Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (1) (2020), 8 – 18
DOI : 10.15672/HJMS.2018.652

Research Article

Some identities on fractional integrals and integral transforms

A. Neşe Dernek*, Gülçin Bozkurt

Department of Mathematics, Marmara University, 34722 Kadıköy, İstanbul, Turkey

Abstract

In this paper, we introduce various theorems that associate the generalized Riemann-Liouville fractional integral operator and the generalized Weyl fractional integral operator with some well-known integral transforms including generalized Laplace transform, Widder potential transform, generalized Widder transform, Hankel transform and Bessel transform. We evaluate certain integrals of some elementary functions and some special functions as applications of these theorems and their results.

Mathematics Subject Classification (2010). 26A33, 44A10, 44A15, 33C10

Keywords. Riemann-Liouville fractional integral, Weyl fractional integral, fractional derivatives, Laplace transform, Widder potential transform, Hankel transform, Bessel transform, error function, exponential integral function

1. Introduction

The name fractional calculus is used for integrals and derivatives of arbitrary order. The idea of the fractional calculus first arises in the 17^{th} century and rapidly improves from that day on. In the last two decades, in addition to theoretical development, fractional integrals and derivatives are widely used in applied science and engineering studies such as viscoelastic systems, signal processing, control processing, fractional stochastic systems, and ecology [10–12].

Unlike the classical calculus, there are various definitions of fractional order integrals and fractional order derivatives. In this paper, we give place to two of the main definitions of fractional integrals which are Riemann-Liouville fractional integral and Weyl fractional integral. Also, we define two new fractional integrals motivated from these definitions and we present some new identities that include fractional integrals, fractional derivatives and some integral transforms such as Laplace transform, $\mathcal{L}_2$ transform, Widder potential transform and Bessel transform.

*Corresponding Author.
Email addresses: ndernek@marmara.edu.tr (A.N. Dernek), gulcinbozkurt1@gmail.com (G. Bozkurt)
Received: 11.06.2018; Accepted: 28.08.2018

2. Preliminaries

Before giving the main results, we introduce some definitions about fractional integrals, fractional derivatives and integral transforms.

Definition 2.1. Weyl fractional integral operator of order μ is defined in the form

$$W^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_y^\infty (x-y)^{\mu-1} f(x) dx, \tag{2.1}$$

where $y \geq 0$, $\mu \in \mathbb{C}$ and $\mathrm{Re}(\mu) > 0$ [11].

Definition 2.2. Riemann-Liouville fractional integral operator of order μ is defined in the form

$$D^{-\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_0^y (y-x)^{\mu-1} f(x) dx, \tag{2.2}$$

where $y \geq 0$ and, $\mu \in \mathbb{C}$ and $\mathrm{Re}(\mu) > 0$ [11, 14].

In the following, we introduce two new fractional integrals namely the generalized Weyl fractional integral and the generalized Riemann-Liouville fractional integral.

Definition 2.3. The generalized Weyl fractional integral can be defined as follows:

$$W_{\mu,2}\{f(x); y\} = \frac{1}{\Gamma(\mu)} \int_y^\infty x(x^2-y^2)^{\mu-1} f(x) dx, \tag{2.3}$$

where $y \geq 0$ and, $\mu \in \mathbb{C}$ and $\mathrm{Re}(\mu) > 0$.

Definition 2.4. The generalized Riemann-Liouville fractional integral is defined as follows:

$$R_{\mu,2}\{f(x); y\} = \frac{1}{\Gamma(\mu)} \int_0^y x(y^2-x^2)^{\mu-1} f(x) dx, \tag{2.4}$$

where $y \geq 0$, $\mu \in \mathbb{C}$ and $\mathrm{Re}(\mu) > 0$.

In the definitions (2.1) – (2.4), $\Gamma(z)$ is the Gamma Euler function given by the following integral,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{2.5}$$

where $z \in \mathbb{C}$ and $\mathrm{Re}(z) > 0$.

Definition 2.5. Weyl fractional derivative of order α is defined by the formula

$$W^\alpha f(x) = \frac{d^n}{dx^n} W^{-(n-\alpha)} f(x), \tag{2.6}$$

where $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ and $n-1 < \mathrm{Re}(\alpha) \leq n$ [11].

Definition 2.6. Riemann-Liouville fractional derivative of order α is defined by the formula

$$_0D_y^\alpha f(x) = \frac{d^n}{dx^n} D^{-(n-\alpha)} f(x), \tag{2.7}$$

where $n \in \mathbb{N}$, $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ and $n-1 < \mathrm{Re}(\alpha) \leq n$ [11].

In the formulas (2.6) and (2.7), fractional derivatives are defined by means of fractional integral operators.

Definition 2.7. The Laplace transform of $f(x)$ is defined by the following formula [1],

$$\mathcal{L}\{f(x); y\} = \int_0^\infty e^{-xy} f(x) dx. \tag{2.8}$$

Definition 2.8. The $\mathcal{L}_2$ transform of $f(x)$ is defined by the following formula [14, 15],

$$\mathcal{L}_2\{f(x); y\} = \int_0^\infty x e^{-x^2 y^2} f(x) dx. \tag{2.9}$$

The $\mathcal{L}_2$ transform and the Laplace transform are related by the equation [3],

$$\mathcal{L}_2\{f(x);y\} = \frac{1}{2}\mathcal{L}\{f(x^{1/2});y^2\}. \tag{2.10}$$

Definition 2.9. The Widder Potential transform of $f(x)$ is defined by [13],

$$\mathcal{P}\{f(x);y\} = \int_0^\infty \frac{xf(x)}{x^2+y^2}dx. \tag{2.11}$$

Definition 2.10. The Stieltjes integral transform is defined by [1,8],

$$\mathcal{S}\{f(x);y\} = \int_0^\infty \frac{f(x)}{x+y}dx. \tag{2.12}$$

The Widder Potential transform and the Stieltjes transform are related by the following relation [7],

$$\mathcal{P}\{f(x);y\} = \frac{1}{2}\mathcal{S}\{f(x^{1/2});y^2\}. \tag{2.13}$$

Definition 2.11. The 2n-Generalized Widder potential transform of $f(x)$ is defined by [2,3],

$$\mathcal{P}_{2n}\{f(x);y\} = \int_0^\infty x^{2n-1}\frac{f(x)}{x^{2n}+y^{2n}}dx, \quad \text{where } n \in \mathbb{N}. \tag{2.14}$$

The $\mathcal{P}_4$ transform, which is the special case of 2n-Generalized Widder potential transform that appears with the choice of $n = 2$, and the Steiltjes transform are related by the following relation [6],

$$\mathcal{P}_4\{f(x);y\} = \frac{1}{4}\mathcal{S}\left\{f(x^{1/4});y^4\right\}. \tag{2.15}$$

Definition 2.12. The generalized Widder potential transform of $f(x)$ is defined by [5],

$$\mathcal{P}_{\nu,2}\{f(x);y\} = \int_0^\infty \frac{xf(x)}{(x^2+y^2)^\nu}dx, \quad \text{where } \nu \in \mathbb{C}. \tag{2.16}$$

The Widder potential transform is the special case of generalized Widder potential transform that appears with the choice of $\nu = 1$.

Definition 2.13. The Hankel transform of $f(x)$ is defined by [4],

$$\mathcal{H}_\nu\{f(x);y\} = \int_0^\infty (xy)^{1/2}J_v(xy)f(x)dx, \tag{2.17}$$

where $J_v(x)$ is the Bessel function of the first kind that has the following series representation,

$$J_\nu(x) = \sum_{n=0}^\infty (-1)^n\left(\frac{x}{2}\right)^{\nu+2n}\frac{1}{n!\Gamma(\nu+n+1)}, \quad \text{where } \mathrm{Re}(\nu) > -(n+1). \tag{2.18}$$

Definition 2.14. The Bessel transform of $f(x)$ is defined by the identity [4],

$$\mathcal{K}_\nu\{f(x);y\} = \int_0^\infty (xy)^{1/2}K_v(xy)f(x)dx \tag{2.19}$$

where $K_v(x)$ is the modified Bessel function of the second kind and defined as

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{sin(\pi\nu)} \tag{2.20}$$

in which $I_\nu(x)$ is the modified Bessel function of the first kind and defined by the series representation

$$I_\nu(x) = \sum_{n=0}^\infty \left(\frac{x}{2}\right)^{\nu+2n}\frac{1}{n!\Gamma(\nu+n+1)}, \quad \text{where } \mathrm{Re}(\nu) > -(n+1). \tag{2.21}$$

Definition 2.15. The error function $\mathrm{Erf}(x)$ is defined by the equation [9],

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{2.22}$$

Definition 2.16. The complementary error function $\mathrm{Erfc}(x)$ is defined by the equation [9],

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \tag{2.23}$$

3. Identities on generalized fractional integrals and the $\mathcal{L}_2$ transform

By the following lemma, we give a relation between the generalized Weyl fractional integral and the generalized Laplace transform.

Lemma 3.1. *If* $\mathrm{Re}(\mu) > 0$ *for* $\mu \in \mathbb{C}$, *the following identity holds true, provided that the integrals involved converge absolutely,*

$$W_{\mu,2}\{\mathcal{L}_2\{f(x); u\}; y\} = \frac{1}{2}\mathcal{L}_2\left\{\frac{f(x)}{x^{2\mu}}; y\right\}. \tag{3.1}$$

Proof. By definitions (2.3) and (2.9), we have

$$W_{\mu,2}\{\mathcal{L}_2\{f(x); u\}; y\} = \frac{1}{\Gamma(\mu)} \int_y^\infty u(u^2 - y^2)^{\mu-1} \left[\int_0^\infty x e^{-u^2x^2} f(x)dx\right] du. \tag{3.2}$$

By making the change of variable $u^2 - y^2 = t^2$ in (3.2), we get that

$$W_{\mu,2}\{\mathcal{L}_2\{f(x); u\}; y\} = \frac{1}{\Gamma(\mu)} \int_0^\infty t(t^2)^{\mu-1} \left[\int_0^\infty x e^{-(t^2+y^2)x^2} f(x)dx\right] dt.$$

Then, by changing the order of integration that is permissible by absolute convergence of the integrals involved, we get

$$\begin{aligned} W_{\mu,2}\{\mathcal{L}_2\{f(x); u\}; y\} &= \frac{1}{\Gamma(\mu)} \int_0^\infty x e^{-x^2y^2} f(x) \left[\int_0^\infty t t^{2(\mu-1)} e^{-t^2x^2} dt\right] dx \\ &= \frac{1}{\Gamma(\mu)} \int_0^\infty x e^{-x^2y^2} f(x) \left[\mathcal{L}_2\{t^{2(\mu-1)}; x\}\right] dx. \end{aligned}$$

Applying the relation (2.10) and using the equation (1) of [7, p. 137], we obtain

$$\begin{aligned} W_{\mu,2}\{\mathcal{L}_2\{f(x); u\}; y\} &= \frac{1}{\Gamma(\mu)} \int_0^\infty x e^{-x^2y^2} f(x) \left[\frac{1}{2}\mathcal{L}\{t^{\mu-1}; x^2\}\right] dx \\ &= \frac{1}{\Gamma(\mu)} \int_0^\infty x e^{-x^2y^2} f(x) \left[\frac{1}{2}\Gamma(\mu)x^{-2\mu}\right] dx \\ &= \frac{1}{2} \int_0^\infty x e^{-x^2y^2} \frac{f(x)}{x^{2\mu}} dx = \frac{1}{2}\mathcal{L}_2\left\{\frac{f(x)}{x^{2\mu}}; y\right\} \end{aligned}$$

which proves the statement. □

Next theorem gives a Parseval-Goldstein type relation between the $\mathcal{L}_2$ transform and the generalized Riemann-Liouville fractional integral defined by (2.4).

Theorem 3.2. *If* $\mathrm{Re}(\mu) > 0$ *for* $\mu \in \mathbb{C}$, *the following Parseval-Goldstein type relation holds true, provided that the integrals involved converge absolutely,*

$$\int_0^\infty t\mathcal{R}_{\mu,2}\{f(x); t\}\, \mathcal{L}_2\{g(u); t\}\, dt = \frac{(-1)^{\mu-1}}{2} \int_0^\infty x f(x)\mathcal{L}_2\left\{\frac{g(u)}{u^{2\mu}}; x\right\} dx. \tag{3.3}$$

Proof. By the definition (2.4), we have

$$I = \int_0^\infty t\mathcal{R}_{\mu,2}\{f(x);t\}\,\mathcal{L}_2\{g(u);t\}\,dt$$
$$= \int_0^\infty t\left[\frac{1}{\Gamma(\mu)}\int_0^t x(t^2-x^2)^{\mu-1}f(x)dx\right]\mathcal{L}_2\{g(u);t\}\,dt. \tag{3.4}$$

By changing the order of integration that is permissible under the absolute convergence assumption, we obtain

$$I = \frac{1}{\Gamma(\mu)}\int_0^\infty xf(x)\left[\int_x^\infty t(t^2-x^2)^{\mu-1}\mathcal{L}_2\{g(u);t\}\,dt\right]dx$$
$$= \frac{(-1)^{\mu-1}}{\Gamma(\mu)}\int_0^\infty xf(x)\left[\int_x^\infty t(x^2-t^2)^{\mu-1}\mathcal{L}_2\{g(u);t\}\,dt\right]dx.$$

Then, by using the definition (2.3) and Lemma (3.1), we get

$$I = (-1)^{\mu-1}\int_0^\infty xf(x)W_{\mu,2}\{\mathcal{L}_2\{g(u);t\};x\}\,dx$$
$$= \frac{(-1)^{\mu-1}}{2}\int_0^\infty xf(x)\mathcal{L}_2\left\{\frac{g(u)}{u^{2\mu}};x\right\}dx$$

which proves the statement. □

Theorem 3.3. *The following Parseval-Goldstein type relation holds true, provided that each of the integrals involved converge absolutely,*

$$\mathcal{L}_2\left\{\frac{1}{u}\mathcal{R}_{1,2}\left\{f(x);\frac{1}{u}\right\};y\right\} = \frac{\sqrt{\pi}}{2y}\int_0^\infty xf(x)\,\mathrm{Erf}\left(\frac{y}{x}\right)dx. \tag{3.5}$$

Proof. By setting $\mu = 1$ and $g(u) = \sin(2uy)$ in (3.3), we obtain

$$\int_0^\infty t\mathcal{R}_{1,2}\{f(x);t\}\,\mathcal{L}_2\{\sin(2uy);t\}\,dt = \frac{1}{2}\int_0^\infty xf(x)\mathcal{L}_2\left\{\frac{\sin(2uy)}{u^2};x\right\}dx. \tag{3.6}$$

Now, by the relation (2.10) and the relation (32) of [7, p.153], we have

$$\mathcal{L}_2\{\sin(2uy);t\} = \frac{1}{2}\mathcal{L}\left\{\sin(2\sqrt{u}y);t^2\right\}$$
$$= \frac{1}{2}\sqrt{\pi}yt^{-3}e^{-y^2/t^2},\ \mathrm{Re}(t^2) > 0. \tag{3.7}$$

Likewise, by the relation (2.10) and the relation (34) of [7, p.154], we have

$$\mathcal{L}_2\left\{\frac{\sin(2uy)}{u^2};t\right\} = \frac{1}{2}\mathcal{L}\left\{\frac{\sin(2\sqrt{u}y)}{u};t^2\right\}$$
$$= \frac{1}{2}\pi\,\mathrm{Erf}\left(\frac{y}{x}\right),\ \mathrm{Re}(t^2) > 0. \tag{3.8}$$

Then, by substituting (3.7) and (3.8) in (3.6), we get

$$\int_0^\infty t\mathcal{R}_{1,2}\{f(x);t\}\left(\frac{1}{2}\sqrt{\pi}yt^{-3}e^{-y^2/t^2}\right)dt = \frac{1}{2}\int_0^\infty xf(x)\left(\frac{1}{2}\pi\,\mathrm{Erf}\left(\frac{y}{x}\right)\right)dx$$
$$\int_0^\infty t\mathcal{R}_{1,2}\{f(x);t\}\,t^{-3}e^{-y^2/t^2}dt = \frac{\sqrt{\pi}}{2y}\int_0^\infty xf(x)\,\mathrm{Erf}\left(\frac{y}{x}\right)dx.$$

Finally, by making change of variable $t = 1/u$ on the left hand side of equation above and using the definition (2.9), we obtain

$$\int_0^\infty \frac{1}{u}\mathcal{R}_{1,2}\left\{f(x);\frac{1}{u}\right\} ue^{-y^2u^2}du = \frac{\sqrt{\pi}}{2y}\int_0^\infty xf(x)\,\mathrm{Erf}\left(\frac{y}{x}\right)dx$$

$$\mathcal{L}_2\left\{\frac{1}{u}\mathcal{R}_{1,2}\left\{f(x);\frac{1}{u}\right\};y\right\} = \frac{\sqrt{\pi}}{2y}\int_0^\infty xf(x)\,\mathrm{Erf}\left(\frac{y}{x}\right)dx,$$

as desired. □

Example 3.4. Following equation holds true under the conditions of Theorem (3.3),

$$\int_0^\infty t^{\nu-2}\,\mathrm{Erf}(t)dt = \frac{1}{\sqrt{\pi}(1-\nu)}\Gamma\left(\frac{\nu}{2}\right), \quad \text{where } 0 < \mathrm{Re}(\nu) < 1. \tag{3.9}$$

Proof. By choosing $f(x) = x^{-\nu-1}$ in Theorem (3.3), we get

$$\mathcal{L}_2\left\{\frac{1}{u}\mathcal{R}_{1,2}\left\{x^{-\nu-1};\frac{1}{u}\right\};y\right\} = \frac{\sqrt{\pi}}{2y}\int_0^\infty x^{-\nu}\,\mathrm{Erf}\left(\frac{y}{x}\right)dx. \tag{3.10}$$

Now, by using the definitions (2.4) and (2.9) on the left hand side of the equation (3.10), we get

$$\begin{aligned}\mathcal{L}_2\left\{\frac{1}{u}\mathcal{R}_{1,2}\left\{x^{-\nu-1};\frac{1}{u}\right\};y\right\} &= \mathcal{L}_2\left\{\frac{1}{u}\left(\int_0^{1/u} x^{-\nu}dx\right);y\right\} \\ &= \mathcal{L}_2\left\{\frac{1}{u}\left(\frac{u^{\nu-1}}{1-\nu}\right);y\right\} \\ &= \frac{1}{(1-\nu)}\mathcal{L}_2\left\{u^{\nu-2};y\right\}.\end{aligned}$$

Then, by using the relation (2.10) and the equation (1) of [7, p.137], we have

$$\begin{aligned}\mathcal{L}_2\left\{\frac{1}{u}\mathcal{R}_{1,2}\left\{x^{-\nu-1};\frac{1}{u}\right\};y\right\} &= \frac{1}{2(1-\nu)}\mathcal{L}\left\{u^{\nu/2-1};y^2\right\} \\ &= \frac{1}{2(1-\nu)}\Gamma\left(\frac{\nu}{2}\right)y^{-\nu}\end{aligned} \tag{3.11}$$

and by making change of variable $x = yt^{-1}$ on the right hand side of the equation (3.10), we get

$$\begin{aligned}\frac{\sqrt{\pi}}{2y}\int_0^\infty x^{-\nu}\,\mathrm{Erf}\left(\frac{y}{x}\right)dx &= \frac{\sqrt{\pi}}{2y}\int_0^\infty (yt^{-1})^{-\nu}\,\mathrm{Erf}(t)(yt^{-2})\,dt \\ &= \frac{\sqrt{\pi}y^{-\nu}}{2}\int_0^\infty t^{\nu-2}\,\mathrm{Erf}(t)\,dt.\end{aligned} \tag{3.12}$$

From the equations (3.11) and (3.12) we have,

$$\frac{1}{2(1-\nu)}\Gamma\left(\frac{\nu}{2}\right)y^{-\nu} = \frac{\sqrt{\pi}y^{-\nu}}{2}\int_0^\infty t^{\nu-2}\,\mathrm{Erf}(t)\,dt$$

which proves the statement. □

4. An identity on fractional integrals and the $2n$-generalized widder potential transform

Theorem 4.1. *If* $\mathrm{Re}(t^2) > |Im(y^2)|$, *then the identity*

$$\mathcal{P}_4\left\{\frac{1}{t^2}\mathcal{R}_{1,2}\left\{f(x);t\right\};y\right\} = \frac{1}{2y^2}\int_0^\infty xf(x)\tan^{-1}\left(\frac{y^2}{x^2}\right)dx \tag{4.1}$$

holds true, provided that each of the integrals involved converges absolutely.

Proof. By choosing $\mu = 1$ and $g(u) = \sin(u^2y^2)$ in the equation (3.3), we obtain

$$\int_0^\infty t\mathcal{R}_{1,2}\left\{f(x);t\right\}\mathcal{L}_2\left\{\sin(u^2y^2);t\right\}dt \tag{4.2}$$
$$= \frac{1}{2}\int_0^\infty xf(x)\mathcal{L}_2\left\{\frac{\sin(u^2y^2)}{u^2};x\right\}dx.$$

Now, by the relation (2.10) and the formula (1) of [7, p.150], we have

$$\mathcal{L}_2\left\{\sin(u^2y^2);t\right\} = \frac{1}{2}\mathcal{L}\left\{\sin(uy^2);t^2\right\} \tag{4.3}$$
$$= \frac{1}{2}\frac{y^2}{y^4+t^4},\ \mathrm{Re}(t^2) > |Im(y^2)|.$$

Likewise, by using the relation (2.10) and the equation (16) of [7, p.152], we have

$$\mathcal{L}_2\left\{\frac{\sin(u^2y^2)}{u^2};x\right\} = \frac{1}{2}\mathcal{L}\left\{\frac{\sin(uy^2)}{u};x^2\right\} \tag{4.4}$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{y^2}{x^2}\right).$$

By substituting (4.3) and (4.4) in (4.2) and using definition (2.14), we get

$$\int_0^\infty \frac{t}{y^4+t^4}\mathcal{R}_{1,2}\left\{f(x);t\right\}dt = \frac{1}{2y^2}\int_0^\infty xf(x)\tan^{-1}\left(\frac{y^2}{x^2}\right)dx$$
$$\mathcal{P}_4\left\{\frac{1}{t^2}\mathcal{R}_{1,2}\left\{f(x);t\right\};y\right\} = \frac{1}{2y^2}\int_0^\infty xf(x)\tan^{-1}\left(\frac{y^2}{x^2}\right)dx$$

□

Example 4.2. If $-3 < \mathrm{Re}(\nu) < 1$, then the following equation holds true under the conditions of Theorem (4.1),

$$\int_0^\infty x^\nu\tan^{-1}\left(\frac{y^2}{x^2}\right)dx = \frac{-\pi y^{\nu+1}}{2(\nu+1)}\csc\left(\frac{\pi(\nu-1)}{4}\right). \tag{4.5}$$

Proof. If we choose $f(x) = x^{\nu-1}$ on the left hand side of equation (4.1), we get

$$\mathcal{P}_4\left\{\frac{1}{t^2}\mathcal{R}_{1,2}\left\{x^{\nu-1};t\right\};y\right\} = \mathcal{P}_4\left\{\frac{1}{t^2}\left(\int_0^t x^\nu dx\right);y\right\}.$$

Then, by using the definitions (2.4) and (2.14) respectively, we obtain

$$\mathcal{P}_4\left\{\frac{1}{t^2}\mathcal{R}_{1,2}\left\{x^{\nu-1};t\right\};y\right\} = \mathcal{P}_4\left\{\frac{1}{t^2}\left(\frac{t^{\nu+1}}{\nu+1}\right);y\right\} \tag{4.6}$$
$$= \frac{1}{\nu+1}\mathcal{P}_4\left\{t^{\nu-1};y\right\},$$

and by using the identity (2.15) and the equation (5) of [8, p.216], we get

$$\mathcal{P}_4\left\{\frac{1}{t^2}\mathcal{R}_{1,2}\left\{x^{\nu-1};t\right\};y\right\} = \frac{1}{4(\nu+1)}\mathcal{S}\left\{t^{(\nu-1)/4};y^4\right\}$$
$$= \frac{-\pi y^{\nu-1}}{4(\nu+1)}\csc\left(\frac{\pi(\nu-1)}{4}\right). \tag{4.7}$$

Finally, by substituting (4.7) in equation (4.1), we obtain

$$\frac{-\pi y^{\nu-1}}{4(\nu+1)}\csc\left(\frac{\pi(\nu-1)}{4}\right) = \frac{1}{2y^2}\int_0^\infty x^\nu \tan^{-1}\left(\frac{y^2}{x^2}\right)dx,$$

which proves the statement. □

5. Identities for various integral transforms and fractional integrals

Lemma 5.1. *If* $\mathrm{Re}(\mu) < 1$, $\mathrm{Re}(t^2) > -\mathrm{Re}(s^2)$ *and* $\mathrm{Re}(x^2) > -\mathrm{Re}(s^2)$, *then the identity*

$$\mathcal{P}\left\{\mathcal{R}_{\mu,2}\left\{f(x);t\right\};s\right\} = \frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\mathcal{P}_{1-\mu,2}\left\{f(x);s\right\}, \tag{5.1}$$

holds true, provided that each of the integrals involved converges absolutely.

Proof. By choosing $g(u) = e^{-s^2u^2}$ in the equation (3.3), we get

$$\int_0^\infty t\mathcal{R}_{\mu,2}\left\{f(x);t\right\}\mathcal{L}_2\left\{e^{-s^2u^2};t\right\}dt \tag{5.2}$$
$$= \frac{(-1)^{\mu-1}}{2}\int_0^\infty xf(x)\mathcal{L}_2\left\{\frac{e^{-s^2u^2}}{u^{2\mu}};x\right\}dx.$$

Now by (2.10) and the relation (1) of [7, p.143], we have

$$\mathcal{L}_2\left\{e^{-s^2u^2};t\right\} = \frac{1}{2}\mathcal{L}\left\{e^{-s^2u};t^2\right\} \tag{5.3}$$
$$= \frac{1}{2}(t^2+s^2)^{-1},\ \mathrm{Re}(t^2) > -\mathrm{Re}(s^2).$$

Likewise, by (2.10) and the relation (3) of [7, p.144], we have

$$\mathcal{L}_2\left\{\frac{e^{-s^2u^2}}{u^{2\mu}};x\right\} = \frac{1}{2}\mathcal{L}\left\{\frac{e^{-s^2u}}{u^\mu};x^2\right\} \tag{5.4}$$
$$= \frac{1}{2}\Gamma(1-\mu)(x^2+s^2)^{\mu-1},\ \mathrm{Re}(x^2) > -\mathrm{Re}(s^2).$$

By substituting results (5.3) and (5.4) into the equation (5.2), we get

$$\int_0^\infty t\mathcal{R}_{\mu,2}\left\{f(x);t\right\}(t^2+s^2)^{-1}dt \tag{5.5}$$
$$= \frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\int_0^\infty xf(x)(x^2+s^2)^{\mu-1}dx.$$

Then, the equation (5.1) can be shown directly by using the definitions (2.11), (2.16) and the relation (5.5) . □

Theorem 5.2. *If* $\mathrm{Re}(\mu) < 1$, $\mathrm{Re}(t^2) > -\mathrm{Re}(s^2)$ *and* $\mathrm{Re}(x^2) > -\mathrm{Re}(s^2)$ *for* $\mu, s, t \in \mathbb{C}$, *then the Parseval-Goldstein type identity*

$$\int_0^\infty t\mathcal{P}\left\{f(x);t\right\}\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt \tag{5.6}$$
$$= \frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\int_0^\infty xf(x)\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\}dx$$

holds true, provided that each of the integrals involved converges absolutely.

Proof. Firstly, if we apply the definition (2.11) on the left-hand side of equation (5.6) and then change the order of integration under the absolute convergence condition, we obtain

$$\begin{aligned}\int_0^\infty t\mathcal{P}\left\{f(x);t\right\}\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt \\ &= \int_0^\infty t\int_0^\infty \frac{xf(x)}{x^2+t^2}\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dx\,dt \\ &= \int_0^\infty xf(x)\int_0^\infty \frac{t}{x^2+t^2}\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt\,dx \\ &= \int_0^\infty xf(x)\mathcal{P}\left\{\mathcal{R}_{\mu,2}\left\{g(u);t\right\};x\right\}dx. \end{aligned} \tag{5.7}$$

On the other hand, we have the following relation by Lemma (5.1),

$$\begin{aligned}\int_0^\infty t\mathcal{P}\left\{f(x);t\right\}\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt \\ &= \int_0^\infty xf(x)\left(\frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\}\right)dx \\ &= \frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\int_0^\infty xf(x)\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\}dx. \end{aligned} \tag{5.8}$$

The relation (5.6) is a direct consequence of equations (5.7) and (5.8). □

Corollary 5.3. *The following equation holds true under the conditions of Theorem* (5.2)*,*

$$\mathcal{L}_2\left\{\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\};y\right\} = \frac{(-1)^{1-\mu}}{\Gamma(1-\mu)}\int_0^\infty te^{y^2t^2}E_1(y^2t^2)\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt, \tag{5.9}$$

where $\mathrm{Re}(\mu) < 1$ *and* $E_1(x)$ *is the exponential integral function which is defined by the following identity [15],*

$$E_1(x) = \int_x^\infty \frac{e^{-u}}{u}\,du. \tag{5.10}$$

Proof. If we choose $f(x) = e^{-x^2y^2}$ in the equation (5.6), we get

$$\begin{aligned}\int_0^\infty t\mathcal{P}\left\{e^{-x^2y^2};t\right\}\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt \\ = \frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\int_0^\infty xe^{-x^2y^2}\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\}dx.\end{aligned} \tag{5.11}$$

By the equation (2.13) and the relation (11) of [8, p.217], we have

$$\begin{aligned}\mathcal{P}\left\{e^{-x^2y^2};t\right\} &= \frac{1}{2}\mathcal{S}\left\{e^{-xy^2};t^2\right\} \\ &= \frac{1}{2}e^{y^2t^2}E_1(y^2t^2).\end{aligned} \tag{5.12}$$

Then, the equation (5.11) can be written as,

$$\begin{aligned}\int_0^\infty t\left(e^{y^2t^2}E_1(y^2t^2)\right)\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt \\ = (-1)^{\mu-1}\Gamma(1-\mu)\mathcal{L}_2\left\{\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\};y\right\},\end{aligned}$$

by using the definition (2.9) and the relation (5.12). □

Corollary 5.4. *The following equation holds true under the conditions of Theorem* (5.2)*,*

$$\mathcal{L}_2\left\{x^{-1}\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\};y\right\}dx \tag{5.13}$$
$$= \frac{\pi(-1)^{\mu-1}}{\Gamma(1-\mu)}\int_0^\infty e^{y^2t^2}\operatorname{Erfc}(yt)\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt,\ \operatorname{Re}(\mu)<1.$$

Proof. If we choose $f(x)=x^{-1}e^{-x^2y^2}$ in the equation (5.6) and use the formula (15) of [8, p.217], we get

$$\mathcal{P}\left\{x^{-1}e^{-x^2y^2};t\right\} = \frac{1}{2}\mathcal{S}\left\{x^{-1/2}e^{-xy^2};t^2\right\}$$
$$= \frac{1}{2}\pi t^{-1}e^{y^2t^2}\operatorname{Erfc}(yt). \tag{5.14}$$

Now, by substituting the equation (5.14) in the equation (5.6) and using the relation (2.9), we obtain

$$\frac{\pi}{2}\int_0^\infty e^{y^2t^2}\operatorname{Erfc}(yt)\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt$$
$$= \frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\int_0^\infty e^{-x^2y^2}\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\}dx$$
$$= \frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\mathcal{L}_2\left\{x^{-1}\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\};y\right\}dx.$$

□

Corollary 5.5. *If* $y>0$, $-k-1<\operatorname{Re}(\nu)<-2k+3/2$ *for* $k=0,1,2,\ldots$ *and* $\operatorname{Re}(\mu)<1$ *then the identity*

$$\frac{2(-1)^{k-\mu+1}}{\Gamma(1-\mu)}\mathcal{K}_\nu\left\{t^{\nu+2k+1/2}\mathcal{R}_{\mu,2}\left\{g(u);t\right\};y\right\} \tag{5.15}$$
$$= \mathcal{H}_\nu\left\{x^{\nu+2k+1/2}\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\};y\right\},$$

holds true, provided that each of the integrals involved converges absolutely.

Proof. By taking $f(x)=x^{\nu+2k}J_\nu(xy)$ in Theorem (5.2), we obtain

$$\int_0^\infty t\,\mathcal{P}\left\{x^{\nu+2k}J_\nu(xy);t\right\}\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt \tag{5.16}$$
$$= \frac{(-1)^{\mu-1}}{2}\Gamma(1-\mu)\int_0^\infty x^{\nu+2k+1}J_\nu(xy)\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\}dx.$$

Now, by the relation (2.13) and the identity (10) of [8, p.225], we have

$$\mathcal{P}\left\{x^{\nu+2k}J_\nu(xy);t\right\} = \frac{1}{2}\mathcal{S}\left\{x^{\nu/2+k}J_\nu(x^{1/2}y);t^2\right\}$$
$$= (-1)^k t^{\nu+2k}K_\nu(yt), \tag{5.17}$$

where $y>0$ and $-k-1<\operatorname{Re}(\nu)<-2k+3/2$ for $k\in\mathbb{N}$. Now, by using the equation (5.17) and the definition (2.19), we obtain

$$\int_0^\infty t\,\mathcal{P}\left\{x^{\nu+2k}J_\nu(xy);t\right\}\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt$$
$$= (-1)^k\int_0^\infty t^{\nu+2k+1}K_\nu(yt)\mathcal{R}_{\mu,2}\left\{g(u);t\right\}dt$$
$$= \frac{(-1)^k}{y^{1/2}}\mathcal{K}_\nu\left\{t^{\nu+2k+1/2}\mathcal{R}_{\mu,2}\left\{g(u);t\right\};y\right\}. \tag{5.18}$$

On the other hand, by using the definition (2.17), we get

$$\int_0^\infty x^{\nu+2k+1} J_\nu(xy)\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\}dx \qquad (5.19)$$
$$= \frac{1}{y^{1/2}}\mathcal{H}_\nu\left\{x^{\nu+2k+1/2}\mathcal{P}_{1-\mu,2}\left\{g(u);x\right\};y\right\}.$$

So, the equation (5.15) follows directly from the equations (5.18) and (5.19). □

References

[1] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications* 2^{nd} *Edition*, Chapman and Hall CRC, Boca Raton, London and New York, 2007.

[2] A.N. Dernek and F. Aylıkçı, *Identities for the* $\mathcal{L}_n$*-Transform, the* $\mathcal{L}_{2n}$*-Transform, and the* $\mathcal{P}_{2n}$*-Transform and Their Applications*, Journal of Inequaities and Special Functions, **5** (4), 1–16, 2014.

[3] A.N. Dernek and F. Aylıkçı, *Some Results on the* $\mathcal{P}_{\nu,2n}$*,* $\mathcal{K}_{\nu,n}$*,* $\mathcal{H}_{\nu,n}$ *Integral transforms*, Turkish J. Math. **41**, 337–349, 2017.

[4] A. Dernek, N. Dernek and O. Yürekli, *Identities for the Henkel Transform and Their Applications*, J. Math. Anal. Appl. **354**, 165–176, 2009.

[5] A.N. Dernek, V. Kurt, Y. Şimşek and O. Yürekli, *A generalization of Widder Potential Transform and Applications*, Integral Transforms Spec. Funct. **22**, 391–401, 2011.

[6] A.N. Dernek, H.M. Srivasta and O. Yürekli, *Parseval-Goldstein Type Identities Involving The* $\mathcal{L}_4$ *Transform and* $\mathcal{P}_4$ *Transform and Their Applications*, Integral Transforms Spec. Funct. **18**, (6), 397–408, 2007.

[7] A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Tables of Integral Transforms Vol: 1*, Mc-Graw-Hill Book Company, New York, Toronto and London, 1954.

[8] A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Tables of Integral Transforms Vol: 2*, Mc-Graw-Hill Book Company, New York, Toronto and London, 1954.

[9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 1996.

[10] C. Li, D. Qian and Y. Chen, *On Riemann Liouville and Caputo Derivatives*, Discrete Dyn. Nat. Soc. **2011**, 2011.

[11] K.S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, John Wiley-Interscience Publication, New York, Toronto, Singapore, 1993.

[12] I. Podlubny, *An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and Some of their Applications*, Academic Press, San Diego and London, 1990.

[13] D.V. Widder, *An Introduction to Transform Theory*, Academic Press, New York, 1971.

[14] O. Yürekli and I. Sadek, *A Parseval-Goldstein Type Theorem on the Widder Potential Transform and Its Applications*, International J. Math. Math. Sci. **14** (3), 517–524, 1991.

[15] O. Yürekli, *Identities on Fractional Integrals and Various Integral Transforms*, Appl. Math. Comput. **187**, 559–566, 2007.