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A Study Over Determination of Asymptotic Deceleration and Absolute Acceleration Points in Logistic Growth Model

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Abstract. Since growth models has generally upper horizontal asymptote, they do not have a maximum point. We wonder about after which point growth can be considered constant, that is, after which point the curve of the growth function is too close to its asymptote. That point is called maximum deceleration point. After this point the deceleration is very slow and the second derivative of the growth function goes to zero as time tends to infinity. After this point it is considered that the amount of the growth is quite small. Moreover, we wonder about which point is an absolute acceleration point so that before that point acceleration is very slow and after that point actual acceleration starts. So we could say that after this point actual growth starts. In this study, the logistic growth model was used to investigate these points, asymptotic deceleration and absolute acceleration points in addition to the other critical and important points such as inflection point, maximum acceleration point, maximum deceleration point. The graphs of the logistic growth model which show all these points mentioned above are also given by using a data set.

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1. Introduction

Growth models having upper horizontal asymptote have often a point where the growth has not a maximum point but a point considered practically constant. By using modelling, critical points of a growth curve can be defined mathematically. One of the points is inflection point of the curve. Gregorczyk [2] worked with Richards growth function and considered three critical points during plant growth: the inflection point and two other points, one of maximum and another one of minimum acceleration. The searching for a ‘maximum growth point’ is an important matter in many research fields. For instance, Chatkin *et al.* [1] fitted a logistic function to verify asthma mortality trends and then compared this adjusted logistic with a second degree polynomial, with the aim of obtaining the maximum point [3].

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In this study, we will introduce 5 critical and important points of the logistic function: absolute acceleration point (PAA), maximum acceleration point (PAM), inflection point (PI), maximum deceleration point (PDM), asymptotic deceleration point (PDA). These points are given in order. For logistic function we will give three graphs. One of them is the growth function, the second one is rate function found by using the first derivative of the growth function. And the third one is acceleration function found by using the second derivative of the growth function.

First critical and important point is PAA which is found by using the acceleration function (y''). Actually this point is found by equalizing the fourth derivative of the growth function to zero. Before that point the acceleration is very slow. That's it could be thought that actual acceleration and growth start after this point. Because of this, this point could be called as the lag time point. After this point, the acceleration is increasing until PAM. PAM is the maximum acceleration point of the acceleration function found by using the second derivative of the growth function.

Actually this point is found by equalizing the third derivative of the growth function to zero. After this maximum acceleration point, acceleration decreases until the inflection point (PI) which is found by equalizing the second or the fourth derivative of the growth function to zero. At the inflection point while the rate function (y') has one maximum value, the value of the acceleration function (y'') is zero. After that inflection point, the acceleration function has negative values. That's the growth decelerates. The acceleration has a minimum value at the maximum deceleration point (PDM) which is found by using the second derivative of the growth function. Actually this point is found by equalizing the third derivative of the growth function to zero, too. And the acceleration goes to asymptotic deceleration point (PDA) by using the second derivative of the growth function. Actually, this point is found by equalizing the fourth derivative of the growth function to zero. After this point the deceleration is very slow and the graph of the acceleration function goes to zero as time goes to infinity. For that reason, after this point it is expected that the increasing in the growth function will be very small.

2. Material and Methods

Determinations of the asymptotic deceleration point- PDA and the absolute acceleration point-PAA

The logistic growth function and its first to fourth order derivatives were considered, all defined within the interval $-\infty < x < \infty$,

$$y = \alpha\,[1 + \exp(-\beta - \gamma x)]^{-1}$$

where x,y is time and growth value, respectively whileα is the average asymptotic value and β, γ are the parameters about growth.

$$\frac{dy}{dx} = \alpha\gamma \exp(-\beta - \gamma x)\,[1 + \exp(-\beta - \gamma x)]^{-2}$$

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$$\frac{d^2y}{dx^2} = \alpha\gamma^2 \exp(-\beta - \gamma x)\,(\exp(-\beta - \gamma x) - 1)\,[1 + \exp(-\beta - \gamma x)]^{-3}$$

$$\frac{d^3y}{dx^3} = \alpha\gamma^3 \exp(-\beta - \gamma x)\left(1 - 4\exp(-\beta - \gamma x) + (\exp(-\beta - \gamma x))^2\right)[1 + \exp(-\beta - \gamma x)]^{-4}$$

$$\tfrac{d^4y}{dx^4} = \alpha\gamma^4 \exp(-\beta - \gamma x)\left(-1 + 11\exp(-\beta - \gamma x) - 11(\exp(-\beta - \gamma x))^2 + (\exp(-\beta - \gamma x))^3\right)[1 + \exp(-\beta - \gamma x)]^{-5}$$

$$with\ \alpha,\beta\ and\ \gamma\ the\ parameters,\ where\ \alpha > 0\ and\ \gamma > 0.$$

The first derivative of the logistic function is always positive. When the second, third and fourth derivatives of the logistic function are equal to zero, respectively, we get the following:

$$\tfrac{d^2y}{dx^2} = 0\ \Rightarrow \exp(-\beta - \gamma x) = 1,\ from\ which$$
$$x_0 = -\beta/\gamma\ ,\ \ y_0 = \alpha/2$$

This point (x_0,y_0) is called the inflection point (PI)

$$\tfrac{d^3y}{dx^3} = 0\ \Rightarrow 1 - 4\exp(-\beta - \gamma x) + (\exp(-\beta - \gamma x))^2 = 0,\ from\ which$$
$$x_1 = -(\ln(2 + \sqrt{3}) + \beta)/\gamma\ ,\ \ y_1 = \alpha(3 - \sqrt{3})/6$$
$$x_2 = -(-\ln(2 - \sqrt{3}) + \beta)/\gamma\ ,\ \ y_2 = \alpha(3 + \sqrt{3})/6$$

The points (x_1,y_1) and (x_2,y_2) are called the maximum acceleration point (PAM) and the maximum deceleration point (PDM), respectively.

$$\frac{d^4y}{dx^4} = 0 \;\Rightarrow -1 + 11\exp(-\beta-\gamma x) - 11(\exp(-\beta-\gamma x))^2 + (\exp(-\beta-\gamma x))^3 = 0,$$

$from\ which$

$$x_3 = -(\ln(5+2\sqrt{6})+\beta)/\gamma\ ,\quad y_3 = \alpha(3-\sqrt{6})/6\quad ,$$
$$x_4 = -\beta/\gamma\ ,\quad y_4 = \alpha/2\quad ,$$
$$x_5 = -(\ln(5-2\sqrt{6})+\beta)/\gamma\ ,\quad y_5 = \alpha(3+\sqrt{6})/6\ .$$

The points (x_3,y_3), (x_4,y_4) and (x_5,y_5) are called the absolute acceleration point (PAA), the inflection point (PI) and the asymptotic deceleration point (PDA), respectively.

In this study, the data taken from the tree, *E. Camaldulensis* Dehn. were used for the logistic growth model in Table 1. The set of data were taken from the study of Yıldızbakan [4].

Table 1. The height growth value of the trees (*E. Camaldulensis* Dehn) according to year

The height growth value of the trees (*E. Camaldulensis* Dehn)										
Planting Age (year)	0	1	2	3	4	5	6	7	8	9
Height Growth (m)	0.41	3.23	7.45	11.41	14.83	18.11	18.95	19.69	21.50	23.40

3. Results and Discussion

By using Table 1 the estimates of the parameters of the Logistic growth model are:

$$\alpha = 21.89\ \ ,\ \beta = -2.37\ \ ,\gamma = \ 0.78$$

The logistic growth function is an increasing function w.r.t. time. The parameter α is the limit of the logistic function when time goes to infinity. That's $y = \alpha$ is the equation of the upper asymptote.

Considering the parameters α, $\beta$and γ of the data taken from *E. camaldulensis* Dehn., we have the figures of the logistic function of growth, growth rate and growth acceleration with PAA, PAM, PI, PDM and PDA.

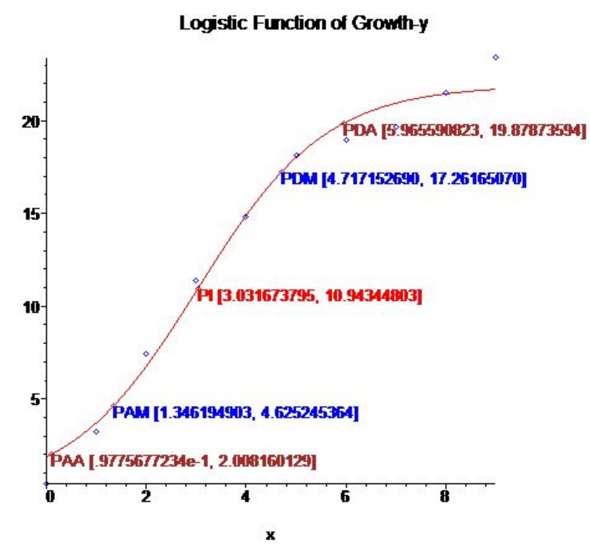

Figure 1. Logistic function of growth (y)

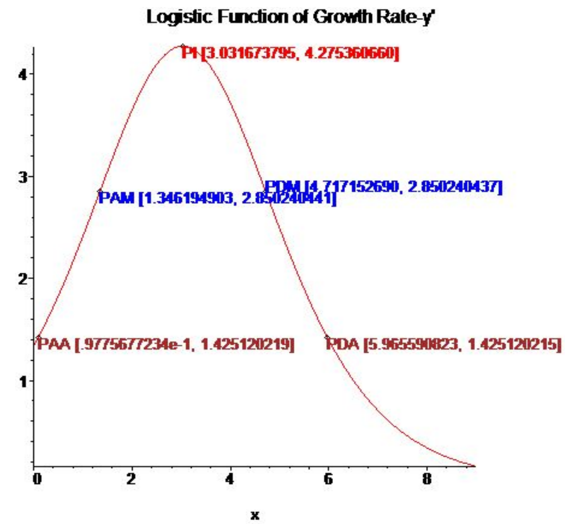

Figure2. Logistic function of growth rate (y')

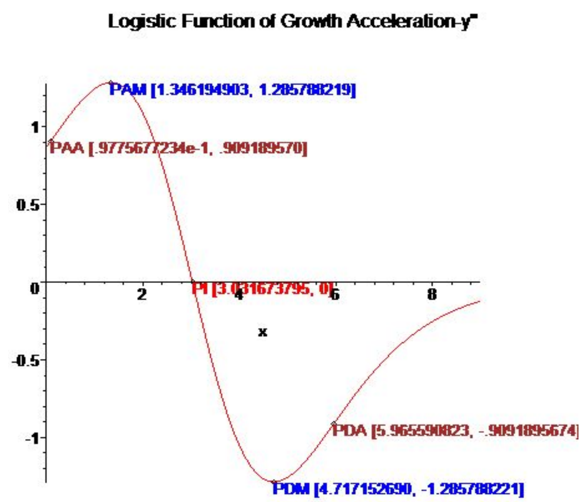

Figure 3. Logistic function of growth acceleration (y'')

In addition to the maximum acceleration point (PAM), the inflection point (PI) and the maximum deceleration point (PDM), the asymptotic deceleration point (PDA) and the absolute acceleration point(PAA) have the advantages of being mathematical points with important biological meaning. So, these points, together with the inflection point (PI), maximum acceleration point (PAM) and maximum deceleration point (PDM) can be used in discussions about biological growth curves.

4. Conclusion

The asymptotic deceleration point of the logistic growth model can be used as a criterion to determine when the growth reaches a value sufficiently close to the asymptote so that following increases can be ignored. Similarly, the absolute acceleration point of the logistic growth model can be used as a criterion to determine when the actual acceleration and growth start after this point so that prior increases can be ignored.

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