Weighted composition operators between Besov-type spaces

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Abstract
In this paper, we study the boundedness and the compactness of weighted composition operators between Besov-type spaces. Also, we give a Carleson measure characterization of weighted composition operators on Besov spaces.

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1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D})$ the class of all complex-valued functions analytic on $\mathbb{D}$. Suppose $\varphi$ and $\psi$ are holomorphic functions defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. The weighted composition operator $W_{\varphi,\psi}$ induced by $\varphi$ and $\psi$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,\psi}f(z) = \psi(z)f(\varphi(z)) = \psi(z)C_\varphi(f),$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. When $\psi(z) \equiv 1$, the composition operator $W_{\varphi,1}$ is denoted by $C_\varphi$, i.e.,

$$W_{\varphi,1}f(z) = f(\varphi(z)) = C_\varphi(f),$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. For the study of composition operators one can refer to [7] and [11].

Fix any $a \in \mathbb{D}$ and let $\sigma_a(z)$ be the Mobius transform defined by

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z}, z \in \mathbb{D}.$$

We denote the set of all Mobius transformations on $\mathbb{D}$ by $G$. The inverse of $\sigma_a$ under composition is again $\sigma_a$ for $a \in \mathbb{D}$. Further, we have

$$|\sigma'_a(z)| = \frac{1 - |a|^2}{|1 - \overline{a}z|^2} \quad (1.1)$$

and

$$1 - |\sigma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2} = (1 - |z|^2)|\sigma'_a(z)|, \quad (1.2)$$

for every $a, z \in \mathbb{D}$.

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For $1 \leq p < \infty$, $L^p(\mathbb{D}, dA)$ will denote the Banach space of Lebesgue measurable functions $f$ on $\mathbb{D}$ with
\[ \|f\|_p = \left( \int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} < +\infty, \]
where $dA(z)$ denote the Lebesgue area measure on $\mathbb{D}$.
For $p = +\infty$, $L^\infty(\mathbb{D}, dA)$ will denote the Banach space of Lebesgue measurable functions $f$ on $\mathbb{D}$ with
\[ \|f\|_\infty = \text{ess sup}\{|f(z)| : z \in \mathbb{D}\} < +\infty. \]
For $1 \leq p < \infty$, the Bergman space $A^p$, is defined to be the subspace of $L^p(\mathbb{D}, dA)$ consisting of analytic functions, i.e. $A^p(\mathbb{D}) = L^p(\mathbb{D}) \cap H^\infty(\mathbb{D})$. The Bergman spaces are Banach spaces.

For $1 \leq p < +\infty$ and $-1 < r < +\infty$, the (weighted) Bergman space $A^p_r = A^p(\mathbb{D})$ of the disc is the space of analytic functions in $L^p(\mathbb{D}, dA_r)$, where
\[ dA_r(z) = (r + 1)(1 - |z|^2)^r dA(z). \]
If $f$ is in $A^p_r$, we write
\[ \|f\|_{A^p_r} = \left( \int_{\mathbb{D}} |f(z)|^p dA_r(z) \right)^{\frac{1}{p}}. \]
When $1 \leq p < +\infty$, the space $A^p_r$ is a Banach space with the above norm.

For $1 < p < +\infty$ and $-1 < r < +\infty$, an analytic function $f$ on $\mathbb{D}$ is said to belong to the Besov-type space $B_{p,r}$ if
\[ \|f\|_{B_{p,r}} = \left( \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^r dA(z) \right)^{\frac{1}{p}} < \infty, \quad (1.3) \]
where $dA(z)$ denote the Lebesgue area measure on $\mathbb{D}$. Also, if we take $1 < p < \infty$ and $r = p - 2$ in (1.3), then we get analytic Besov space, simply denoted by $B_p$. We can see that $\|f(0)\| + \|f\|_{B_{p,r}}$ is a norm on $B_{p,r}$, that makes it a Banach space. Moreover, we can observe that, for $f$ to be in $B_{p,r}$, it is necessary that the derivative of $f$ belong to the weighted Bergman spaces $A^p_r$.

**Definition 1.1.** Let $\mu$ be a positive measure on $\mathbb{D}$. Then the space $\mathbb{D}_p(\mu)$ is defined as the space of all holomorphic functions $f \in H(\mathbb{D})$ for which $f' \in L^p(\mathbb{D}, \mu)$. Also, the norm on $\mathbb{D}_p(\mu)$ is defined as
\[ \|f\|_{\mathbb{D}_p(\mu)} = \int_{\mathbb{D}} |f'(z)|^p d\mu(z). \]

Take $0 < p < \infty$. A positive measure $\mu$ on $\mathbb{D}$ is called a p-Carleson measure in $\mathbb{D}$ if
\[ \sup_{I \subseteq \partial \mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty, \quad (1.4) \]
where $|I|$ denotes the arc length of $I$ and $S(I)$ denotes the Carleson square based on $I$,
\[ S(I) = \{ z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I \}. \]
Again, $\mu$ is called a vanishing p-Carleson measure if
\[ \lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^p} = 0. \quad (1.5) \]
Take $h \in (0, 1)$ and $\theta \in [0, 2\pi)$. If we set
\[ S(h, \theta) = \{ z \in \mathbb{D} : |z - e^{i\theta}| < h \}, \]
then we can see that (1.4) and (1.5) are equivalent to
\[ \sup_{h \in (0,1), \theta \in [0,2\pi)} \frac{\mu(S(h, \theta))}{h^p} < \infty \quad (1.6) \]
Lemma 1.4. Let \( \psi \in \mathcal{B}_{q,r} \) and \( g \in L^q(\mathbb{D}, dA_r) \). We define the measures \( \mu_{q,r} \) and \( \nu_{q,r} \) on \( \mathbb{D} \) by

\[
\mu_{q,r}(E) = \int_{\mathbb{D}} |\psi(z)\varphi'(z)|^q(1 - |z|^2)^rdA(z) \tag{1.8}
\]

and

\[
\nu_{q,r}(E) = \int_{\mathbb{D}} |\psi'(z)|^q(1 - |z|^2)^rdA(z), \tag{1.9}
\]

where \( E \) is a measurable subset of the unit disc \( \mathbb{D} \).

If \( \psi \in A_p^q \), then we can define the measure \( \nu_{q,\psi,r} \) on \( \mathbb{D} \) by

\[
\nu_{q,\psi,r}(E) = \int_{\mathbb{D}} |\psi(z)|^q(1 - |z|^2)^rdA(z). \tag{1.10}
\]

**Definition 1.2.** Take \( 1 < p < +\infty \) and \(-1 < r < \infty \). Let \( \mu \) be a positive measure on \( \mathbb{D} \). Then the measure \( \mu \) is \((p, r)\)-Carleson measure for \( B_{p,r} \) if there is a constant \( K > 0 \) such that

\[
\int_{\mathbb{D}} |f'(w)|^pd\mu(w) \leq K\|f\|_{B_{p,r}}^p,
\]

for all \( f \in B_{p,r} \). That is, the inclusion operator \( i : B_{p,r} \rightarrow \mathbb{D}_p(\mu) \) is bounded. Further, the measure \( \mu \) is a vanishing \( p \)-Carleson measure for \( B_{p,r} \) if the inclusion operator \( i : B_{p,r} \rightarrow \mathbb{D}_p(\mu) \) is compact.

The following characterization of \((p, r)\)-Carleson measures can be obtained easily from [1].

**Theorem 1.3.** Take \( 1 < p < \infty \) and \(-1 < r < \infty \). Let \( \mu \) be a positive measure on \( \mathbb{D} \). Then the following statements are equivalent:

1. The measure \( \mu \) is a \((p, r)\)-Carleson measure for \( B_{p,r} \).
2. There exists a constant \( K < \infty \) such that

\[
\mu(S(h, \theta)) \leq Kh^p
\]

for all \( \theta \in [0, 2\pi) \) and \( h \in (0, 1) \).
3. There exists a constant \( C < \infty \) such that

\[
\int_{\mathbb{D}} |\sigma'_a(z)|^pd\mu(z) \leq C
\]

for all \( a \in \mathbb{D} \).

Using ([6], Lemma 2.1) and ([8], page 163), the following lemma can be proved easily.

**Lemma 1.4.** Let \( \varphi \) be a holomorphic mapping defined on \( \mathbb{D} \) such that \( \varphi(\mathbb{D}) \subseteq \mathbb{D} \). Take \( \psi \in B_{q,r} \) such that \( \psi(z)\varphi'(z) \in L^q(\mathbb{D}, dA_r) \). Then

\[
\int_{\mathbb{D}} g d\mu_{q,r} = \int_{\mathbb{D}} |\psi(z)\varphi'(z)|^q(g \circ \varphi)(z)(1 - |z|^2)^rdA(z) \tag{1.11}
\]

and

\[
\int_{\mathbb{D}} g d\nu_{q,r} = \int_{\mathbb{D}} |\psi'(z)|^q(g \circ \varphi)(z)(1 - |z|^2)^rdA(z). \tag{1.12}
\]

where \( g \) is an arbitrary measurable positive function in \( \mathbb{D} \).
We use the following lemma for compactness of the weighted composition operators on Besov-type spaces. The proof of this lemma follows by similar lines as in the case of composition operators on Besov spaces ([12], Lemma 3.8).

**Lemma 1.5.** Given $1 \leq p, q < \infty$, $-1 < r < \infty$, let $\varphi$ be a holomorphic mapping defined on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in B_{q,r}$ be such that $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is bounded. Then $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is compact (weakly compact) if and only if whenever $\{f_n\}$ is a bounded sequence in $B_{p,r}$ converging to zero uniformly on compact subsets of $\mathbb{D}$, then $\|W_{\varphi,\psi}(f_n)\|_{B_{q,r}} \to 0$ (respectively, $\{W_{\varphi,\psi}(f_n)\}$ is a weak null sequence in $B_{q,r}$).

Boundedness and compactness of the weighted composition operators on spaces of analytic functions has been studied by many authors. For example we refer to [2-5,9,10,13].

In this article, by using the Carleson measure, we characterize the boundedness and compactness of $W_{\varphi,\psi}$ on Besov-type spaces, in section 2. The Carleson measure characterization of $W_{\varphi,\psi}$ acting on Besov spaces is given in section 3.

2. Bounded and compact weighted composition operators on Besov-type spaces

In this section, we characterize the boundedness and compactness of $W_{\varphi,\psi}$ on Besov-type spaces by using Carleson measures.

**Theorem 2.1.** Take $1 < p \leq q < \infty$ and $-1 < r < \infty$. Let $\varphi \in B_{p,r}$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, $\psi \in A^q_\mathbb{D}$ and the measure $\nu_{q,\psi,r}$ is a vanishing $(q, r)$-Carleson measure for $B_{q,r}$. Then $W_{\varphi,\psi}$ defines a bounded operator from $B_{p,r}$ into $A^q_\mathbb{D}$. Moreover, $W_{\varphi,\psi} : B_{p,r} \to A^q_\mathbb{D}$ is compact.

**Proof.** We prove the compactness only. Let $\{f_n\}$ be a bounded sequence in $B_{p,r}$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since the measure $\nu_{q,\psi,r}$ is a vanishing $(q, r)$-Carleson measure for $B_{q,r}$, the inclusion map $i : B_{q,r} \to L^q(\mathbb{D}, \nu_{q,\psi,r})$ is compact. Since $B_{p,r} \subset B_{q,r}$, we have $\|f_n\|_{L^q(\mathbb{D}, \nu_{q,\psi,r})} \to 0$ as $n \to \infty$. Therefore, by Lemma 1.4, we have

$$
\|W_{\varphi,\psi}(f_n)\|_{A^q_\mathbb{D}} = \int_{\mathbb{D}} |\psi(z)|^q |(f_n \circ \varphi)(z)|^q (1 - |z|^2)^r dA(z) = \int_{\mathbb{D}} |f_n(z)|^q d\nu_{q,\psi,r}(z) \to 0, \quad \text{as} \quad n \to \infty. \quad (2.1)
$$

Thus, $W_{\varphi,\psi} : B_{p,r} \to A^q_\mathbb{D}$ is compact. \hfill \Box

**Theorem 2.2.** Take $1 < p \leq q < \infty$ and $-1 < r < \infty$. Let $\varphi, \psi \in B_{p,r}$ be such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and the measure $\nu_{q,\psi,r}$ is a vanishing $(q, r)$-Carleson measure for $B_{q,r}$. Then $W_{\varphi,\psi}$ is a bounded operator from $B_{p,r}$ into $B_{q,r}$ if and only if $W_{\varphi,\psi'}$ is a bounded operator from $A^p_\mathbb{D}$ into $A^q_\mathbb{D}$.

**Proof.** Suppose that $W_{\varphi,\psi} : B_{p,r} \to B_{q,r}$ is bounded. Then there exists a constant $C > 0$ such that

$$
\|W_{\varphi,\psi}(g)\|_{B_{q,r}} \leq C \|g\|_{B_{p,r}}
$$

for all $g \in B_{p,r}$. Also, by Theorem 2.1, we can find a constant $M > 0$ such that

$$
\|W_{\varphi,\psi'}(g)\|_{A^q_\mathbb{D}} \leq M \|g\|_{B_{p,r}}, \quad g \in B_{p,r}.
$$
Take $f \in A^p_r$ and let the function $g \in B_{p,r}$ be such that $g' = f$ and $g(0) = 0$. Then,
\[
\|W_{\varphi,\psi'}(f)\|_{A^q_r} = \|\psi'f \circ \varphi\|_{A^q_r} = \|\psi'f \circ \varphi + \psi'g \circ \varphi - \psi'g \circ \varphi\|_{A^q_r} \\
\leq \|\psi'g \circ \varphi\|_{A^q_r} + \|\psi'g \circ \varphi\|_{A^q_r} = \|\psi'g \circ \varphi\|_{A^q_r} + \|\psi'g \circ \varphi\|_{A^q_r} = C\|g\|_{B_{p,r}} + M\|g\|_{B_{p,r}} = (C + M)\|g\|_{B_{p,r}}.
\]

Thus, $W_{\varphi,\psi'} : A^p_r \rightarrow A^q_r$ is bounded. Conversely, suppose $W_{\varphi,\psi'} : A^p_r \rightarrow A^q_r$ is bounded. Again, by Theorem 2.1,
\[
W_{\varphi,\psi'} : B_{p,r} \rightarrow A^q_r
\]
is bounded. Take $f \in B_{p,r}$ such that $f(0) = 0$. Then, we have
\[
\|W_{\varphi,\psi}(f)\|_{B_{q,r}} = \|(\psi \circ \varphi)'\|_{A^p_r} = \|\psi'f \circ \varphi + \psi'g \circ \varphi\|_{A^p_r} \leq \|W_{\varphi,\psi'}(f')\|_{A^q_r} + \|W_{\varphi,\psi'}(f)\|_{A^q_r} < +\infty.
\]
The theorem is proved.

**Theorem 2.3.** Take $1 < p < q < \infty$ and $-1 < r < \infty$. Let $\varphi, \psi \in B_{p,r}$ be such that $\varphi(D) \subseteq D$ and the measure $\nu_{q,r}$ is a vanishing $(q, r)$-Carleson measure for $B_{q,r}$. Then $W_{\varphi,\psi}$ is a compact operator from $B_{p,r}$ into $B_{q,r}$ if and only if $W_{\varphi,\psi'}$ is a bounded operator from $A^p_r$ into $A^q_r$.

**Proof.** Suppose that $W_{\varphi,\psi} : B_{p,r} \rightarrow B_{q,r}$ is compact. Let $\{f_n\}$ be a bounded sequence in $A^p_r$ such that $f_n \rightarrow 0$ uniformly on compact subsets of $D$. For each $n$, there exists a function $g_n \in B_{p,r}$ such that $g_n = f_n$ and $g_n(0) = 0$. The sequence $\{g_n\}$ also converges to zero uniformly on compact subsets of $D$ as $n \rightarrow \infty$. Further, since $W_{\varphi,\psi} : B_{p,r} \rightarrow B_{q,r}$ is compact, so $\|W_{\varphi,\psi}(g_n)\|_{B_{q,r}} \rightarrow 0$ as $n \rightarrow \infty$. Again, by Theorem 2.1, $W_{\varphi,\psi'} : B_{p,r} \rightarrow A^q_r$ is compact, so $\|W_{\varphi,\psi'}(g_n)\|_{A^q_r}$ also converges to zero as $n \rightarrow \infty$. We have
\[
\|W_{\varphi,\psi'}(f_n)\|_{A^q_r} = \|\psi'f_n \circ \varphi\|_{A^q_r} \leq \|\psi'f_n \circ \varphi + \psi'g_n \circ \varphi\|_{A^q_r} = \|\psi'g_n \circ \varphi\|_{A^q_r} + \|W_{\varphi,\psi'}(g_n)\|_{A^q_r} = \|W_{\varphi,\psi'}(g_n)\|_{B_{q,r}} + \|W_{\varphi,\psi'}(g_n)\|_{A^q_r} \rightarrow 0, \text{ as } n \rightarrow \infty.
\]
Therefore, $W_{\varphi,\psi'} : A^p_r \rightarrow A^q_r$ is compact.

Conversely, suppose $W_{\varphi,\psi'} : A^p_r \rightarrow A^q_r$ is compact. Again, by Theorem 2.1, $W_{\varphi,\psi'} : B_{p,r} \rightarrow A^q_r$ is compact. Let $\{g_n\}$ be the same sequence as in the direct part. Then,
\[
\|W_{\varphi,\psi}(g_n)\|_{B_{q,r}} = \|(\psi g_n \circ \varphi)'\|_{A^p_r} = \|\psi'g_n \circ \varphi + \psi'g_n \circ \varphi\|_{A^p_r} = \|W_{\varphi,\psi'}(g_n)\|_{A^q_r} \rightarrow 0, \text{ as } n \rightarrow \infty.
\]
Thus, $W_{\varphi,\psi} : B_{p,r} \rightarrow B_{q,r}$ is compact.

**Theorem 2.4.** Take $1 < p < \infty$ and $-1 < r < \infty$. Let $\varphi, \psi \in B_{p,r}$ be such that $\varphi(D) \subseteq D$ and the measure $\nu_{p,r}$ is a vanishing $(p, r)$-Carleson measure for $B_{p,r}$. Then $W_{\varphi,\psi}$ is a bounded (compact) operator from $B_{p,r}$ into $B_{p,r}$ if and only if $\mu_{p,r}$ is a bounded (vanishing) $(p, r)$-Carleson measure for $B_{p,r}$.

**Proof.** We only prove the boundedness. Suppose first that $W_{\varphi,\psi} : B_{p,r} \rightarrow B_{p,r}$ is bounded. Then by Theorem 2.2, $W_{\varphi,\psi'}$ is a bounded operator on $A^p_r$. Let $f \in B_{p,r}$ be such that
The Mobius invariant measure on $\mathcal{B}_p$. Carleson measure characterization of the weighted composition operators on Besov spaces

By using (2.2), (2.3) and (2.4),

$$
\int |\psi(z)|^p |f'(z)|^p |(1 - |z|^2)^r| dA(z)
$$

Hence,

$$
\int |f'(w)|^p d\mu_{p,r}(w).
$$

Since $W_{\psi,\varphi'}$ is bounded on $\mathcal{B}_p$, therefore we can find a constant $C > 0$ such that

$$
\|W_{\psi,\varphi'}(f')\|^p_{\mathcal{B}_p} \leq C \|f'\|^p_{\mathcal{B}_p}.
$$

Hence,

$$
\int |f'(w)|^p d\mu_{p,r}(w) \leq C \|f\|^p_{\mathcal{B}_p}.
$$

That is, the inclusion operator $i : B_{p,r} \to \mathcal{D}_{p,r}(\mu)$ is bounded. Thus the measure $\mu_{p,r}$ is a bounded $(p, r)$-Carleson measure for $B_{p,r}$.

Conversely, suppose that $\mu_{p,r}$ is a bounded $(p, r)$-Carleson measure for $B_{p,r}$. We want to show that $W_{\psi,\varphi} : B_{p,r} \to B_{p,r}$ is bounded. We have

$$
(\psi(f \circ \varphi')) = \psi\varphi'(f' \circ \varphi) + \varphi'(f \circ \varphi).
$$

(2.2)

Take $f \in B_{p,r}$. So by Lemma 1.4,

$$
\int |\psi(z)|^p |f'(z)|^p |(1 - |z|^2)^r| dA(z) = \int |f'(w)|^p d\mu_{p,r}(w) < +\infty.
$$

(2.3)

Also, by using Theorem 2.1, we get

$$
\int |\psi(z)|^p |f'(z)|^p |(1 - |z|^2)^r| dA(z) = \|W_{\psi,\varphi'}(f')\|^p_{\mathcal{B}_p} < +\infty.
$$

(2.4)

By using (2.2), (2.3) and (2.4), $W_{\psi,\varphi} : B_{p,r} \to B_{p,r}$ is bounded.

Compactness of $W_{\psi,\varphi}$ can be proved by using the Theorems 2.1 and 2.3, which we omit its proof.

3. Carleson measure characterization of the weighted composition operators on Besov spaces

In this section, we give a Carleson measure characterization of $W_{\psi,\varphi}$ on Besov space.

Let $1 < p, q < \infty$, $\varphi$ be a holomorphic mapping defined on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi \in B_q$ be such that $\psi(z)\varphi'(z)(1 - |z|^2)^2 \in L^q(\mathbb{D}, d\lambda)$ (where $d\lambda(z) = (1 - |z|^2)^{-2} dA(z)$ is the Mobius invariant measure on $\mathbb{D}$). For $f \in B_p$ there exists a constant $C_q$ such that

$$
\|W_{\varphi,\psi}(f)\|_{B_q}^q = \int_{\mathbb{D}} |(\psi C_{\varphi} f)'(z)|^q (1 - |z|^2)^{q-2} dA(z)
$$

$$
\leq C_q \int_{\mathbb{D}} |\psi(z)|^q |(C_{\varphi} f)(z)|^q (1 - |z|^2)^{q-2} dA(z)
$$

$$
+ C_q \int_{\mathbb{D}} |\psi(z)|^q |\varphi'(z)|^q |f'(\varphi(z))|^q (1 - |z|^2)^{q-2} dA(z).
$$

By using Lemma 1.4, we have

$$
\|W_{\psi,\varphi}(f)\|_{B_q}^q \leq C_q \int_{\mathbb{D}} |f(w)|^q d\mu_q(w) + C_q \int_{\mathbb{D}} |f'(w)|^q d\mu_q(w).
$$

(3.1)

Since $W_{\psi,\varphi} : B_p \to B_q$ is a bounded operator if and only if there exists a positive constant $C$ such that

$$
\|W_{\psi,\varphi}(f)\|_{B_q}^q \leq C \|f\|_{B_p}^q;
$$

so, the following theorem holds.
**Theorem 3.1.** Let \(1 < p < \infty\), \(\varphi\) be a holomorphic mapping defined on \(\mathbb{D}\) such that \(\varphi(\mathbb{D}) \subseteq \mathbb{D}\) and \(\psi \in B_p\) be such that \(\psi(z)\varphi'(z)(1 - |z|^2) \in L^p(\mathbb{D}, d\lambda)\). If the measure \(\mu_p\) is a \(p\)-Carleson measure and \(\nu_p\) is a vanishing \(p\)-Carleson measure for \(B_p\), then \(W_{\varphi, \psi} : B_p \to B_p\) is a bounded operator.

**Proof.** Suppose that \(\mu_p\) is a \(p\)-Carleson measure for \(B_p\). By Definition 1.2, there exists a constant \(C_1\) such that

\[
\int_D |f'(w)|^p d\mu_p(w) \leq C_1 \|f\|_{B_p}^p. \tag{3.2}
\]

Let \(\nu_p\) be a vanishing \(p\)-Carleson measure for \(B_p\). By using Theorem 2.1 for \(r = p - 2\), there exists a constant \(C_2\) such that

\[
\|W_{\varphi, \psi}(f)\|_{A_p}^p = \int_D |\psi'(z)|^p |f \circ \varphi|^p (1 - |z|^2)^{p-2} dA_z
= \int_D |f(w)|^p d\nu_p(w)
\leq C_2 \|f\|_{B_p}^p. \tag{3.3}
\]

By using (3.2) and (3.3), from (3.1) the theorem is proved. \(\square\)

**Theorem 3.2.** Suppose \(1 < p \leq q < \infty\) and \(\varphi\) is a holomorphic mapping defined on \(\mathbb{D}\). Let \(\varphi(\mathbb{D}) \subseteq \mathbb{D}\) and \(\psi \in B_q\) be such that \(\psi(z)\varphi'(z)(1 - |z|^2) \in L^q(\mathbb{D}, d\lambda)\). If the measures \(\nu_q\) and \(\mu_q\) are vanishing \(q\)-Carleson measures for \(B_q\), then \(W_{\varphi, \psi} : B_p \to B_q\) is a compact operator.

**Proof.** Let \(\{f_n\}\) be a bounded sequence in \(B_p\), such that \(f_n \to 0\) uniformly on compact subsets of \(\mathbb{D}\). Then the mean value property for the holomorphic function yields

\[
f'_n(w) = \frac{4}{\pi(1 - |w|^2)^2} \int_{|w - z| < \frac{1 - |w|}{2}} f'_n(z) dA(z). \tag{3.4}
\]

Therefore by Jensen’s inequality,

\[
|f'_n(w)|^q \leq \frac{4}{\pi(1 - |w|^2)^2} \int_{|w - z| < \frac{1 - |w|}{2}} |f'_n(z)|^q dA(z). \tag{3.5}
\]

Since the measure \(\nu_q\) is a vanishing \(q\)-Carleson measure for \(B_q\), by using Theorem 2.1 (relation (2.1)), we have

\[
\int_D |f_n(w)|^q d\nu_q(w) \to 0, \text{ as } n \to \infty. \tag{3.6}
\]

By using (3.1), (3.5), (3.6) and Fubini’s Theorem,

\[
\|W_{\varphi, \psi}(f_n)\|_{B_q}^q \leq C_q \int_D |f_n(w)|^q d\nu_q(w) + C_q \int_D |f'_n(w)|^q d\mu_q(w)
\leq C_q \int_D \frac{4}{\pi(1 - |w|^2)^2} \left( \int_{|w - z| < \frac{1 - |w|}{2}} |f'_n(z)|^q dA(z) \right) d\mu_q(w)
\leq C_q \frac{4}{\pi} \int_D |f'_n(z)|^q \left( \int_D \frac{1}{(1 - |w|)^2} |z|^{1 - |w|} \right) d\mu_q(w) \, dA(z).
\]

Note that if \(|w - z| < \frac{1 - |w|}{2}\), then \(w \in S(2(1 - |z|), \theta)\), where \(z = |z|e^{i\theta}\), since

\[
|w - |z|| \leq |z - w| + |e^{i\theta} - z| < \frac{1 - |w|}{2} + \frac{|z|}{|z|} - z < 2(1 - |z|).
\]

Moreover, if \(|w - z| < \frac{1 - |w|}{2}\) then

\[
\frac{1}{(1 - |w|)^2} \leq const. \frac{1}{(1 - |z|)^2}.
\]
Hence,
\[ \|W_{\varphi,\psi}(f_n)\|_{B_q}^q \leq \text{const.} \int_D \frac{|f'_n(z)|^q}{(1-|z|)^2} \left( \int_{S(2|1-|z||,\theta)} d\mu_q(w) \right) dA(z) \]
\[ = \text{const.} \left( \int_{|z|>1-\frac{\delta}{2}} + \int_{|z|\leq1-\frac{\delta}{2}} \frac{|f'_n(z)|^q}{(1-|z|)^2} \left( \int_{S(2|1-|z||,\theta)} d\mu_q(w) \right) dA(z) \right) \]
\[ = I + II \quad (3.7) \]
for any \(0 < \delta < 1\).

Fix \(\epsilon > 0\) and let \(\delta > 0\) be such that for any \(\theta \in [0,2\pi]\) and any \(h < \delta\),
\[ \mu_q(S(h,\theta)) < \epsilon h^q, \quad (3.8) \]
and so
\[ \int_{S(h,\theta)} d\mu_q < \epsilon h^q. \]

By (3.8),
\[ I \leq \text{const.} 2^{q}\epsilon \int_{|z|>1-\frac{\delta}{2}} \frac{|f'_n(z)|^q}{(1-|z|)^2} (1-|z|^2)^q dA(z) \]
\[ \leq \text{const.} \epsilon \|f'_n\|_{B_q}^q < \text{const.} \epsilon. \quad (3.9) \]

For \(n\) large enough, since \(f'_n \to 0\) uniformly on compact sets, we have
\[ II \leq \text{const.} \int_{|z|\leq1-\frac{\delta}{2}} |f'_n(z)|^q \left( \int_{D} d\mu_q \right) dA(z) < \text{const.} \epsilon. \quad (3.10) \]

Therefore, from (3.7), (3.9) and (3.10) we obtain
\[ \|W_{\varphi,\psi}(f_n)\|_{B_q}^q < \text{const.}\epsilon. \]

Thus \(\|W_{\varphi,\psi}(f_n)\|_{B_q}^q \to 0\) as \(n \to \infty\), and from Lemma 1.5, \(W_{\varphi,\psi}\) is compact. \(\square\)

References
