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Numerical solution of linear integro-differential equation by using modified Haar wavelets

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Abstract

In this paper, we introduce a numerical method for solving linear Fredholm integro-differential equations of the first order. To solve these equations, we consider the equation solution approximately from rationalized Haar (RH) functions.

The numerical solution of a linear integro-differential equation reduces to solving a linear system of algebraic equations. Also, Some numerical examples are presented to illustrate the efficiency of the method.

Keywords: Block-Pulse Functions, Operational matrix, Volterra integral equations, Integro-differential equations, RHFs.

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1. Introduction

Some important problems in science and engineering can usually be reduced to a system of integral and integro-differential equations. Integro-differential equations have attracted much attention and solving such equations has been one of the interesting tasks for mathematicians. Several methods have been proposed for numerical solution of these equations (see, e.g., [12]). One technique is the collocation method; of numerous research papers about this approach we cite here ([6], [18]). Since 1991 the wavelet method has been applied to solving integral equations. Various wavelet bases have been employed. In addition to the conventional Daubechies wavelets [12], the Hermite-type trigonometric wavelets [8], linear B-splines [2], Walsh functions [9], Cohen [8] and Fariborzi [10] wavelets have been used. These solutions are often quite complicated, therefore simplifications are welcome. One possibility is to make use of Haar wavelets, which are mathematically the simplest wavelets. For linear integral equations this approach has

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been realized in ([5], [13]). In this paper we examine the rate of convergence of the modified rationalized method using Haar functions for solving Fredholm integro-Differential equations combined with finite difference methods.

Solving the algebraic system obtained by the (RH) functions method allows one to obtain first derivative approximations using a central difference scheme. We apply the proposed method on some test problems to show its accuracy and efficiency. Also, the error evaluation of this method is presented. Before starting, let us recall some definitions.

1.1. Definition. ([5]) The Haar wavelet is the function defined on the real line $\mathbb{R}$ as:

$$H(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

The Haar wavelet $H(t)$ can be used to define a sequence of one-dimensional (RH)functions on $[0, 1)$ as follows:

1.2. Definition. ([5]) The (RH)functions $h_n(t)$, for $n = 2^i + j$ with $i \in \mathbb{Z}$ and $j = 0, 1, \ldots, 2^{i-1}$, are the functions defined on the interval $[0, 1)$ as:

$$h_n(t) = H(2^n t - j)_{|[0,1)} \tag{1.2}$$

Also, we define $h_0(t) = 1$ for all $t \in [0, 1)$.

In Eq.(1.2), are the orthogonal set of rationalized Haar functions and can be defined on the interval $[0, 1)$ as [17]:

$$RH(r,\ t) = h_r(t) = \begin{cases} 1, & \text{if } J_1 \leq t < J_{(\frac{1}{2})}, \\ -1, & \text{if } J_{(\frac{1}{2})} \leq t < J_0, \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

where, $J_u = \dfrac{j-u}{2^i}$, $u = 0,\ \dfrac{1}{2}, 1$.

The value of r is defined by two parameters i and j as:

$$r = 2^i + j - 1,\ i = 0, 1, 2 \ldots,\ j = 1, 2,\ \ldots,\ 2^i$$

$h_0(t)$ is defined for $i = j = 0$ and given by:

$$h_0(t) = 1,\ 0 \leq t < 1 \tag{1.4}$$

$h_0(t)$ is also included to make this set complete. The orthogonality property is given by:

$$\int_0^1 RH(r,\ t) RH(v,\ t) dt = \begin{cases} 2^{-i}, & r = v \\ 0 & r \neq v \end{cases} \tag{1.5}$$

where

$$v = 2^n + m - 1,\ n = 0, 1, 2, 3,\ \ldots,\ m = 1, 2, 3,\ \ldots,\ 2^n$$

2. Function Approximation

Any function $f(t)$ defined over the interval $[0,1)$, which is $\mathbb{L}^2([0,1))$, can be expanded in (RH) functions as([23]);

$$f(t) = \sum_{r=0}^{+\infty} \alpha_r RH(r,\ t),\ r = 0,1,2,\ \ldots.. \tag{2.1}$$

where the (RH) function coefficients α_r are given by:

$$\alpha_r = \frac{\langle f(t), RH(r,t)\rangle}{\langle RH(r,t), RH(r,t)\rangle} = 2^i \int_0^1 f(t)RH(r,\ t)dt,\ r = 0,1,2,\ \ldots \tag{2.2}$$

with $r = 2^j + i - 1,\ i = 0,1,2,3,\ \ldots,\ j = 1,2,3,\ \ldots,\ 2^n$ and $r = 0$ for $i = j = 0$.

Usually, the series expansion of Eq. (2.1) contains infinite terms. If $f(t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then Eq. (2.1) will be terminated at finite terms. Otherwise, it is truncated up to its first m terms as:

$$f(t) \approx \sum_{r=0}^{k-1} a_r RH(r,\ t) = A^T \phi(t) \tag{2.3}$$

where $k = 2^{\alpha+1}$, and $\alpha = 0,1,2,3,\ \ldots$.

The (RH) function coefficients vector $\phi(t)$ and (RH) functions vector $h(t)$ are defined as;

$$A = [a_0, a_1, a_2,\ \ldots, a_{k-1}]^T \tag{2.4}$$

and

$$\phi(t) = [h_0,\ h_1,\ h_3,\ \ldots..,\ h_{k-1}]^T \tag{2.5}$$

where

$$h_r(t) = RH(r,\ t),\ r = 0,1,2,\ \ldots,\ k-1 \tag{2.6}$$

Babolian et al. proved in [21] that:

$$\begin{aligned} &\|f(t)\ - \sum_{r=0}^{k-1} a_r RH(r,\ t)\ \|_{\mathbb{L}^2}^2 == \|\sum_{r=k}^{+\infty} a_r h_r(t)\ \|_{\mathbb{L}^2}^2 \\ &\leq \sum_{r=k}^{+\infty} |a_r|^2 \|h_r(t)\ \|_{\mathbb{L}^2}^2 \\ &= \sum_{r=k}^{+\infty} |a_r|^2 \\ &\sim \sum_{r=k}^{+\infty} 2^{-r} \sim 2^{-k} = O(2^{-k}) \leq C2^{-k} \end{aligned} \tag{2.7}$$

where C is a constant of integration.

Now, let $k(t,\ s)$ be a function of two independent variables defined for $t \in [0,1)$ and $s \in [0,1)$. Then $k(t,\ s)$ can be expanded in (RH) functions as:

$$k(t,\ s) = \sum_{u=0}^{m-1}\sum_{v=0}^{m-1} h_{uv}h_v(t)h_u(s) \tag{2.8}$$

In Eq. (2.8) h_{uv}, for $u = 0,1,2,..,m-1$ and $v = 0,1,2,...,m-1$, is given as:

$$h_{uv} = 2^{i+q}\int_0^1\int_0^1 k(s,\ t)h_v(t)h_u(s)dtds \tag{2.9}$$

where

$u = 2^i + j,\ i \geq 0$ and $0 \leq j < 2^i v = 2^q + r,\ q \geq 0$ and $0 \leq r < 2^q$ hence we have

$$k(t,\ s) = \phi^T(t)H\phi(s) \tag{2.10}$$

where

$$H = (\hat{\Phi}_{k\times k}^{-1})^T\hat{H}\hat{\Phi}_{k\times k}^{-1} \tag{2.11}$$

with

$$\hat{H} = (h_{uv})_{k\times k}^T \tag{2.12}$$

Where $\hat{H}$ is an $k \times k$ matrix such that:

$$h_{ij} = \frac{\langle RH(i,t), \langle k(t,s), RH(j,s)\rangle\rangle}{\langle RH(i,t), RH(i,t)\rangle\langle RH(j,t), RH(j,t)\rangle} \tag{2.13}$$

Take the Newton-Côtes nodes as:

$$t_i = \frac{2i-1}{2k},\ i = 1,2,..,k \tag{2.14}$$

$$\hat{h}_{lp} = k(\frac{2l-1}{2k},\ \frac{2p-1}{2k}),\ p,\ l = 1,2,....,k. \tag{2.15}$$

2.1. Operational matrix of integration. Discrete Haar functions of order k represented by $2^k \times 2^k$ matrix $\hat{\Phi}_{k\times k}$, in the sequency ordering are given by the following recurrence relation ([23]):

$$\hat{\Phi}_{k\times k} = \left\{\begin{array}{ll} \hat{\Phi}_{\frac{k}{2}\times\frac{k}{2}} & \otimes[1 \quad 1] \\ I_{\frac{k}{2}\times\frac{k}{2}} & \otimes[1 \quad -1] \end{array}\right\} \tag{2.16}$$

$$\Phi_{1\times 1} = [1] \tag{2.17}$$

where $I_{\frac{k}{2}\times\frac{k}{2}}$ is the identity matrix of dimension k and $\otimes$ is the Kronecker product.

The integration of (RH) functions can be expanded into Haar series with Haar coefficient matrix P as follows:

$$\int_0^1 t\phi(x)dx = P\phi(t) \tag{2.18}$$

The $k \times k$ square matrix $P = P_k$ is called the operational matrix of integration and is given in [7] as:

$$P_k = \frac{1}{2k}\begin{pmatrix} 2kP_{\frac{k}{2}} & -\hat{\Phi}_{\frac{k}{2}} \\ -\hat{\Phi}_{\frac{k}{2}} & 0 \end{pmatrix} \tag{2.19}$$

where $\hat{\Phi}_1^{-1} = [1],\ P_1 = [\frac{1}{2}], \hat{\Phi}_k$ is given by Eq. (2.16) and

$$\hat{\Phi}_k^{-1} = \frac{1}{k}\hat{\Phi}_k^T\, diag\left(1, 1, 2, 2, \underbrace{2^2, .., 2^2}_{2^2}, ..., \underbrace{2^{\alpha-1}, .., 2^{\alpha-1}}_{2^{\alpha-1}}\right) \tag{2.20}$$

Also, the integration of the cross-product of two (RH) function vector is:

$$\int_0^1 \phi(t)\phi^T(t)dt = D \tag{2.21}$$

where D is a diagonal matrix given by:

$$D = diag\left(1, 1, 2, 2, \underbrace{2^2, .., 2^2}_{2^2}, ..., \underbrace{2^{\alpha-1}, .., 2^{\alpha-1}}_{2^{\alpha-1}}\right) \tag{2.22}$$

2.2. The product operational matrix. ([23])

Let the product of $\phi(t)$ and $\phi^T(t)$ be called the (RH) product matrix $\psi_{k\times k}(t)$. That is:

$$\phi(t)\phi^T(t) = \psi_{k\times k}(t) \tag{2.23}$$

The basic multiplication properties of (RH) functions are as:

$$h_0(t)h_i(t) = h_i(t\ ),\ i = 0, 1,\ \ldots,\ m-1 \tag{2.24}$$

and for $i < j$, we have

$$h_i(t)h_j(t) = \begin{cases} h_j(t), & \text{if } h_j \textit{ occurs during the positive } \text{half-}\textit{wave of } h_i \\ -h_j(t), & \text{if } h_j \textit{ occurs during the negative } \text{half-}\textit{wave of } h_i \\ 0 & \text{otherwise.} \end{cases} \tag{2.25}$$

Also, the square of any (RH) functions is a block-pulse, with magnitude unity during both the positive and negative half-waves of (RH) functions.

For notation simplification, let us define:

$$\hat{\phi}_a(t) = [h_0(t),\ \ldots,\ h_{k/2-1}(t)]^T \tag{2.26}$$

$$\hat{\phi}_b(t) = [h_{k/2}(t),\ \ldots,\ h_{k-1}(t)]^T \tag{2.27}$$

The matrix $\psi_{k\times k}(t)$ in Eq. (2.23) can be derived easily as follows from ([7]):

$$\psi_{k\times k}(t)=\begin{bmatrix}\psi_{k/2}(t) & D_{k/2}diag[\hat{\phi}_b(t)]\\ diag[\hat{\phi}_b(t)]D_{k/2}^T & di\alpha g[D_{k/2}^{-1}\hat{\phi}_a(t)]\end{bmatrix}\tag{2.28}$$

where

$$\psi_1(t)=[h_0(t)]\tag{2.29}$$

With the above recursive formulas, we can evaluate $\psi_k(t)$ for any $k=2^\alpha$, where α is a positive integer. Furthermore, by multiplying the matrix $\psi_k(t)$ in Eq. (2.23) by the vector A in Eq. (2.3) we obtain:

$$\psi_k(t)A=\tilde{A}_k\phi(t)\tag{2.30}$$

Where $\tilde{A}_k$ is a $k\times k$ given by [7]:

$$\tilde{A}_k=\begin{bmatrix}\tilde{A}_{k/2}(t) & D_{k/2}diag[\tilde{c}_b]\\ diag[\tilde{c}_b]D_{k/2}^{-1}(t) & diag[\tilde{c}_b^T D_{k/2}]\end{bmatrix}\tag{2.31}$$

where $C_1=c_0$, and

$$\tilde{c}_a=[c_0,\ \ldots,\ c_{k/2-1}]^T\tag{2.32}$$

$$\tilde{c}_b=[c_{k/2},\ \ldots,\ c_{k-1}]^T\tag{2.33}$$

3. Application of HAAR wavelet method

3.1. Solution of the Linear Fredholm Integro-Differential Equation. Consider the linear Fredholm integro-differential equation given by:

$$\begin{cases} q(t)y'(t)=\int_0^1 k(t,\ s)y(s)ds\ +r(t)y(t)\ +x(t)\\ y(0)=y_0\end{cases}\tag{3.1}$$

where the functions $x,\ q,\ r\in\mathbb{L}^2([0,1))$, the kernel $k\in\mathbb{L}^2([0,1)\times[0,1))$ are known and $y(t)$ is the unknown function to be determined.

We approximate $x,\ q,\ r,\ y'$ and k using Haar wavelet space as follows:

$$\begin{cases} y(t)\ =Y^T\phi(t)=\phi^T(t)Y\\ y'(t)=Y'^T\phi(t)=\phi^T(t)Y'\\ y(0)=Y_0^T\phi(t)=\phi^T(t)Y_0\\ x(t)\ =X^T\phi(t)=\phi^T(t)X\\ k(t,\ s)\ =\psi^T(t)K\psi(s)=\psi^T(s)K^T\psi(t)\\ r(t)\ =R^T\phi(t)=\phi^T(t)R\\ q(t)\ =Q^T\phi(t)=\phi^T(t)Q\end{cases}\tag{3.2}$$

where $\phi(t)$ is given by Eq. (2.5) and Y is an unknown $m\times 1$ vector.

k is a known $m\times m$ dimensional matrix given by Eq. (2.8) and X is a known $m\times 1$ vector given by Eq. (2.3).

Substituting Eq. (3.2) into (3.1) we have:

$$
\begin{aligned}
Q^T\phi(t)\phi^T(t)Y' &= \int_0^t \phi^T(t)H\phi(s)\phi^T(s)(P^TY'+Y_0)ds \\
&\quad +R^T\phi(t)\phi^T(t)(P^TY'+Y_0)+X^T\phi(t) \qquad (3.3)
\end{aligned}
$$

we have $\phi(t)\phi^T(t)=\psi_{k\times k}(t)$

$$Q^T\psi_{k\times k}(t)Y' = \int_0^t \phi^T(t)H\psi_{k\times k}(s)(P^TY'+Y_0)ds \qquad (3.4)$$

$$+R^T\psi_{k\times k}(t)(P^TY'+Y_0)+X^T\phi(t) \qquad (3.5)$$

$$Q^T\psi_{k\times k}(t)Y' = \phi^T(t)H\int_0^t \psi_{k\times k}(s)(P^TY'+Y_0)ds \qquad (3.6)$$

$$+R^T\psi_{k\times k}(t)(P^TY'+Y_0)+X^T\phi(t) \qquad (3.7)$$

by Eq. (2.21) and by Eq. (2.23), we have $Q^T\psi_{k\times k}(t)=\psi_{k\times k}(t)Q=\widetilde{Q}\phi(t)$

$$\phi^T\widetilde{Q}Y' = \phi^T(t)HD(P^TY'+Y_0)+\phi^T(t)\widetilde{R}(P^TY'+Y_0)+\phi^T(t)X \qquad (3.8)$$

or

$$(\widetilde{Q}\ -HDP^T-RP^T)Y' = HDPY_0+\widetilde{R}Y_0+X \qquad (3.9)$$

By solving this linear system we can obtain the vector Y'. Thus,

$$y^{'}(t)=Y'^T\phi(t)=\phi^T(t)Y' \qquad (3.10)$$

Eq. (3.9) can be solved for the unknown vector Y'.

The numerical solution y_k is obtained by using finite differences formulas to approximate the first time derivative. In general, the first order derivative of second order error central difference formula can be derived from the Taylor series expansion as follows:

The Algorithm

Step 1:

Put $h=\dfrac{1}{k},\ k\in\mathbb{N},\ ,\ y(0)=y_0$ (initial condition is given)

Step 2:

Set $t_i=ih$, with $t_0=0$ and $t_k=1,\ i=0,1,\ \ldots,\ k$.

Step 3:

for $i=1,2,\ \ldots,k-1$

$$Y'(t_i)\approx\frac{Y(t_{i+1})-Y(t_{i-1})}{2h} \qquad (3.11)$$

for $i=k$

$$Y'(t_i)\approx\frac{3Y(t_i)-4Y(t_{i-1})+Y(t_{i-2})}{2h} \qquad (3.12)$$

Use step 1 and step 2, 3 to find the approximate value of y_k. Where $h\approx\dfrac{1}{k}$ is interval length between nodes.

4. Numerical Examples

In this section, we consider three integro-differential equations. We apply the system of equations in (3.8) and (3.9-3.10). The programs have been provided by **MATLAB 7.8**.

The $\mathbb{L}^2$, $\mathbb{L}^\infty$ error and rate of convergence are defined to be, respectively:

$$e_2 = \|y_k(t) - y_{ex}(t)\|_2 = (\int_0^1 (y_k(t) - y_{ex}(t))^2 dx)^{\frac{1}{2}} \tag{4.1}$$

$$e_\infty = \max_{1 \le i \le 2M} |y_k(t_i) - y_{ex}(t_i)| \tag{4.2}$$

$$\rho_{2,\infty} = \frac{\log[e_{2,\infty}(\frac{k}{2})/e_{2,\infty}(k)]}{\log(2)} \tag{4.3}$$

where $y_{ex}(t)$ is the exact solution and $y_k(t)$ is the approximate solution obtained by Eq. (3.11-3.12).

4.1. Example. Consider the following linear Fredholm integro-differential equation:

$$y'(t) \quad = \quad \int_0^1 e^{ts} y(s) ds + y(t) + \frac{1 - e^{t+1}}{1 + t}, \tag{4.4}$$

with initial condition $y(0) = 1$.

The exact solution is as follows: $y(t) = e^t$.

The numerical results are shown in table (1) and in figures (1, 2). Table (1) shows the behaviour of the error for the norm $\mathbb{L}^2$ and norm $\mathbb{L}^\infty$ in function of the parameter of discretization h for different values of k. Note that as h approaches zero, the numerical solution converges to the analytical solution $y(t)$.

k	$\mathbf{L}_2$	$\mathbf{L}_\infty$
8	$8.6353e-002$	$5.4116e-002$
16	$2.9590e-002$	$1.4152e-002$
32	$1.0405e-002$	$3.8126e-003$
64	$3.6778e-003$	$9.9140e-004$
128	$1.3009e-003$	$2.5290e-004$
256	$4.6012e-004$	$6.3875e-005$
512	$1.6271e-004$	$1.6051e-005$
1024	$5.7535e-005$	$4.0231e-006$
2048	$2.0343e-005$	$1.0071e-006$
4096	$7.1925e-006$	$2.5193e-007$
Convergence rate	2.8284	3.9975

Table 1. The errors estimates $\mathbb{L}^2$, $\mathbb{L}^\infty$ and convergence rates $\rho_{2,\infty}$

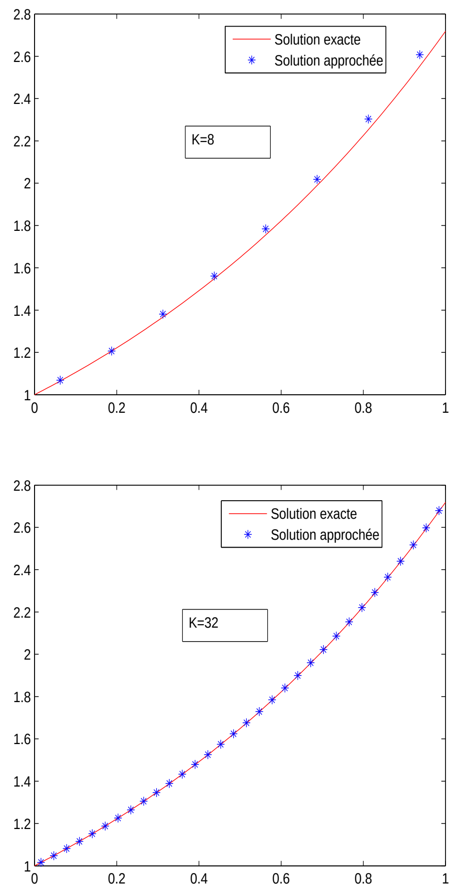

Figure 1. Comparison between approximate solution y_k and exact solution y_{ex}

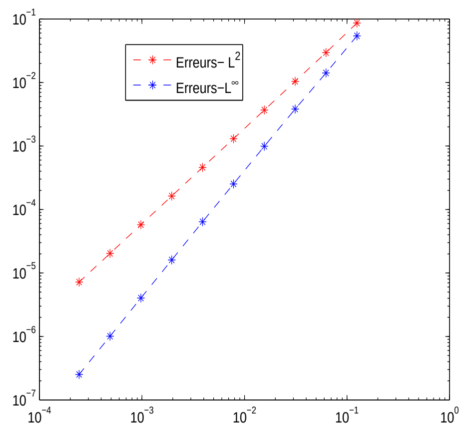

Figure 2. The errors $\mathbb{L}^2$ and $\mathbb{L}^\infty$ with different values of k.

We have also calculated the experimental rate of convergence $\rho_{2,\infty}$. We notice from Table (1) that the convergence rates are 2.8284 in $\mathbb{L}^2$ norm and 3.9975 in norm $\mathbb{L}^\infty$, which is approximately $2\sqrt{2}$ and 4 respectively.

4.2. Example. Consider the following linear Fredholm integro-differential equation:

$$y'(t) = 1 - \frac{1}{3}t \ + \int_0^1 tsy(s)ds \tag{4.5}$$

with initial condition $y(0) = 0$.

The exact solution is as follows: $y(t) = t$.

The numerical results are shown in table (2) and in figures (3, 4). Table (2) shows the behaviour of the error for the norm $\mathbb{L}^2$ and norm $\mathbb{L}^\infty$ in function of the parameter of discretization h for different values of k. Note that as h approaches zero, the numerical solution converges to the analytical solution $y(t)$.

k	$\mathbf{L}_2$	$\mathbf{L}_\infty$
8	$9.2671e-004$	$6.4523e-004$
16	$4.7431e-004$	$2.1068e-004$
32	$1.1754e-004$	$4.5027e-005$
64	$4.1585e-005$	$1.1443e-005$
128	$1.4705e-005$	$2.8836e-006$
256	$5.1991e-006$	$7.2377e-007$
512	$1.8382e-006$	$1.8130e-007$
1024	$6.4990e-007$	$4.5369e-008$
2048	$2.2977e-007$	$1.1348e-008$
4096	$8.1237e-008$	$2.8376e-009$
Convergence rate	2.8284	3.9992

Table 2. The errors estimates $\mathbb{L}^2$, $\mathbb{L}^\infty$ and convergence rates $\rho_{2,\infty}$

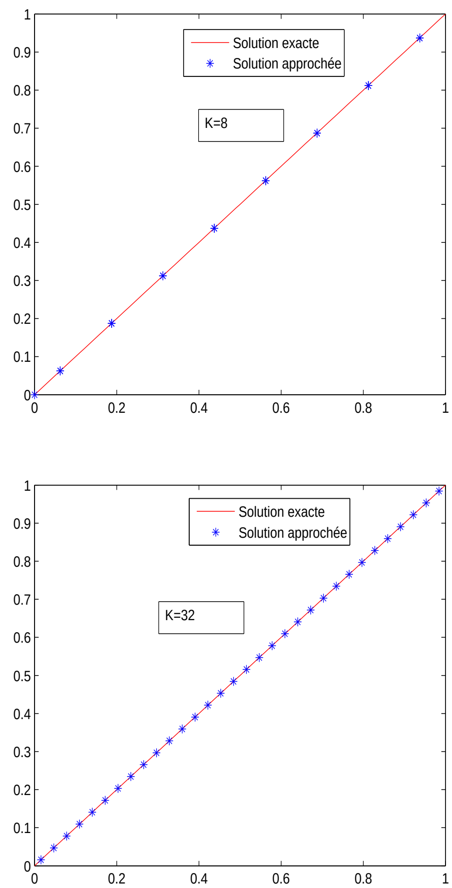

Figure 3. Comparison between approximate solution y_k and exact solution y_{ex}

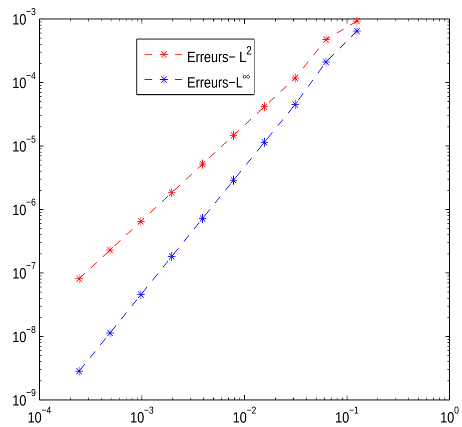

Figure 4. The errors $\mathbb{L}^2$ and $\mathbb{L}^\infty$ with different values of k.

We have also calculated the experimental rate of convergence $\rho_{2,\infty}$. We notice from Table (2) that the convergence rates are 2.8284 in $\mathbb{L}^2$ norm and 3.9975 in norm $\mathbb{L}^\infty$, which is approximately $2\sqrt{2}$ and 4 respectively.

4.3. Example. Consider the following linear Fredholm integro-differential equation:

$$y'(t) \quad = \quad \int_0^1 \sin(4\pi t + 2\pi s)y(s)ds + y(t) - \cos(2\pi t) - 2\pi\sin(2\pi t) - \frac{1}{2}\sin(4\pi x), \tag{4.6}$$

with initial condition $y(0) = 1$.

The exact solution is: $y(t) = \cos(2\pi t)$. The numerical results are shown in Table (3) and in figures (5, 6). Table (3) shows the behaviour of the error for the norm $\mathbb{L}^2$ and norm $\mathbb{L}^\infty$ in function of the parameter of discretization h for different values of k. Note that as h approaches zero, the numerical solution converges to the analytical solution $y(t)$.

k	$\mathbf{L}_2$	$\mathbf{L}_\infty$
8	4.9567e-001	3.0091e-001
16	1.7633e-001	7.5816e-002
32	6.2562e-002	1.8990e-002
64	2.2160e-002	4.7500e-003
128	7.8420e-003	1.1877e-003
256	2.7739e-003	2.9696e-004
512	9.8094e-004	7.4244e-005
1024	3.4686e-004	1.8561e-005
2048	1.2264e-004	4.6402e-006
4096	4.3361e-005	1.1601e-006
Convergence rate	2.8284	3.9992

Table 3. The errors estimates $\mathbb{L}^2$, $\mathbb{L}^\infty$ and convergence rates $\rho_{2,\infty}$

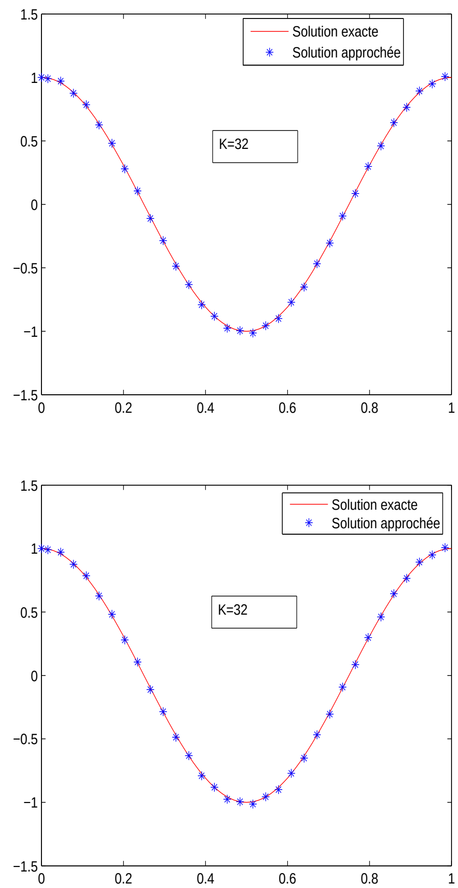

Figure 5. Comparison between approximate solution y_k and exact solution y_{ex}

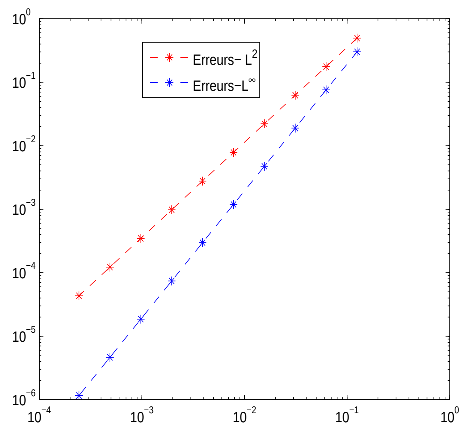

Figure 6. The errors $\mathbb{L}^2$ and $\mathbb{L}^\infty$ with different values of k.

We have also calculated the experimental rate of convergence $\rho_{2,\infty}$. We notice from Table (3) that the convergence rates are 2.8284 in $\mathbb{L}^2$ norm and 3.9975 in norm $\mathbb{L}^\infty$, which is approximately $2\sqrt{2}$ and 4 respectively.

4.4. Example. Consider the Fredholm integral equation of the second kind:

$$y'(t) \quad = \quad \frac{1}{(log2)^2}\int_0^1 (\frac{t}{1+s})y(s)ds + y(t) \ - \frac{1}{2}t \ + \frac{1}{1+t} - log(1+t) \tag{4.7}$$

with initial condition: $y(0) = 0$. The exact solution is: $y(t) = log(1+t)$. The numerical results are shown in Table (4) and in figures (7, 8). Table (4) shows the behaviour of the error for the norm $\mathbb{L}^2$ and norm $\mathbb{L}^\infty$ in function of the parameter of discretization h for different values of k. Note that as h approaches zero, the numerical solution converges to the analytical solution $y(t)$.

k	$\mathbf{L}_2$	$\mathbf{L}_\infty$
8	1.0262e-002	6.3139e-003
16	4.0064e-003	1.8118e-003
32	1.4848e-003	4.8413e-004
64	5.3714e-004	1.2506e-004
128	1.9207e-004	3.1777e-005
256	6.8289e-005	8.0088e-006
512	2.4211e-005	2.0103e-006
1024	8.5720e-006	5.0359e-007
2048	3.0328e-006	1.2602e-007
4096	1.0726e-006	3.1522e-008
Convergence rate	2.8284	3.9992

Table 4. The errors estimates $\mathbb{L}^2$, $\mathbb{L}^\infty$ and convergence rates $\rho_{2,\infty}$

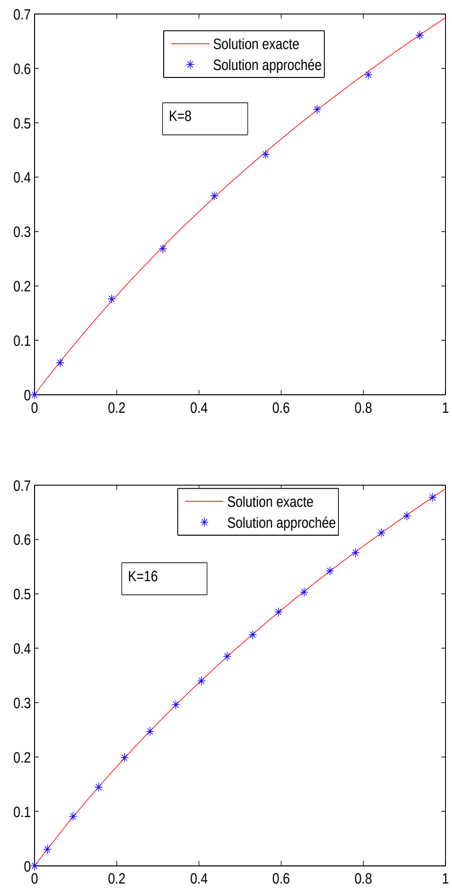

Figure 7. Comparison between approximate solution y_k and exact solution y_{ex}

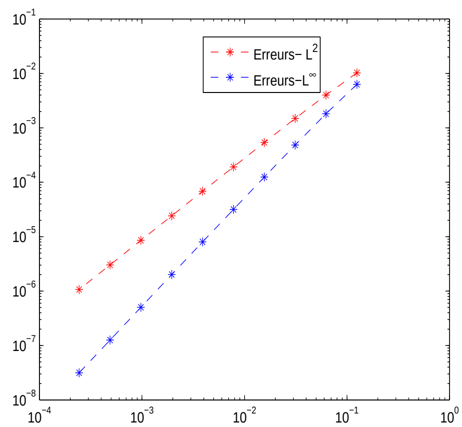

Figure 8. The errors $\mathbb{L}^2$ and $\mathbb{L}^\infty$ with different values of k.

We have also calculated the experimental rate of convergence $\rho_{2,\infty}$. We notice from Table (4) that the convergence rates are 2.8284 in $\mathbb{L}^2$ norm and 3.9975 in norm $\mathbb{L}^\infty$, which is approximately $2\sqrt{2}$ and 4 respectively.

5. CONCLUSION

The proposed method is a powerful procedure for solving linear Fredholm integro-differential. The examples analyzed illustrate the efficiency and reliability of the method presented and show that the method is very simple and effective. The obtained numerical solutions are very accurate, in comparison with the exact solutions. Results also indicate that the convergence rate is fast, and lower order approximations can achieve high accuracy.

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