Introduction to Timelike Uniform B-spline Curves in Minkowski-3 Space

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Abstract

The intention of this article is to study on timelike uniform B-spline curves in Minkowski-3 space. In our paper, we take the control points of uniform B-spline curves as a timelike point in Minkowski-3 space. Then we calculate some geometric elements for this new curve in Minkowski-3 space.

1. Introduction

B-spline curves were described by Schoenberg who was worked on B-spline curves for statistical data collection in [1]. The B-spline curves was constructed for computing a convolution of some probability distributions. Moreover, de Boor and Hollig considered a different approach to B-spline curves in [2]. Recently, in Computer Aided Geometric Design (CAGD), B-spline curves have been commonly used for designing an automobile, a boat, an aircraft, [3] and [4]. There are many studies on the B-spline curves, see some of them in [5], [6], [7] and [8]. Although degree $d$ of a Bezier curve has $d+1$ control points, degree $d$ of a B-spline curves can have any number of control points supplied a sufficient number of knots are defined in [9]. In addition, the control points of the Bezier curves provide a global change on the curve, while the control points of the B-spline curves provide a local change on the curve. For this reason, B-spline curves can be given additional freedom by increasing the number of control points in order to define complex curve shapes without increasing the degree of the curve, [10]. Minkowski space was introduced by H. Minkowski. In our paper, we try to investigate some geometric properties of the B-spline curves in Minkowski 3-space. We present the curvature and torsion of the B-spline curves in Minkowski 3-space.

2. Preliminaries

In this section the B-spline curves are defined and some preliminaries are given. Then some basics of Minkowski space is given.

**Definition 2.1.** Let $t_0, t_1, ..., t_n$ be knot vectors of the B-spline basis function of degree $d$. The B-spline basis function denoted $N_{i,d}(t)$ is defined by

$$N_{i,0}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

$$N_{i,d}(t) = \frac{t-t_i}{t_{i+d}-t_i}N_{i,d-1}(t) + \frac{t_{i+d+1}-t}{t_{i+d+1}-t_{i+1}}N_{i+1,d-1}(t) \tag{2.2}$$

for $i = 0, ..., n$ and $d \geq 1$. 

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Definition 2.2. If the B-spline curve of degree $d$ with control points $b_0, \ldots, b_n$ and knots $t_0, t_1, \ldots, t_m$ is defined on the interval $[a, b] = [t_d, t_{m-d}]$, then the curve can be written in the form

$$B(t) = \sum_{i=0}^{n} b_i N_{i,d}(t).$$

When the B-spline curves are in the rational form, they are often called integral B-spline curves. Moreover, if the knots are equally spaced, then the B-spline curve has the convex hull. If $p$ shows us that the B-spline curve is achieved by the local control. If $\nu$ is the multiplicity of the breakpoint $t = u_i$ then $B(t)$ is $C^{d-p}$ (or greater) at $t = u_i$.

3. Main result

Definition 3.1. Let $X = \{b_0, b_1, \ldots, b_n\}$ be a timelike points set in $\mathbb{R}^3_1$. The

$$TCH\{X\} = \left\{ \lambda_0 b_0 + \ldots + \lambda_n b_n | \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0 \right\}$$

set formed by these $X$ points are called timelike convex hull of a timelike uniform B-spline curve.

Definition 3.2. If the control points $b_0, \ldots, b_n \in TCH\{X\}$ and the knots $t_0, t_1, \ldots, t_m$ on the interval $[a, b] = [t_d, t_{m-d}]$ are equally spaced, then the timelike uniform B-spline curve of degree $d$ in Minkowski 3-space is defined by

$$B(t) = \sum_{i=0}^{n} b_i N_{i,d}(t),$$

where $N_{i,d}(t)$ are the basis functions.

Example: Let consider the timelike uniform B-spline curve $B(t)$ of degree $d = 2$ defined on the knots $t_0 = 0, t_1 = 1, t_2 = 2, t_3 = 3, t_4 = 4, t_5 = 5, t_6 = 6, t_7 = 7$ and with control points $b_0(2, 3), b_1(-1, 7), b_2(2, 5), b_3(4, 5), b_4(1, 3)$. The basis graphic and the curve shape are in the following figures.

![Figure 3.1: a) Basis function graphic b) A timelike uniform B-spline curve](image-url)
Let $B(t)$ be a timelike uniform B-spline curve of degree $d$ with the knot vector $t_0, \ldots, t_m$ in Minkowski 3-space. The second and third derivative of the control points $b_i$ are calculated by

$$
\begin{align*}
    b_i^{(2)} &= (d-1) m_i \Delta b_i^{(1)} \\
    b_i^{(3)} &= (d-1)(d-2) p_i (n_i \Delta b_i^{(1)} - m_i \Delta b_i^{(1)})
\end{align*}
$$

where $m_i, n_i, p_i$ are some constants of $t_i$.

**Proof.** Using the Eq.(2.1) and Eq.(2.2) the control points can be written as

$$
\begin{align*}
    b_i^{(2)} &= (d-1) \frac{b_i^{(1)} - b_i^{(1)}}{t_{i+d} - t_{i+2}} \\
    b_i^{(3)} &= (d-1) m_i \Delta b_i^{(1)} + (d-2) \frac{b_i^{(2)} - b_i^{(2)}}{t_{i+d} - t_{i+3}} \\
    &= \frac{(d-1)(d-2)}{t_{i+d} - t_{i+3}} \left( (d-1) n_i (b_i^{(2)} - b_i^{(2)}) - (d-1) m_i (b_i^{(1)} - b_i^{(1)}) \right)
\end{align*}
$$

where $m_i = \frac{1}{n_i + 1 - t_{i+2}}$, $n_i = \frac{1}{n_i + 1 - t_{i+3}}$ and $p_i = \frac{1}{n_i + 1 - t_{i+3}}$.

**Theorem 3.5.** Let $B(t)$ be a timelike uniform B-spline curve of degree $d$ with the knot vector $t_0, \ldots, t_m$ in Minkowski 3-space. The derivatives of $B(t)$ are computed by

$$
\begin{align*}
    B^{(1)}(t) &= \sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \\
    B^{(2)}(t) &= (d-1) \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)} \\
    B^{(3)}(t) &= (d-1)(d-2) \sum_{i=0}^{n-3} p_i (n_i \Delta b_i^{(1)} - m_i \Delta b_i^{(1)}) N_{i,d-3}^{(3)}.
\end{align*}
$$

**Proof.** Substituting the above results in Eq.(2.2), the proof is obvious.

**Theorem 3.6.** Let $B(t)$ be an arbitrary timelike uniform B-spline curve and $\{ T, N, B \}_{i=0}^n$ be the Serret-Frenet frame of $B(t)$, where $T$ is timelike, $N$ and $B$ are spacelike. Then the following conditions are satisfied

$$
\begin{align*}
    g(T,T) &= -1, g(N,N) = 1, g(B,B) = 1 \\
    g(T,N) &= 0, g(T,B) = 0, g(N,B) = 0.
\end{align*}
$$

The Serret-Frenet frame of the timelike uniform B-spline curve $B(t)$ is obtained by

$$
\begin{align*}
    T &= \frac{\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t)}{\| \sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \|} \\
    B &= \frac{\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)}}{\| \sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)} \|} \\
    N &= \frac{-g \left( \sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t), \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)} \right) \oplus g \left( \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)}, \sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \right)}{\| \sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)} \|}.
\end{align*}
$$
Proof. Let consider the B-spline curve $B(t)$ is non unit speed curve in Minkowski 3-space. Using the scalar and vector product in Minkowski 3-space, the tangent vector of the timelike uniform B-spline curve $B(t)$ is calculated as

$$ T = \frac{B^{(1)}(t)}{\|B^{(1)}(t)\|} = \sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) $$

and the binormal vector of the timelike B-spline curve is

$$ B = \frac{B^{(1)}(t) \wedge B^{(2)}(t)}{\|B^{(1)}(t) \wedge B^{(2)}(t)\|} = \sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \wedge (d-1) \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)} $$

The principal normal can be obtained as

$$ N = -B \wedge T = \frac{-\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)}}{\|B^{(1)}(t) \wedge B^{(2)}(t)\|} $$

Theorem 3.7. If the B-spline curve of degree $d$ with control points $b_0,...,b_m$ and knots $t_0,t_1,...,t_m$ is defined on the interval $[a,b] = [t_d,t_{m-d}]$, the curvatures of timelike uniform B-spline curve $B(t)$ is found as

$$ \kappa = |d-1| \frac{\|\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)}\|}{\|\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t)\|^3} $$

Proof. From the definition of curvature of the non-unit speed curve, we have

$$ \kappa = \frac{\|B^{(1)}(t) \wedge B^{(2)}(t)\|}{\|B^{(1)}(t)\|^3} $$

$$ = \frac{\|\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \wedge (d-1) \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)}\|}{\|\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t)\|^3} $$

$$ = |d-1| \frac{\|\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_{i,d-2}^{(2)}\|}{\|\sum_{i=0}^{n-1} b_i^{(1)} N_{i,d-1}^{(1)}(t)\|^3} $$

\qed
Theorem 3.8. If $B(t)$ is a timelike uniform B-spline curve of degree $d$ with the knot vector $t_0, \ldots, t_m$ in Minkowski 3-space, the torsion of a timelike uniform B-spline curve $B(t)$ is computed by

$$
\tau = -(d-2) \frac{\det \left( \sum_{i=0}^{n} b_i^{(1)} N_i^{(0)}(t), \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_i^{(2)}_{d-2} \sum_{i=0}^{n-3} p_i, \left( n_i, \Delta b_i^{(1)} - m_i \Delta b_i^{(1)} \right) N_i^{(3)}_{d-3} \right)}{\left\| \sum_{i=0}^{n-1} b_i^{(1)} N_i^{(0)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_i^{(2)}_{d-2} \right\|^2} 
$$

Proof. Using the definition of torsion, we have the following equations:

$$
\tau = \frac{\left( B^{(1)}(t) B^{(2)}(t) B^{(3)}(t) \right)}{\left\| B^{(1)}(t) \wedge B^{(2)}(t) \right\|^2} 
$$

$$
= \frac{\left( \sum_{i=0}^{n} b_i^{(1)} N_i^{(0)}(t), (d-1) \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_i^{(2)}_{d-2}. (d-1)(d-2) \sum_{i=0}^{n-3} p_i, \left( n_i, \Delta b_i^{(1)} - m_i \Delta b_i^{(1)} \right) N_i^{(3)}_{d-3} \right)}{\left\| \sum_{i=0}^{n-1} b_i^{(1)} N_i^{(0)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_i^{(2)}_{d-2} \right\|^2} 
$$

$$
= -(d-2) \frac{\det \left( \sum_{i=0}^{n-1} b_i^{(1)} N_i^{(0)}(t), \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_i^{(2)}_{d-2} \sum_{i=0}^{n-3} p_i, \left( n_i, \Delta b_i^{(1)} - m_i \Delta b_i^{(1)} \right) N_i^{(3)}_{d-3} \right)}{\left\| \sum_{i=0}^{n-1} b_i^{(1)} N_i^{(0)}(t) \wedge \sum_{i=0}^{n-2} m_i \Delta b_i^{(1)} N_i^{(2)}_{d-2} \right\|^2} 
$$

4. Conclusion

In this paper, we present a theoretical work about the timelike uniform B-spline curves in Minkowski-3 space. The timelike B-spline curve in Minkowski 3-space at first time is introduced. The derivatives of control points are calculated. Later Serret-Frenet frame of the timelike uniform B-spline curve is given. Moreover, the curvature and torsion of the B-spline curve are computed.

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References