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Flow Exergy as a Lagrangian for the Navier-Stokes Equations for Incompressible Flow

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Abstract

A novel variational derivation of the Navier-Stokes equations for incompressible flows is presented and discussed. The Lagrangian density is obtained from the exergy balance equation written for both the (Lagrangian) steady and quasi-stationary isothermal flows of an incompressible fluid. The exergy of a fluid mass (composed of a kinetic, a pressure-work, a diffusive, and a dissipative portion, the latter being the result of viscous irreversibility) is derived first, and it is then shown that a formal minimisation of the exergy variation (i.e. destruction) generates, without recurring to "local potentials", the Navier-Stokes equations of motion under the given assumptions. The acceleration being held constant, the proposed variational method can be classified as a "restricted" principle.

The problem is also briefly discussed both in its historical perspective and in its possible formal and practical implications.

Keywords: Navier-Stokes variational, Navier-Stokes Lagrangian, Exergy-based Lagrangian

1. The Problem of the Variational Formulation of the Equations of Fluid Motion

The quest for a variational principle regulating the motion of a fluid is certainly not a recent one: a first formulation may be derived directly from the D'Alembert-Lagrange principle and leads, for an inviscid fluid, to the statement that "*A perfect fluid moves in such a way that*

$$\delta w_v - \int_V \rho \frac{dU}{dt} dx\, dU = 0$$

is satisfied for all virtual displacements dx that satisfy continuity" (see Serrin, 1959). This Lagrangian formulation applies to compressible fluids as well, provided due account is taken for the varying density (i.e., if variations are taken with respect to the density as well).

The theoretical usefulness of this kind of statement for the field of fluid dynamics is limited though: as long as the integrand is in the form of a generic "internal force", no new insight is gained about the phenomenology of the flow. But if the integrand could be shown by independent (and *a priori*) reasoning to represent a physical quantity, then the optimization of such a quantity would lead to a deepening of our phenomenological interpretation of the fluid motion, because it would provide a direction in which the motion develops (e.g., maximising entropy, or minimising kinetic energy, etc.).

Unfortunately, the search has proven to be an elusive one: not only the existence of a simple general Lagrangian of motion for a viscous fluid as asserted by some and negated by other authors but even the methods adopted for a correct positing of the problem do not enjoy universal acceptance. A complete critical review of the abundant literature on this topic exceeds the limits of this paper, and interested readers are referred to Sciubba (2004) and Sieniutycz (1994), both of which include previous reviews by Serrin (1959) and Schecther (1967). A somewhat restricted view will be adopted here, and only a brief account of the debate on the specific issues related to whether or not the Navier-Stokes equations admit of a variational formulation will be provided. In other words, we will not delve into any technicality related to the admissibility of the proposition that "*the general equation of motion of a viscous fluid is derived from a variational formulation in which a functional $\int_V \mathcal{L}\, dV$ is minimised under the proper boundary and accessory conditions*". We shall assume that the above question is admissible (i.e. it makes sense from a physical point of view), and report only the debate related to the problem of existence and uniqueness.

It is well known that the problem of the existence of a Lagrangian of motion for ideal fluids was solved long ago: already in 1766 Lagrange [as quoted by Serrin (1959) but also see http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Lagrange.html] obtained a solution for a compressible perfect fluid by means of a "multiplier", λ, an initially unknown function that was to be determined as a part of the solution and that turned out to be the pressure. A more modern derivation was offered by Herivel (1955), who showed that a suitable Lagrangian could be what we now call the stagnation enthalpy. Of course, in the realm of ideal fluids, there are two other well-known results. The first is the so-called Kelvin principle ("*the irrotational motion of a perfect incompressible fluid has the least kinetic energy of all possible motions for a specified mass flow rate*"). The second is a much less known formulation due to Clebsch (1857) who, by means of a coordinate transformation, obtained a statement (that he presented in a rather obscure form) more general than the one derived a century later by Herivel. Clebsch uses a "potential" to express the velocity field, and this potential can be restated in terms of entropy and rothalpy: thus, his Lagrangian applies to rotational fluids as well.

To this author's knowledge, the first general answer on the matter of the possible existence of a variational form of the viscous equations of motion was provided by C. B. Millikan (1929). He noted that if the integrand $\mathcal{L}$ is a functional of only the velocity components u_i and their spatial derivatives $\partial u_i / \partial x_j$, then it can at best represent a variational formulation for the class of steady flows in which inertia terms ($u_i u_j$) may be neglected. Perhaps not independently[1], Bateman (1929) showed that the integral of the functional $p+\rho|\mathbf{U}^*\mathbf{U}|$, under the constraints of a prescribed mass flux and of the continuity equation, is minimised over a certain domain $\mathcal{D}$ if and only if **U** is irrotational.

Both of these statements in reality constitute generalisations of the so-called minimum dissipation theorem derived by Helmholtz (1869) half a century earlier and later reframed by Rayleigh (1913), which states that "*If in the motion of an incompressible viscous fluid the rotor of the vorticity admits of a potential (i.e., $\nabla\times\omega = \nabla\Phi$), then that motion has the minimal dissipation of any other motion consistent with the same boundary conditions*". We can now take this statement as a starting point to more precisely define the object of the present quest: *is it possible to extend the Helmholtz-Rayleigh theorem to all viscous flows?* We can anticipate here the conclusions that we shall draw after critically reviewing some of the existing literature and state the general answer to this question seems to be affirmative. However, useful and concretely applicable formulations have been obtained only by "restricted" formulations (in a sense that will be clarified below).

Let us begin though with an opposite opinion: Finlayson and Scriven (1967), in a very aggressively worded paper, begin by stating an undisputable fact, namely, that the existence of a variational formulation for an operator $\mathcal{F}$ is linked to its self-adjointness. If $\mathcal{F}$ is self-adjoint, $\mathcal{G}$ ($\mathcal{F}$'s stationary functional) exists "by definition"; if $\mathcal{F}$ is not self-adjoint, the existence of a stationary $\mathcal{G}$ is not guaranteed (it may though exist for "special" forms of $\mathcal{F}$ and the boundary conditions of the problem). This premise is correct: but Finlayson and Scriven stretched it somewhat to negate the validity of what they call "restricted" or "pseudo-" variational methods proposed for dissipative systems. Specifically, they negate the general validity of the Onsager (1931), Glansdorff and Prigogine (1954, 1962, 1964), Rosen (1953), and Biot (1970) approaches and insist that all these "approximate" methods are in effect weaker variations of a general Galerkin-like method. We shall return soon to the problem of "restricted" variations.

This radically negative position (which was taken also by other researchers, see for instance Gage et al., 1965), was soon to be disproved: about a decade later, Tonti succeeded in devising a general procedure to transform any non-self-adjoint operator into a self-adjoint one by means of another operator acting as an "integrating factor". Tonti's results (1984) went virtually unnoticed, but they are very relevant for the problem examined in this paper: he indeed proved that it is always possible to formally derive any non-linear operator from a properly constructed functional. The problem with Tonti's approach is that in most cases the phenomenology is completely lost in his mathematical derivation: the integrating factor does not necessarily resemble any physical quantity; and indeed, even if it is true from a formal point of view that "*what is essential is not the form of the equation but the solution*", in the case of the Navier-Stokes equations, we are really interested in physical principles and not

[1] Millikan writes in his paper that the "idea" for his derivation had been suggested to him by Bateman. But in Bateman's paper published some six months later there is no mention of Millikan's work.

only in the specific form of the solution. Tonti's work, therefore, serves only to reassure us that our search has (at least) one necessary solution: but it does not provide us with a link to the relevant physical principles.

Let us step back for a moment and reconsider the physical issue: in three of the formulations that would (and two of them did!) fall under Finlayson's and Scriven's criticism, the underlying principle is the principle of minimal entropy production. The problem was addressed by Prigogine and co-workers (Glansdorff et al., 1954; Rosen, 1953; and later by Gyarmaty, 1970): assuming the validity of the Onsager relations, these authors did indeed derive a Lagrangian of motion for a viscous fluid from the formal, "restricted" minimisation of the entropy generation. "Restricted" is a word coined by Rosen to denote a variation in which one of the relevant parameters (e.g., a force) is kept constant during the variation of the other (a flux). Onsager (1931) had already used this procedure, when in his famous work on entropy generation, he varied only with respect to the fluxes and kept the forces constant without elaborating on the procedure. Rosen did not as well but simply stated that results similar to Onsager's could be obtained by varying the forces instead, while keeping the fluxes constant, and that the principle of "restricted" variations can be used to construct an endless number of integral functionals.

In contrast to Onsager and Rosen, it is also clear that Glansdorff and Prigogine see their own "method of local potentials" (in which they keep one of the relevant variables, say φ, fixed at its yet unknown "minimising value" φ_o, and then relax φ_o τo φ after the formnal integral derivation is completed) as a "consistent scheme of successive approximations" (Glansdorff and Prigogine, 1964).

Gyarmati (1970) goes a step further than Glansdorff-Prigogine in that he derives a "general equation of motion" in variational form for a very broad class of flows and fluids. However, his derivation also calls for separate variations with respect to fluxes and forces (holding forces constant while varying with respect to fluxes and vice versa) and, thus, employs, in effect, a restricted variational approach.

The importance of the Glansdorff-Prigogine and Gyarmati formulations lies beyond their mathematical formalism in the fact that they both subsume a very basic assumption of the greatest physical significance: that any "motion" of any system is realised under the fundamental constraint of minimum entropy production (or its equivalent, of minimum energy dissipation).

This aspect is recovered in Sieniutycz's approach (Sieniutycz, 1994). He suggests that, though it is clear that the approaches by Glansdorff-Prigogine and Gyarmati yield "restricted" variational formulations, the physical principles on which their derivation is founded are so relevant to the known phenomenology that it is important to try to capitalise on them and proceed to develop similar functional methods in a more mathematically sound way. Sieniutycz adopts a standard Lagrangian approach (one in which integrating factors are used) and uses the entropy generation as the basic functional (the integrand $\mathcal{L}$) with (some of) the constitutive equations appearing as constraints, i.e.

$$\int_V \mathcal{L}\, dV - \int_V \Sigma\, \lambda\, f\, dV = \text{minimal in V} \qquad (1)$$

where the λ are now the lagrangian multipliers that must be also obtained in the course of the minimisation procedure.

Such an approach was also independently proposed by Ecer (1980), who was indeed able to obtain a complete set of solutions valid for the Navier-Stokes equations and by Geskin (1989), who developed Gyarmati's ideas under a standard Lagrangian formalism. Sieniutycz's formulation is more general though, and it also tackles the more difficult problem of including non-stationary considerations, i.e. his quest goes a step further than the one posed here in that he seeks a truly general equation of motion that would regulate both the behaviour of systems near equilibrium and that of systems "far" from equilibrium relaxing to it.

We can now conclude this short (and evidently incomplete) review by stating that, in view of the above, it is certain

1) That a general equation of motion can indeed be formulated in variational form for viscous compressible and incompressible flows (Tonti, 1984; Sieniutycz, 2004; Ecer, 1980); and
2) That - at least in a restricted sense (Glansdorff et al., 1962; Gyarmati, 1970; Sieniutycz, 1994) - the functional contains the entropy generation, i.e. that the underlying physical principle is that a fluid moves in such a way as to minimise its entropy production under the given external constraints (including boundary conditions).

What is missing is a step definitely secondary from a substantial point of view but of great importance from a physical point of view: an explicit derivation in which the known equations (continuity, momentum, and energy) are not employed as lagrangian constraints, but are derived in the course of the procedure. This is the purpose of the present paper.

We must clearly state that the derivation presented here is limited to a certain sub-class of motions (see *Section 2* below) and that its extension to other types of flow (for instance, compressible) is neither straightforward nor certain and is not implied here in any sense. However, for the realm of viscous incompressible Newtonian flows, the present derivation has the merit of explicitly linking a measurable and well-known thermodynamic function (flow exergy) to the standard form of the Navier-Stokes equations: some phenomenological consequences of this connection are outlined in the *Conclusions*. Furthermore, it is reasonable to assume that it should also be possible to extend a similar derivation (based on minimum exergy destruction) to compressible flows and, with the necessary adjustments, to non-isothermal flows as well.

2. Exergy Accounting for a Fluid in Motion

Consider the flow of a viscous fluid in a channel of known geometry and in which the physical parameters of the fluid are constant. In the most general situation, the act of motion is driven by a set of well-defined external fields (pressure, external force, and temperature) and by the inertia of the mass under examination and is affected by some "dissipative" effects related to the real viscosity and thermal conductivity of the fluid. Dissipation is associated with entropy production or, as in the considerations that follow, to the exergy destruction of the flow. Flow exergy is an extensive thermodynamic state function defined as (Kotas, 1985; Moran, 1989)

$$e = h - h_0 - T_0\,(s - s_0) \tag{2}$$

where T_0 is a properly chosen reference state temperature, usually that of a large "environment" with which the system - here, the flowing fluid - may eventually come into thermodynamic equilibrium. A representation of the work and heat interactions of a system in terms of exergy has the advantage of unifying both work/heat interactions and dissipative effects into a unified framework. Thus, for any dissipative system, a theorem of "exergy destruction" applies, which states that if the system undergoes an irreversible process, its specific exergy content is destroyed (annihilated) at a rate given by:

$$\dot{e}_{\lambda} = T_0\,\dot{s}_{irr} \tag{3}$$

The interpretation of equation (3) is straightforward: every real (irreversible) process destroys exergy at a rate proportional to the irreversible entropy generation. For a general exposition of the paradigm of Exergy Analysis, see Kotas (1985) and Moran and Shapiro (2000).

Consider now a unit mass of fluid undergoing a completely specified act of motion. In a small interval of time *dt*, the exergetic content of the unit mass will be modified by four different contributions:

1) an exergy variation rate equal to the exchanged mechanical power (can be positive or negative) [W/kg]:

$$\dot{e}_w = \dot{w}_{rev} = \left[U \cdot \dot{U} + U \frac{\nabla p}{\rho} + U \cdot B \right] \tag{4}$$

2) an exergy destruction rate proportional to the viscous dissipation function of the flow field (always negative, i.e. corresponding to an exergy sink) [W/kg]:

$$\dot{e}_{\lambda,visc} = -\nu\, D_{visc} \tag{5}$$

3) an exergy variation rate proportional to the reversible thermal entropy exchange (positive or negative) [W/kg]:

$$\dot{e}_{Q,rev} = (T - T_0)\,\dot{s}_{rev} \tag{6}$$

4) an exergy destruction rate proportional to the irreversible thermal entropy production (always negative) [W/kg]:

$$\dot{e}_{l,therm} = -T_0\,\dot{s}_{irr,therm} \tag{7}$$

Thus, the total exergy change per unit mass of the fluid in time *dt* is [J/kg]

$$\begin{aligned} \Delta e_{fluid} = dt \sum \dot{e}_j = dt\,[\, & U \cdot \dot{U} + U \frac{\nabla p}{\rho} \\ & + U \cdot B - \nu\, D_{visc} \\ & + (T - T_0)\,\dot{s}_{rev} - (T - T_0)\,\dot{s}_{irr}\,] \end{aligned} \tag{8}$$

Notice that once the flow variables are exactly known at each time *t* and at each point in the flow domain, the quantity defined by equation (8) can be computed exactly locally and, if necessary, integrated over the entire domain to yield the global variation of the exergy of the flow. This is indeed often done to assess the efficiency of technical flows like turbine nozzles, turbine and compressor blades (Iandoli and Sciubba, 2003). The reverse is obviously not true, as infinitely many flow fields may display the same value of the exergy destruction rate at any instant of time. Our goal is to show that *if we assume that, at every instant of time, the fluid motion is governed by the minimisation of the exergy destruction given by equation (8), the* *resulting equations of motion are indeed the Navier-Stokes equations*. Notice that such an assumption is in line with the statement made by Serrin (1959) that a credible Hamiltonian for a viscous fluid ought to include the energy equation in some form.

3. Variational Derivation of the Flow Field

Consider the steady (in a Lagrangian sense) and isothermal flow of a viscous homogeneous fluid with constant properties. As stated above, our basic assumption is that *the fluid moves in such a way that its exergy destruction is at its minimum at each instant of time, compatible with the assigned external constraints*. It can be easily shown by means of the so called "Gouy-Stodola lost-work" theorem (Bejan, 1982), that this assumption corresponds to the minimum entropy generation principle. The "external constraints" are the imposed boundary conditions and the specified work and heat interactions. Neglecting for the moment the boundary conditions (we shall assume that "natural" boundary conditions apply), it is clear that the external energy exchanges are completely specified in a quantitative and qualitative manner by the expression for the exergy variation of the fluid mass (equation (8) above). That is, imposing the condition of constrained minimum exergy destruction is equivalent to searching for the minimisation of a functional whose integrand is the total exergy change of the unit fluid mass given by equation (8). Therefore, we can write that

$$\mathcal{L} = \left[U \cdot \dot{U} + U \frac{\nabla p}{\rho} + U \cdot B - \nu D \right] \qquad (9)$$

and

$$\int_V \mathcal{L}\, dV = \text{minimal in } V \qquad (10)$$

We see, thus, that the exergy formulation offers (at no extra computational cost) the advantage of the formal positing of an unconstrained problem. If different boundary conditions are to be imposed, the integral in equation (10) must be augmented as needed (via a surface integral extended to the domain boundary).

The Euler-Lagrange equations for the problem so posed are

$$\frac{\partial \mathcal{L}}{\partial u_k} - \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial(\partial u_k / \partial x_j)} \right) = 0 \qquad (11)$$

Using for the viscous dissipation function the standard expression:

$$D_{visc} = \frac{1}{2\,\mathrm{Re}} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 \qquad (12)$$

system (11) results in the following set of equations:

$$\frac{du_k}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_k} + B_k - \nu \left(\frac{\partial^2 u_k}{\partial x_i \partial x_i} \right) = 0 \qquad (13)$$

which are indeed the Navier-Stokes equations of motion. A detailed derivation in a three dimensional Cartesian domain is presented in *Section 3.1* below. Notice that, aside from the flow deformation work that is kept constant along the variation (see equation (15)), no "local potential" is employed here. The method is "restricted", however, in the sense that the minimisation is performed in space, but not in time, i.e. the time derivative of the velocity is not subjected to the variation (this corresponds to the "steadiness" assumption listed at the beginning of this section).

4. Explicit Variational Derivation of the Navier-Stokes Equations

The present derivation is carried out in a three-dimensional Cartesian, Galilean frame. For more generality, the Lagrangian of equation (9) is made dimensionless by dividing the right-hand side by U^3/L, where U and L are representative velocity and length scales for the flow domain. The procedure develops along the following steps:

1) Expand $\mathcal{L}$ in the three components x, y, and z (the suffix indicates the relevant component), namely,

$$\mathcal{L}^x = u\frac{du}{dt} + up_x + D/3 \qquad (14a)$$

$$\mathcal{L}^y = v\frac{dv}{dt} + vp_y + D/3 \qquad (14b)$$

$$\mathcal{L}^z = w\frac{dw}{dt} + wp_z + D/3 \qquad (14c)$$

2) Augment the Lagrangian density by including the deformation work (here, power), which is not varied during the derivation, i.e.

$$\mathcal{L}^{aug} = \mathbf{U.\,(Mo.Uo)} = \begin{vmatrix} u \\ v \\ w \end{vmatrix} \times \begin{vmatrix} 0 & u_{yo} & u_{zo} \\ v_{xo} & 0 & v_{zo} \\ w_{xo} & w_{yo} & 0 \end{vmatrix} \times \begin{vmatrix} u_o \\ v_o \\ w_o \end{vmatrix}$$

$$= \begin{vmatrix} uv_o u_y + uw_o u_z \\ u_o vv_x + vw_o v_z \\ u_o ww_x + v_o ww_y \end{vmatrix} \qquad (15)$$

3) The explicit expression for the dissipation function is given by

$$D=\frac{2}{Re}\left(u_x^2+v_y^2+w_z^2\right)+ \frac{1}{Re}\left[\left(u_y+v_x\right)^2+\left(v_z+w_y\right)^2\left(u_z+w_x\right)^2\right] \tag{16}$$

4) We first expand equations (14) and (15) (incorporating the nonessential factor of 1/3 into Re) so that

$$\begin{aligned}\mathcal{L}^x &= uu_t+u^2u_x+uvu_y+uwu_z+up_x \\ &+\frac{2}{Re}u_x^2+\frac{2}{Re}v_y^2+\frac{2}{Re}w_z^2 \\ &+\frac{(u_y+v_x)^2}{Re}+\frac{(w_y+v_z)^2}{Re}+\frac{(u_z+w_x)^2}{Re}\end{aligned} \tag{17a}$$

$$\begin{aligned}\mathcal{L}^x &= vv_t+uvv_x+v^2v_y+vwv_z+vp_y \\ &+\frac{2}{Re}u_x^2+\frac{2}{Re}v_y^2+\frac{2}{Re}w_z^2 \\ &+\frac{(u_y+v_x)^2}{Re}+\frac{(w_y+v_z)^2}{Re}+\frac{(u_z+w_x)^2}{Re}\end{aligned} \tag{17b}$$

$$\begin{aligned}\mathcal{L}^z &= ww_t+uww_x+vww_y+w^2w_z+wp_z \\ &+\frac{2}{Re}u_x^2+\frac{2}{Re}v_y^2+\frac{2}{Re}w_z^2 \\ &+\frac{(u_y+v_x)^2}{Re}+\frac{(w_y+v_z)^2}{Re}+\frac{(u_z+w_x)^2}{Re}\end{aligned} \tag{17c}$$

5) Then we separately compute the terms in the Euler-Lagrange ("E/L") equations (equa-tions (11)) to yield

$$\mathbf{L}^x{}_u = u_t+2uu_x+vu_y+wu_z+p_x \tag{18a}$$

$$\mathbf{L}^x{}_{ux} = u^2+\frac{4}{Re}u_x \tag{18b}$$

$$\mathbf{L}^x{}_{ux,x} = 2\,uu_x+\frac{4}{Re}u_{xx} \tag{18c}$$

$$\mathbf{L}^x{}_{uy} = uv+\frac{2}{Re}u_y+\frac{2}{Re}v_x \tag{18d}$$

$$\mathbf{L}^x{}_{uy,y} = uv_y+vu_y+\frac{2}{Re}u_{yy}+\frac{2}{Re}v_{xy} \tag{18e}$$

$$\mathbf{L}^x{}_{uz} = uw+\frac{2}{Re}u_z+\frac{2}{Re}w_x \tag{18f}$$

$$\mathbf{L}^x{}_{uz,z} = uw_z+wu_z+\frac{2}{Re}u_{zz}+\frac{2}{Re}w_{xz} \tag{18g}$$

(the remaining terms are symmetrical with the "*x*" terms derived above). After some manipulation and making use of the incompressible continuity equation (imposed as a constraint), we obtain the following:

$$\begin{aligned}(E/L)^x &= u_t-uv_y-uw_z+p_x \\ &-\frac{2}{Re}(u_{xx}+u_{yy}+u_{zz})\end{aligned} \tag{19a}$$

which, augmented with the Euler-Lagrange equation for the corresponding line of (15), becomes[2]

$$\begin{aligned}(E/L)^x+(\mathbf{U}\cdot\mathbf{M_o}\cdot\mathbf{U_o})^x &= \\ &= u_t-uv_y-uw_z+p_x \\ &-\frac{2}{Re}(u_{xx}+u_{yy}+u_{zz})+vu_y+wu_z \\ &= \dot{u}+uu_x+vu_y+wu_z+p_x \\ &-\frac{2}{Re}(u_{xx}+u_{yy}+u_{zz})\end{aligned} \tag{20a}$$

i.e. the *x*-component of the Navier-Stokes equation for an incompressible isothermal fluid. In the other directions, we have that

$$\begin{aligned}(E/L)^y &= v_t-vu_x-vw_z+p_y \\ &-\frac{2}{Re}(v_{xx}+v_{yy}+v_{zz})\end{aligned} \tag{19b}$$

and

$$\begin{aligned}(E/L)^z &= w_t-wu_x-wv_y+p_z \\ &-\frac{2}{Re}(w_{xx}+w_{yy}+w_{zz})\end{aligned} \tag{19c}$$

which, respectively, augmented with the corresponding lines of equation (15), become

$$\begin{aligned}(E/L)^y+(\mathbf{U}\cdot\mathbf{M_o}\cdot\mathbf{U_o})^y &= \\ &= v_t-vu_x-vw_z+p_y \\ &-\frac{2}{Re}(v_{xx}+v_{yy}+v_{zz})+uv_x+vv_z \\ &= v_t+uv_x+vv_y+wv_z+p_y \\ &-\frac{2}{Re}(v_{xx}+v_{yy}+v_{zz})\end{aligned} \tag{20b}$$

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[2] Notice that this is the formal step that makes our procedure "restricted" in the Finlayson's sense: we are here in effect substituting the "free varying" variables u, v, and w for their "frozen" values u_o, v_o, and w_o appearing in equation (15). We justify this substitution though using the same reasoning offered by Glansdorff and Prigogine (1964): once a first approximation to the "extremal solution" has been found, its values can be replaced into equation (15), and the procedure iterated to convergence, i.e. until $\left|\begin{matrix}\|u-u_o\| \\ \|v-v_o\| \\ \|w-w_o\|\end{matrix}\right| < \varepsilon$, with ε arbitrarily small.

and

$$(E/L)^z + (\mathbf{U}\cdot\mathbf{M_o}\cdot\mathbf{U_o})^z =$$
$$= w_t - wu_x - wv_y + p_z - \frac{2}{Re}(w_{xx}+w_{yy}+w_{zz}) + uv_x + vw_y$$
$$= w_t + uw_x + vw_y + ww_z + p_z - \frac{2}{Re}(w_{xx}+w_{yy}+w_{zz}) \quad (20c)$$

These are, of course, the sought after Navier-Stokes equations.

5. Conclusions

The novelty of the derivation presented here resides in the physical meaning attached to the Lagrangian functional. Since the entropy minimisation (equivalent to exergy destruction minimisation) principle has a clear and univocal physical meaning, it may be used not only to interpret the results of our derivation but also to suggest possible theoretical and practical applications. If the principle is that every flow is driven by an entropy-minimising paradigm, some of its implications are:

1) *Realisable flows*
Not every flow field is a permissible one: if a certain analytically or numerically specified flow field respects both the mass- and the energy equations, but its velocity vector **U**(**x**,t) does not satisfy equation (10), it will not occur in nature nor it can be maintained in practice unless an external force field is imposed such that the final equation of motion is consistent with equation (10);

2) *Asymptotic bounding*
If the boundary conditions are varied after an initial transient, the flow will approach a configuration that satisfies equation (10). This can be used to set upper and lower bounds to some of the flow-derived quantities (heat transfer, deliverable work, etc.);

3) *"Physically correct" time marching*
Once an initial flow field is known, equation (10) can be solved for the (Lagrangian) velocity **U**(**x**,t+dt*)*, and the Eulerian field marched in time by successive application of the minimum exergy destruction principle. Under the quasi-stationary assumption implicit here, such a procedure would guarantee stability irrespective of the numerical method adopted.

To put the procedure proposed in this paper to a formal and substantial test, applications will have to be developed to specific cases and tested o n known flow fields (experimental, analytical, and numerical). An even more important step would be that of extending its validity to non-isothermal and compressible flows.

Nomenclature

B=(b_x,b_y,b_z)	body force vector, N
D	flow dissipation function, m^2/s^3
$\mathcal{D}$	fluid domain
e	specific exergy, J/kg
h	enthalpy, J/kg
$\mathcal{L}$	Lagrangian density
p	pressure, Pa
s	entropy, J/(kg*K)
t	time, s
T	temperature, K
U=(u,v,w)	velocity vector, m/s
V	integration volume
w	specific work, J/kg
x=(x,y,z)	cartesian coordinate set, m
λ	Lagrange multiplier
ν	kinematic viscosity, m^2/s
ρ	density, kg/m^3
Φ	potential function
ω=(ω_x,ω_y,ω_z)	vorticity vector, s^{-1}

Suffixes

irr	irreversible
λ	loss
o	reference state
rev	reversible
v	virtual displacement
visc	viscous

A vector is identified by a bold case letter (**U**). The substantial time derivative is indicated by a dot ($\dot{s}$), the eulerian by $\partial/\partial t$ or u_t. Spatial derivatives are written either explicitly, as $\partial/\partial x_i$, or as u_{x_i}. ∇ indicates the gradient operator and "*" the scalar product.

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