]
\[ u_6(x,t) = \frac{1}{2} \left[ \text{Cosec} \left( x - wy + \beta z - wt \right) + \text{Cot} \left( x - wy + \beta z - wt \right) \right] \] 

(2.11)

**Solution 3:**

\[ u_7(x,t) = \frac{i}{2} \left[ \text{Tanh} \left( \frac{i}{2} x - \frac{i}{2} wy + \frac{i}{2} \beta z - \frac{i}{2} wt \right) \right] \] 

(2.12)

\[ u_8(x,t) = \frac{i}{2} \left[ \text{Coth} \left( \frac{i}{2} x - \frac{i}{2} wy + \frac{i}{2} \beta z - \frac{i}{2} wt \right) \right] \] 

(2.13)

**Solution 4:**

\[ u_9(x,t) = \frac{1}{2} \left[ \text{Tan} \left( \frac{1}{2} x + \frac{1}{2} wy - \frac{1}{2} \beta z + \frac{1}{2} wt \right) \right] \] 

(2.14)

**Solution 5:**

\[ u_{10}(x,t) = \frac{1}{2} \left[ \text{Cot} \left( \frac{1}{2} x - \frac{1}{2} wy + \frac{1}{2} \beta z - \frac{1}{2} wt \right) \right] \] 

(2.15)

3. **Explanations and graphical presentments of some of the solutions obtained**

The graphical demonstrations of some obtained solutions are shown in Figures 1-3.

![Graphical Demonstrations](image.png)

**Figure 3.1:** a) The 3D surfaces of Eq. (2.9) for the values \( y = 1, \ z = 0 \) and \( w = 5 \) within the interval \(-5 \leq x \leq 5, \ -1 \leq t \leq 1\). b) The 2D surfaces of Eq. (2.9) for the values \( y = 1, \ z = 0, \ w = 5 \) and \( t = 1 \) within the interval \(-5 \leq x \leq 5\).
Figure 3.2: a) The 3D surfaces of Eq.(2.14) for the values $y = 1$, $z = 0$, and $w = 5$ within the interval $-5 \leq x \leq 5$, $-1 \leq t \leq 1$. b) The 2D surfaces of Eq.(2.14) for the values $y = 1$, $z = 0$, $w = 5$, and $t = 1$ within the interval $-5 \leq x \leq 5$.

Figure 3.3: a) The 3D surfaces of Eq.(2.15) for the values and within the interval b) The 2D surfaces of Eq.(2.15) for the values and within the interval

4. Conclusion

We used the improved tanh function method to find the multiple soliton solutions of new coupled Konno-Oono equation and extended (3+1)-dimensional KdV-type equation. This method has been successfully applied to solve some nonlinear wave equations and can be used to many other nonlinear equations or coupled ones.

References
