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Hermite-Hadamard Type Integral Inequalities for Strongly GA-convex Functions

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Abstract. In this paper we obtain the Hermite-Hadamard Inequality for strongly GA-convex function. Using this strongly GA-convex function we get the new theorem and corollary.

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1. Introduction

In recent years, several integral inequalities related to various classes of convex functions. Convex functions have played an important role in the development of various fields in pure and applied sciences. A significant class of convex functions is strongly convex functions. The strongly convex functions also play an important role in optimization theory and mathematical economics, see [7].

In this paper, we firstly list several definitions. Then, we discuss some properties of strongly GA-convex functions.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ the following inequalities holds.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

This inequality is known in the literature as Hermite-Hadamard inequality for convex functions. Note that some of the classical inequalities for means can be derived from this inequality for suitable specific selections of the mapping f. If both inequalities hold in the reversed, f is concave function ([8], [3]).

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Definition 1.1 ([7]). Let $I \subseteq \mathbb{R} \to \mathbb{R}$ be an interval and c be a positive number. A function $f : I = [a, b] \subset \mathbb{R}$ is called strongly convex function with modulus $c > 0$, if

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y) - ct(1-t)\|y - x\|^2 \in I$$

for all $x, y \in I, t \in [0, 1]$.

Some studies related to strongly convex functions and several different types of them can be found in the literature, for example , in [1, 2, 4, 9].

Definition 1.2 ([5, 6]). A function $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be GA-convex function on I if

$$f(x^t y^{1-t}) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$,where $x^t y^{1-t}$ and $tf(x) + (1-t)f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

Theorem 1.3. *Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a GA-convex function and $a, b \in I$ with $a < b$. If $f \in L[a, b]$ the following inequalities holds.*

$$f(\sqrt{ab}) \leq \frac{1}{\ln b - \ln a} \int_a^b f(x) \frac{1}{x} dx \leq \frac{f(a) + f(b)}{2}$$

This inequality known as Hermite-Hadamard inequality for GA-convex function.

2. Main Results

In this section, we derive Hermite-Hadamard inequalities for strongly GA-convex function.

Definition 2.1. Let I be a interval, $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ is said to be strongly GA-convex function with modulus $c > 0$, if

$$f(x^{1-t} y^t) \leq (1-t)f(x) + tf(y) - ct(1-t)\|\ln y - \ln x\|^2 \tag{2.1}$$

for all $x, y \in I$ ve $t \in [0, 1]$.

Lemma 2.2. *A function $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ is strongly GA-convex function with modulus $c > 0$, if and only if the function $g(x) = f(x) - c\|\ln x\|^2$ is GA-convex function.*

Proof. Assume that f is strongly GA-convex function with modulus $c > 0$. Using properties of the inner product, we have

$$\begin{aligned}
g(x^t y^{1-t}) &= f(x^t y^{1-t}) - c\|\ln(x^t y^{1-t}\|^2 \\
&\leq tf(x) + (1-t)f(y) - ct(1-t)\|\ln y - \ln x\|^2 - c\|\ln x^t + \ln y^{1-t}\|^2 \\
&\leq tf(x) + (1-t)f(y) - c(t(1-t)\|\ln y\|^2 - 2t(1-t)\ln y \ln x \\
&\quad + t(1-t)\|\ln x\|^2 + \|\ln x^t\|^2 + 2\ln x^t \ln y^{1-t} + \|\ln y^{1-t}\|^2) \\
&\leq tf(x) + (1-t)f(y) - c(t\|\ln x\|^2 + (1-t)\|\ln y\|^2) \\
&\leq tf(x) - ct\|\ln x\|^2 + (1-t)f(y) - c(1-t)\|\ln y\|^2 \\
&= tg(x) + (1-t)g(y)
\end{aligned}$$

which gives that g is GA-convex function. Conversely, if g is GA-convex function, then we have

$$\begin{aligned}
f(x^t y^{1-t}) &= g(x^t y^{1-t}) + c\|\ln x^t y^{1-t}\|^2 \\
&\leq tg(x) + (1-t)g(y) + c\|\ln x^t + y^{1-t}\|^2 \\
&\leq tg(x) + (1-t)g(y) + c(1-t)\|\ln y\|^2 - ct(1-t)\|\ln y\|^2 + 2ct(1-t)\ln x \ln y \\
&\quad + ct\|\ln x\|^2 - ct(1-t)\|\ln x\|^2 \\
&= tg(x) + ct\|\ln x\|^2 + (1-t)g(y) + c(1-t)\|\ln y\|^2 - ct(1-t)\|\ln y - \ln x\|^2 \\
&= tf(x) + (1-t)f(y) - ct(1-t)\|\ln y - \ln x\|^2
\end{aligned}$$

which shows that f is strongly GA-convex function with modulus $c > 0$. □

Theorem 2.3. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a strongly GA-convex function with modulus $c > 0$. If $f \in L[a, b]$, then*

$$f(\sqrt{ab}) + \frac{c}{12}\|\ln b - \ln a\|^2 \leq \frac{1}{\ln b - \ln a}\int_a^b f(x)\frac{1}{x}dx \leq \frac{f(a) + f(b)}{2} - \frac{c}{6}\|\ln b - \ln a\|^2 \tag{2.2}$$

Proof. Since $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a strongly GA-convex function, we have, $\forall x, y \in I$, (with $t = \frac{1}{2}$ in the inequality (2.1)).

$$f(\sqrt{xy}) \leq \frac{f(x) + f(y)}{2} - \frac{c}{4}\|\ln y - \ln x\|^2$$

Choosing $x = a^{1-t}b^t, y = a^t b^{1-t}$, we get

$$f(\sqrt{ab}) \leq \frac{f(a^{1-t}b^t) + f(a^t b^{1-t})}{2} - \frac{c}{4}\|\ln a^t b^{1-t} - \ln a^{1-t}b^t\|^2$$

By integrating for $t \in [0, 1]$, we have

$$f(\sqrt{ab}) \leq \frac{1}{2}\Big[\int_0^1 f(a^{1-t}b^t)dt + \int_0^1 f(a^t b^{1-t})dt\Big]$$

$$-\frac{c}{4}\|\ln b - \ln a\|^2\int_0^1 (1 - 2t)^2 dt$$

$$f(\sqrt{ab}) + \frac{c}{12}\|\ln b - \ln a\|^2 \leq \frac{1}{\ln b - \ln a}\int_a^b f(x)\frac{1}{x}dx$$

We get the left hand side of the inequality (2.2). Furthermore, we observe that $\forall t \in [0, 1]$

$$f(a^{1-t}b^t) \leq (1 - t)f(a) + tf(b) - ct(1 - t)\|\ln b - \ln a\|^2$$

By integrating this inequality with respect to t over $[0, 1]$, we have the right-hand side of the inequality (2.2).

$$\leq \int_0^1 ((1 - t)f(a) + tf(b))dt - c\|\ln b - \ln a\|^2\int_0^1 t(1 - t)dt$$

$$\leq \frac{f(a) + f(b)}{2} - \frac{c}{6}\|\ln b - \ln a\|^2$$

□

Theorem 2.4. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a strongly GA-convex function with modulus $c > 0$, and $\forall x, y \in I, t \in [0, 1]$. Then*

$$f(\sqrt{ab}) + \frac{c}{12}\|\ln b - \ln a\|^2 \leq \phi(x)$$

$$\leq \frac{1}{\ln b - \ln a}\int_a^b f(x)\frac{1}{x}dx$$

$$\leq \psi(x) \leq \frac{f(a) + f(b)}{2} - \frac{c}{6}\|\ln b - \ln a\|^2$$

where

$$\phi(x) = \frac{1}{2}\Big[f(a^{\frac{3}{4}}b^{\frac{1}{4}}) + f(a^{\frac{1}{4}}b^{\frac{3}{4}})\Big] + \frac{c}{48}\|\ln b - \ln a\|^2$$

$$\psi(x) = \frac{1}{2}\Big[f(\sqrt{ab}) + \frac{f(a) + f(b)}{2}\Big] - \frac{c}{24}\|\ln b - \ln a\|^2$$

Proof. By applying (2.1) on each of the interval $\left[a, \sqrt{ab}\right]$ ve $\left[\sqrt{ab}, b\right]$, we have

$$f\left(\sqrt{a\sqrt{ab}}\right) + \frac{c}{12}\|\ln\sqrt{ab} - \ln a\|^2$$

$$\leq \frac{1}{\ln\sqrt{ab} - \ln a}\int_a^{\sqrt{ab}} f(x)\frac{1}{x}dx \leq \frac{f(a) + f(\sqrt{ab})}{2} - \frac{c}{6}\|\ln\sqrt{ab} - \ln a\|^2$$

$$f(a^{\frac{3}{4}}b^{\frac{1}{4}}) + \frac{c}{48}\|\ln b - \ln a\|^2$$

$$\leq \frac{2}{\ln b - \ln a}\int_a^{\sqrt{ab}} f(x)\frac{1}{x}dx \leq \frac{1}{2}\left[f(a) + f(\sqrt{ab})\right] - \frac{c}{24}\|\ln b - \ln a\|^2$$

and similarly,

$$f(a^{\frac{1}{4}}b^{\frac{3}{4}}) + \frac{c}{48}\|\ln b - \ln a\|^2$$

$$\leq \frac{2}{\ln b - \ln a}\int_{\sqrt{ab}}^{b} f(x)\frac{1}{x}dx \leq \frac{1}{2}\left[f(\sqrt{ab}) + f(b)\right] - \frac{c}{24}\|\ln b - \ln a\|^2$$

Summing up side by side, we obtain

$$\phi(x) = \frac{1}{2}\left[f(a^{\frac{3}{4}}b^{\frac{1}{4}}) + f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right] + \frac{c}{48}\|\ln b - \ln a\|^2$$

$$\leq \frac{1}{\ln b - \ln a}\int_a^{b} f(x)\frac{1}{x}dx \leq \frac{1}{2}\left[f(\sqrt{ab}) + \frac{f(a)f(b)}{2}\right] - \frac{c}{24}\|\ln b - \ln a\|^2$$

$$\leq \frac{1}{2}\left[\frac{f(a) + f(b)}{2} + \frac{f(a) + f(b)}{2} - \frac{c}{4}\|\ln b - \ln a\|^2\right] - \frac{c}{24}\|\ln b - \ln a\|^2$$

$$\leq \frac{f(a) + f(b)}{2} - \frac{c}{6}\|\ln b - \ln a\|^2$$

□

Theorem 2.5. *Let $f, g : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a strongly GA-convex function with modulus $c > 0$. If $f, g \in L[a, b]$, then*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)g\left(\frac{ab}{x}\right)\frac{1}{x}dx$$

$$\leq \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) - \frac{c}{12}\|\ln b - \ln a\|^2 S(a,b) - \frac{c^2}{30}\|\ln b - \ln a\|^4$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b) \tag{2.3}$$

$$N(a,b) = f(a)g(b) + f(b)g(a) \tag{2.4}$$

$$S(a,b) = f(a) + f(b) + g(a) + g(b) \tag{2.5}$$

Proof. Let f, g be strongly GA-convex functions with modulus $c > 0$. Then

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)g\Big(\frac{ab}{x}\Big)\frac{1}{x}dx = \int_0^1 f(a^{1-t}b^t)g(a^t b^{1-t})dt$$

$$\leq \int_0^1 \Big[(1-t)f(a) + tf(b) - ct(1-t)\|\ln b - \ln a\|^2\Big]\Big[tg(a) + (1-t)g(b) - ct(1-t)\|\ln b - \ln a\|^2\Big]dt$$

$$= f(a)g(b)\int_0^1 (1-t)^2dt + f(b)g(a)\int_0^1 t^2dt + [f(a)g(a) + f(b)g(b)]\int_0^1 t(1-t)dt$$

$$-c\|\ln b - \ln a\|^2[f(a) + g(b)]\int_0^1 t(1-t)^2dt - c\|\ln b - \ln a\|^2[f(b) + g(a)]\int_0^1 t^2(1-t)dt$$

$$-c^2\|\ln b - \ln a\|^4\int_0^1 t^2(1-t)^2dt$$

$$= \frac{f(a)g(b) + f(b)g(a)}{3} + \frac{f(a)g(a) + f(b)g(b)}{6}$$

$$-\frac{c}{12}\|\ln b - \ln a\|^2[f(a) + f(b) + g(a) + g(b)] - \frac{c^2}{30}\|\ln b - \ln a\|^4$$

$$= \frac{1}{6}M(a,b) + \frac{1}{3}N(a,b) - \frac{c}{12}\|\ln b - \ln a\|^2 S(a,b) - \frac{c^2}{30}\|\ln b - \ln a\|^4$$

□

If $f = g$ in Theorem 2.5, then it reduces to the following result.

Corollary 2.6. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a strongly GA-convex function with modulus $c > 0$. If $f \in L[a, b]$, then*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)f\Big(\frac{ab}{x}\Big)\frac{1}{x}dx$$

$$\leq \frac{2[f(a)f(b)]}{3} + \frac{f^2(a) + f^2(b)}{6} - \frac{c}{6}\|\ln b - \ln a\|^2[f(a) + f(b)] - \frac{c^2}{30}\|\ln b - \ln a\|^4$$

Theorem 2.7. *Let $f, g : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a strongly GA-convex function with modulus $c > 0$. If $fg \in L[a, b]$, then*

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)g(x)\frac{1}{x}dx$$

$$\leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) - \frac{c}{12}\|\ln b - \ln a\|^2 S(a,b) - \frac{c^2}{30}\|\ln b - \ln a\|^4$$

where $M(a,b), N(a,b), S(a,b)$ are given by (2.3), (2.4) and (2.5), respectively.

Proof. Let f, g be strongly GA-convex functions with modulus $c > 0$. Then

$$\frac{1}{\ln b - \ln a}\int_a^b f(x)g(x)\frac{1}{x}dx = \int_0^1 f(a^{1-t}b^t)g(a^{1-t}b^t)dt$$

$$\leq \Big[(1-t)f(a) + tf(b) - ct(1-t)\|\ln b - \ln a\|^2\Big]\Big[(1-t)g(a) + tg(b) - ct(1-t)\|\ln b - \ln a\|^2\Big]dt$$

$$= f(a)g(a)\int_0^1 (1-t)^2dt + f(b)g(b)\int_0^1 t^2dt + [f(a)g(b) + f(b)g(a)]\int_0^1 t(1-t)dt$$

$$-c\|\ln b - \ln a\|^2[f(a) + g(a)]\int_0^1 t(1-t^2)dt - c\|\ln b - \ln a\|^2[f(b) + g(b)]\int_0^1 t^2(1-t)dt$$

$$-c^2\|\ln b - \ln a\|^4\int_0^1 t^2(1-t)^2dt$$

$$= \frac{f(a)g(a) + f(b)g(b)}{3} + \frac{f(a)g(b) + f(b)g(a)}{6} - \frac{c}{12}\| \ln b - \ln a\|^2[f(a) + f(b) + g(a) + g(b)]$$
$$-\frac{c^2}{30}\| \ln b - \ln a\|^4$$
$$\frac{1}{3}M(a,b) + \frac{1}{6}N(a,b) - \frac{c}{12}\| \ln b - \ln a\|^2 S(a,b) - \frac{c^2}{30}\| \ln b - \ln a\|^4$$

□

If $f = g$ in Theorem 2.7, then it reduces to the following result.

Corollary 2.8. *Let $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be a strongly GA-convex function with modulus $c > 0$. If $f \in L[a,b]$, then*

$$\frac{1}{\ln b - \ln a}\int_a^b f^2(x)\frac{1}{x}dx$$
$$\leq \frac{[f(a)f(b)]}{3} + \frac{f^2(a) + f^2(b)}{3} - \frac{c}{6}\| \ln b - \ln a\|^2[f(a) + f(b)] - \frac{c^2}{30}\| \ln b - \ln a\|^4$$

References

[1] Angulo, H., Gimenez, J., Moros, A.M. and Nikodem, K., *On strongly h-convex functions*, Annals of Functional Analysis, **2**(2011), 85–91. 1

[2] Erdem, Y., Ögünmez, H. and Budak, H., *Some generalized inequalities of Hermite-Hadamard type for strongly s-convex functions*, New Trends in Mathematical Sciences, **3**(2017), 22–32. 1

[3] Işcan, I., *Hermite-Hadamard type inequaities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistic, **43**(2014), 935–942. 1

[4] Nikodem, K., Sanchez, J.L. and Sanchez, L., *Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps*, Mathematica Aeterna, **4**(2014), 979–987. 1

[5] Niceluscu, P.C., *Convexity according to the geometric mean*, Mathematical Inequalities and Applications, **2**(2000), 155–167. 1.2

[6] Niculescu, P.C., *Convexity according to means*, Mathematical Inequalities and Applications, **4**(2003), 571–579. 1.2

[7] Noor, M.A., Noor, K.I. and Iftikhar, S., *Hermite-Hadamard inequalities for strongly harmonic convex functions*, Journal of Inequalities and Special Functions, **7**(2016), 99–113. 1, 1.1

[8] Turhan, S., Işcan, I. and Kunt, M., *Hermite-Hadamard type inequalities for $M_\varphi A$ convex functions*, 2016. 1

[9] Turhan, S, Okur, N. and Maden, S., *Hermite-Hadamard type inequality for strongly convex functions via sugeno integrals*, Sigma J Eng. and Nat. Sci, **8**(2016), 1–10. 1