Characterization of automorphisms of Hom-biprodcts

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Abstract

We study certain subgroups of the full group of monoidal Hom-Hopf algebra automorphisms of a Hom-biproduct, which gives a Hom-version of Radford’s results.

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1. Introduction

In the theory of the classical Hopf algebras, Radford’s biproducts are very important Hopf algebras, which play a central role in the theory of classification of pointed Hopf algebra [1] and account for many examples of semisimple Hopf algebra. There has been many generalizations of Radford’s biproducts such as [2] for quasi-Hopf algebra case, [9] for multiplier Hopf algebra case and [13] for monoidal Hom-Hopf algebra.

Let $B \times H$ be the Radford’s biproduct, where $B$ is both a left $H$-module algebra and a left $H$-comodule coalgebra. Define $\pi : B \times H \to H$, $\pi(b \times h) = \varepsilon_B(b)h$ and $j(h) = 1_B \times h$, let $\text{Aut}_{\text{Hopf}}(B \times H, \pi)$ be the set of Hopf algebra automorphisms $F$ of $B \times H$ satisfying $\pi \circ F = \pi$. Radford [17] characterized the element of $\text{Aut}_{\text{Hopf}}(B \times H, \pi)$, and factorized $F \in \text{Aut}_{\text{Hopf}}(B \times H, \pi)$ into two suitable maps. Motivated by the idea in [17], the study of automorphisms of Radford’s Hom-biproducts introduced in [13] is the focus of this paper.

This paper is organized as follows. In Section 2, we recall some definitions and basic results related to monoidal Hom-algebras, Hom-coalgebras, Hom-bialgebras (Hopf algebras), Hom-(co)module, Hom-module algebras, Hom-smash (co)products and Hom-biproducts.

In Section 3, we study the automorphisms of Radford’s Hom-biproducts and show that the automorphism has a factorization closely related to the factors $B$ and $H$ of Radford’s Hom-biprodct $B \times H$ in [13]. Finally, we characterize the automorphisms of a concrete example.

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2. Preliminaries

Throughout this paper, $k$ will be a field. More materials about monoidal Hom-(co)algebra, monoidal Hopf Hom-algebra, etc. can be found in ([3–8, 10–16, 18–20]). We denote $id_M$ for the identity map from $M$ to $M$.

Let $M = (M, \otimes, k, a, l, r)$ be the monoidal category of vector spaces over $k$. We can construct a new monoidal category $\tilde{\mathcal{H}}(M)$ whose objects are ordered pairs $(M, \mu)$ with $M \in \mathcal{M}$ and $\mu \in \text{Aut}(M)$ and morphisms $f: (M, \mu) \to (N, \nu)$ are morphisms $f: M \to N$ in $\mathcal{M}$ satisfying $\nu \circ f = f \circ \mu$. The monoidal structure is given by $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$ and $(k, id_k)$. All monoidal Hom-structures are objects in the tensor category $\tilde{\mathcal{H}}(\mathcal{M}) = (\mathcal{H}(\mathcal{M}), \otimes, (k, id_k), \tilde{\varepsilon}, \tilde{\mu})$ introduced in [3] with the associativity and unit constraints given by

$$\tilde{\varepsilon}_{M,N,C}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \gamma^{-1}(c)),$$

$$\tilde{\mu}(x \otimes m) = \tilde{\varepsilon}(m \otimes x) = x\mu(m),$$

for $(M, \mu), (N, \nu)$ and $(C, \gamma)$. The category $\tilde{\mathcal{H}}(\mathcal{M})$ is termed Hom-category associated to $\mathcal{M}$. In the following, we recall some definitions about Hom-structures from [3] and [13].

2.1. Monoidal Hom-algebra

A monoidal Hom-algebra is an object $(A, \alpha) \in \tilde{\mathcal{H}}(\mathcal{M})$ together with linear maps $m_A: A \otimes A \to A$, $m_A(a \otimes b) = ab$ and $\eta_A: k \to A$ such that

$$\alpha(ab) = \alpha(a)\alpha(b), \alpha(a)(bc) = (ab)\alpha(c),$$

(2.1)

$$\alpha(\eta(1)) = \eta(1), \quad \alpha(\eta(1)a) = \eta(1)a,$$

(2.2)

for all $a, b, c \in A$. We shall write $\eta_A(1) = 1_A$.

A left $(A, \alpha)$-Hom-module consists of an object $(M, \mu) \in \tilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $\psi: A \otimes M \to M$, $\psi(a \otimes m) = a \cdot m$ satisfying the following conditions:

$$(ab) \cdot \mu(m) = \alpha(a) \cdot (b \cdot m), 1_A \cdot m = \mu(m),$$

(2.3)

for all $m \in M$ and $a, b \in A$. For $\psi$ to be a morphism in $\tilde{\mathcal{H}}(\mathcal{M})$, one needs

$$\mu(a \cdot m) = \alpha(a) \cdot \mu(m).$$

(2.4)

We call that $\psi$ is a left Hom-action of $(A, \alpha)$ on $(M, \mu)$.

Let $(M, \mu)$ and $(M', \mu')$ be two left $(A, \alpha)$-Hom-modules. We call a morphism $f: M \to M'$ right $(A, \alpha)$-linear, if $f \circ \mu = \mu' \circ f$ and $f(a \cdot m) = a \cdot f(m)$. $\tilde{\mathcal{H}}(\mathcal{AM})$ denotes the category of all left $(A, \alpha)$-Hom-modules.

2.2. Monoidal Hom-coalgebras

A monoidal Hom-coalgebra is an object $(C, \gamma) \in \tilde{\mathcal{H}}(\mathcal{M})$ together with two linear maps $\Delta_C: C \to C \otimes C$, $\gamma^{-1}(c) \otimes \Delta_C(c) = c(1) \otimes c(2)$ (summation implicitly understood) and $\varepsilon_C: C \to k$ such that

$$\gamma^{-1}(c(1)) \otimes \Delta_C(c(2)) = c(1) \otimes (c(1) \otimes \gamma^{-1}(c(2))), \Delta_C(\gamma(c)) = \gamma(c(1)) \otimes \gamma(c(2)),$$

(2.5)

$$\varepsilon_C(\gamma(c)) = \varepsilon_C(c), \quad c(1)\varepsilon_C(c(2)) = \gamma^{-1}(c) = \varepsilon_C(c(1))c(2),$$

(2.6)

for all $c \in C$.

A left $(C, \gamma)$-Hom-comodule consists of an object $(M, \mu) \in \tilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $\rho_M: M \to C \otimes M$, $\rho_M(m) = [m[-1] \otimes m[0]]$ (summation implicitly understood) satisfying the following conditions:

$$\Delta_C(m[-1]) \otimes \mu^{-1}(m[0]) = \gamma^{-1}(m[-1]) \otimes (m[0][-1] \otimes m[0][0]),$$

(2.7)

$$\varepsilon_C(m[-1])m[0] = \mu^{-1}(m),$$

(2.8)

$$\rho_M(\mu(m)) = \gamma(m[-1]) \otimes \mu(m[0]),$$

(2.9)
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for all \( m \in M \). We call that \( \rho_M \) is a left Hom-coaction of \((A, \alpha)\) on \((M, \mu)\).

Let \((M, \mu)\) and \((M', \mu')\) be two left \((C, \gamma)\)-Hom-comodules. We call a morphism \( f : M \to M'\) left \((C, \gamma)\)-colinear, if \( f \circ \mu = \mu' \circ f \) and \( f(m)_{[0]} \otimes f(m)_{[-1]} = f(m_{[0]}) \otimes m_{[-1]} \).

\( \mathcal{H}(C, \mathcal{M}) \) denotes the category of all left \((C, \gamma)\)-Hom-comodules.

2.3. Monoidal Hom-Hopf algebra

A monoidal Hom-bialgebra \( H = (H, \beta, m_H, 1_H, \Delta_H, \varepsilon_H) \) is a bialgebra in the category \( \mathcal{H}(M) \). This means that \((H, \beta, m_H, 1_H)\) is a monoidal Hom-algebra and \((H, \beta, \Delta_H, \varepsilon_H)\) is a monoidal Hom-coalgebra such that \( \Delta_H \) and \( \varepsilon_H \) are Hom-algebra maps, that is, for any \( h, g \in H \),

\[
\Delta_H(hg) = \Delta_H(h)\Delta_H(g), \quad \Delta_H(1_H) = 1_H \otimes 1_H, \quad (2.10)
\]

\[
\varepsilon_H(hg) = \varepsilon_H(h)\varepsilon_H(g), \quad \varepsilon_H(1_H) = 1. \quad (2.11)
\]

A monoidal Hom-bialgebra \((H, \beta)\) is called a monoidal Hom-Hopf algebra, if there exists a morphism (called the Hom-antipode) \( S_H : H \to H \) in \( \mathcal{H}(M) \) such that

\[
S_H(h(1))h(2) = \varepsilon_H(h)1_A = h(1)S(h(2)), \quad (2.12)
\]

for all \( h \in H \).

2.4. Hom-module algebra

Let \((H, \beta)\) be a monoidal Hom-bialgebra. A monoidal Hom-algebra \((B, \alpha)\) is called a left \((H, \beta)\)-Hom-module algebra, if \((B, \alpha)\) is a left \((H, \beta)\)-Hom-module with the action \( \cdot \) obeying the following axioms:

\[
h \cdot (ab) = (h(1) \cdot a)(h(2) \cdot b), \quad h \cdot 1_B = \varepsilon_H(h)1_B, \quad (2.13)
\]

for all \( a, b \in A \) and \( h \in H \).

Let \((B, \alpha)\) be a left \((H, \beta)\)-Hom-module algebra. The Hom-smash product \((B \sharp H, \alpha \sharp \beta)\) of \((B, \alpha)\) and \((H, \beta)\) is defined as follows, for all \( a, b, h, g \in H \),

- as \( k \)-space, \( B \sharp H = B \otimes H \),
- Hom-multiplication is given by

\[
(a \sharp h)(b \sharp g) = a(h(1) \cdot \alpha^{-1}(b))\beta(h(2))g.
\]

\((B \sharp H, 1_B \sharp 1_H, \alpha \otimes \beta)\) is a monoidal Hom-algebra.

2.5. Hom-comodule coalgebra

Let \((H, \beta)\) be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra \((B, \alpha)\) is called a left \((H, \beta)\)-Hom-comodule coalgebra, if \((B, \alpha)\) is a left \((H, \beta)\)-Hom-comodule with the coaction \( \rho_B(b) = b_{[-1]} \otimes b_{[0]} \) obeying the following axioms:

\[
b_{[-1]} \otimes \Delta_B(b_{[0]}) = b(1)_{[-1]} b(2)_{[-1]} \otimes b(1)_{[0]} \otimes b(2)_{[0]}, \quad b_{[-1]} \varepsilon_B(b_{[0]}) = \varepsilon_B(b)1_H, \quad (2.14)
\]

for all \( b \in B \).

Let \((B, \alpha)\) be a left \((H, \beta)\)-Hom-comodule cocomodule. The Hom-smash coproduct \((B \sharp H, \alpha \sharp \beta)\) of \((B, \alpha)\) and \((H, \beta)\) is defined as follows, for all \( a, b, h, g \in H \),

- as \( k \)-space, \( B \sharp H = B \otimes H \),
- Hom-coproduct is given by

\[
\Delta(b \sharp h) = (b(1) \sharp b(2)_{[-1]} \beta^{-1}(h(1))) \otimes (\alpha(b(2)_{[0]}) \sharp h(2)).
\]

\((B \sharp H, \Delta, \varepsilon_B \otimes \varepsilon_H, \alpha \otimes \beta)\) is a monoidal Hom-coalgebra.
2.6. Hom-comodule algebra

Let \((H, \beta)\) be a monoidal Hom-bialgebra. A monoidal Hom-algebra \((B, \alpha)\) is called a left \((H, \beta)\)-Hom-comodule algebra, if \((B, \alpha)\) is a left \((H, \beta)\)-Hom-comodule algebra with the coaction \(\rho_B\) obeying the following axioms:

\[
\rho_B(ab) = a_{[-1]}b_{[-1]} \otimes a_{[0]}b_{[0]}, \quad \rho_B(1_B) = 1_H \otimes 1_B, \tag{2.15}
\]

for all \(a, b \in B\).

2.7. Hom-module coalgebra

Let \((H, \beta)\) be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra \((B, \alpha)\) is called a left \((H, \beta)\)-Hom-module coalgebra, if \((B, \alpha)\) is a left \((H, \beta)\)-Hom-module with the action \(\cdot\) obeying the following axioms:

\[
\Delta_B(h \cdot b) = h_{(1)} \cdot b_{(1)} \otimes h_{(2)} \cdot b_{(2)}, \quad \varepsilon_B(h \cdot b) = \varepsilon_H(h) \varepsilon_B(b), \tag{2.16}
\]

for all \(b \in B\) and \(h \in H\).

2.8. Radford’s Hom-biproduct

Recall from [Theorem 3.5] that the vector space \(B \otimes H\) with the Hom-smash product structure and the Hom-smash coproduct structure is a monoidal Hom-bialgebra if and only if the following conditions hold:

- \(\varepsilon_B\) is an algebra map and \(\Delta_B(1_B) = 1_B \otimes 1_B\),
- \((B, \alpha)\) is a left \((H, \beta)\)-Hom-module coalgebra,
- \((B, \alpha)\) is a left \((H, \beta)\)-Hom-comodule algebra,
- for \(a, b \in B\),

\[
\Delta_B(ab) = a_{(1)}(a_{[2]}b_{[-1]} \cdot \alpha^{-1}(b_{(1)})) \otimes \alpha(a_{(2)}b_{[0]}b_{(2)}) \tag{2.17}
\]

for all \(h \in H\) and \(b \in B\),

\[
(h_{(1)} \cdot \alpha^{-1}(b))_{[-1]}h_{(2)} \otimes \alpha((h_{(1)} \cdot \alpha^{-1}(b))_{[0]}) = h_{(1)}b_{[-1]} \otimes h_{(2)} \cdot b_{[0]}. \tag{2.18}
\]

Under the assumption that \((H, S_H)\) is a monoidal Hom-Hopf algebra and \(id_B\) has a convolution inverse in \(\text{End}(B)\), \(B \otimes H\) is a monoidal Hom-Hopf algebra. The monoidal Hom-Hopf algebra \((B \otimes H, \alpha \otimes \beta)\) is called the Radford’s Hom-biproduct and is denoted by \(B \times H\).

3. Factorization of certain biproduct endomorphisms

Let \((B \times H, \alpha \otimes \beta)\) be the Radford’s Hom-biproduct. We define \(\pi : B \times H \to H\) by \(\pi(b \times h) = \varepsilon_B(b)h\) for \(b \in B\) and \(h \in H\) and \(j : H \to B \times H\) by \(j(h) = 1_B \times h\) for \(h \in H\) are monoidal Hom-Hopf algebra maps which satisfy \(\pi \circ j = id_H\). Let \(\text{End}_{\text{Hom-Hopf}}(B \times H, H, \pi)\) be the set of all monoidal Hom-Hopf algebra endomorphisms \(F\) of \(A\) such that \(\pi \circ F = \pi\) and let \(\text{Aut}_{\text{Hom-Hopf}}(B \times H, H, \pi)\) be its set of units. Thus \(\text{Aut}_{\text{Hom-Hopf}}(B \times H, H, \pi)\) is the group of monoidal Hom-Hopf algebra automorphisms \(F\) of \(B \times H\) such that \(\pi \circ F = \pi\) under composition. We will write \(\text{End}_{\text{Hom-Hopf}}(B \times H, \pi)\) for \(\text{End}_{\text{Hom-Hopf}}(B \times H, H, \pi)\), and \(\text{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)\) for \(\text{Aut}_{\text{Hom-Hopf}}(B \times H, H, \pi)\). The purpose of this section is to show that \(F\) has a factorization closely related to the factors \(B\) and \(H\) of \(B \times H\).

We define \(\Pi : B \times H \to B\) and \(J : B \to B \times H\) by \(\Pi(b \times h) = be_H(h)\), for all \(b \in B\), \(h \in H\) and \(J(b) = b \times 1_H\), for all \(b \in B\). There is a fundamental relationship between these four maps given by:

\[
J \circ \Pi = id_{B \times H} \ast (j \circ S_H \circ \pi). \tag{3.1}
\]

The factorization of \(F\) is given in terms of \(F_l : B \to B\) and \(F_r : H \to B\) defined by:

\[
F_l = \Pi \circ F \circ J \quad \text{and} \quad F_r = \Pi \circ F \circ j. \tag{3.2}
\]

First, we shall reveal the relationships among \(F, F_l\) and \(F_r\) in the following lemma.
Lemma 3.1. Let $F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then:

\[
F_l(b) \times 1_H = F(b \times 1_H), \tag{3.3}
\]
\[
F_r(h) \times 1_H = F(1_B \times h(1))(1_B \times S_H(h(2))), \tag{3.4}
\]
\[
F(b \times h) = F_l(\alpha^{-1}(b))F_r(\alpha(h(1))) \times \beta(h(2)), \tag{3.5}
\]
for all $b \in B$ and $h \in H$.

**Proof.** We need to calculate $J \circ \Pi \circ F$. For $b \in B$ and $h \in H$, we use (3.1) to compute

\[
(J \circ \Pi)(F(b \times h)) = F((b \times h)(1))(j \circ S_H \circ \pi)(F((b \times h)(2)))
\]
\[
= F((b \times h)(1))(j \circ S_H \circ \pi)((b \times h)(2))
\]
\[
= F(b(1) \times b(2)[1\beta^{-1}(h(1))](j \circ S_H \circ \pi)(\alpha(b(2)[0]) \times h(2))
\]
\[
= F(\alpha^{-1}(b) \times h(1))(1_A \times S_H(h(2))).
\]

It follows that

\[
(J \circ \Pi \circ F)(b \times h) = F(\alpha^{-1}(b) \times h(1))(1_B \times S_H(h(2))),(\tag{3.6}
\]
for all $b \in B$ and $h \in H$. Equations (3.3) and (3.4) follow from the above equation. As for (3.5), we calculate

\[
F(b \times h)
\]
\[
= F(\alpha^{-1}(b) \times 1_H)F(1_B \times \beta^{-1}(h))
\]
\[
= F(\alpha^{-1}(b) \times 1_H)[F(1_B \times \beta^{-1}(h(1)))(1_B \times S_H(\beta^{-1}(h(2))))(1_B \times \beta^{-1}(h(2)))]
\]
\[
= F(\alpha^{-1}(b) \times 1_H)[(F(1_B \times \beta^{-2}(h(1)))(1_B \times S_H(\beta^{-1}(h(2)))))(1_B \times \beta^{-1}(h(2)))]
\]
\[
= F(\alpha^{-2}(b) \times 1_H)[F(1_B \times \beta^{-1}(h(1)))(1_B \times S_H(\beta^{-1}(h(2))))(1_B \times \beta^{-1}(h(2)))]
\]
\[
= F(\alpha^{-2}(b) \times 1_H)(F_r(\beta^{-1}(h(1))) \times 1_H)(1_B \times h(2))
\]
\[
= F_l(\alpha^{-1}(b))F_r(\alpha(h(1))) \times \beta(h(2)),
\]
as desired. \hfill \blacksquare

By (3.3) and (3.4) of Lemma 3.1:

\[
(id_{B \times H})_l = id_B \text{ and } (id_{B \times H})_r = \eta_B \circ \varepsilon_H. \tag{3.6}
\]

Since $F_l(1_B) = 1_B$ by (3.3) of Lemma 3.1. By (3.5) of Lemma 3.1:

\[
F(1_B \times h) = F_l(\beta(h(1))) \times \beta(h(2)), \tag{3.7}
\]
for all $h \in H$. We are now able to compute the factors of a composite.

**Corollary 3.2.** Let $F, G \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then

1. $(F \circ G)_l = F_l \circ G_l$,
2. $(F \circ G)_r = (F_l \circ G_r) \ast F_r$

**Proof.** For $b \in B$, by (3.3) of Lemma 3.1, we have

\[
(F \circ G)_l(b) \times 1_H = (F \circ G)(b \times 1) = F(G_l(b) \times 1_H) = (F_l \circ G_l)(b) \times 1_H.
\]

Thus, it follows that part (1) holds. Let $h \in H$. Using (3.7), the fact that $F$ is multiplicative, and part (1) of (3.3), we obtain that:

\[
(F \circ G)_r(\beta(h(1))) \times \beta(h(2))\]
\[
= F((F \circ G)(1_B \times h))\]
\[
= F((F \circ G)(1_B \times h))\]
\[
= F(G_r(\beta(h(1))) \times \beta(h(2)))
\]
\[
= F(G_r(\beta(h(1))) \times 1_H)F(1_B \times h(2))
\]
\( (3.3) = (F_l G_r h_{(1)}) \times 1_H (F_r (\beta(h_{(2)(1)})) \times \beta(h_{(2)(2)})) \\
= F_l G_r h_{(1)} F_r (\beta(h_{(2)(1)})) \times \beta^2(h_{(2)(2)}) \\
= F_l G_r (\beta(h_{(1)(1)})) F_r (\beta(h_{(1)(2)})) \times \beta(h_{(2)}), \)

i.e.,
\( (F \circ G)_r (\beta(h_{(1)})) \times \beta(h_{(2)}) = F_l G_r (\beta(h_{(1)(1)})) F_r (\beta(h_{(1)(2)})) \times \beta(h_{(2)}). \)

Applying \( id_B \otimes \varepsilon_H \) to both sides of the above equation, we can get part (2).

By virtue of Lemma 3.1, to characterize \( F \) is a matter of characterizing \( F_l \) and \( F_R \).

Note in particular part (5) of the following describes a commutation relation between \( F_l \) and \( F_R \). First, we shall characterize \( F_l \) in the following lemma.

**Lemma 3.3.** Let \( F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi) \). Then:

1. \( F_l : B \rightarrow B \) is a monoidal Hom-algebra endomorphism.
2. \( \varepsilon_B \circ F_l = \varepsilon_B \).
3. For all \( b \in B \),
\[
\Delta(F_l(b)) = F_l(\alpha^{-1}(b_{(1)})) F_r(b_{(2)[1]}) \otimes F_l(\alpha(b_{(2)[0]})),
\]
4. For all \( b \in B \),
\[
\rho(F_l(b)) = b_{[-1]} \otimes F_l(b_{[0]}),
\]
5. For all \( b \in B \) and \( h \in H \),
\[
F_l(\beta(h_{(1)})) F_r(h_{(2)}) = F_r(\beta(h_{(1)})) (h_{(2)} \cdot F_l(b)).
\]

**Proof.** In order to prove (1), we need to check three aspects. From the above discussion, we have known that \( F_l(1_B) = 1_B \). It is easy to check that \( F_l \circ \alpha = \alpha \circ F_l \). Finally, we shall check that \( F_l \) preserves the multiplication. In fact, for \( a, b \in B \), we have
\[
F_l(ab) = (\Pi \circ F \circ \beta)(ab)
\]
\[
= \Pi((ab \times 1_H)) = \Pi(F(a \times 1_H)F(b \times 1_H))
\]
\[
= \Pi((F_l(a) \times 1_H)(F_l(b) \times 1_H))
\]
\[
= F_l(ab).
\]

It is easy to check part (2). Next, we shall check that parts (3) and (4) hold. As a matter of fact, we compute the coproduct of \( F_l(b) \times 1_H = F(b \times 1_H) \) in two ways. First of all,
\[
\Delta(F_l(b) \times 1_H) = (F_l(b_{(1)}) \times \beta(F_l(b_{(2)[1]})) \otimes (\alpha(F_l(b_{(2)[0]})) \times 1_H)
\]
and secondly, since \( F \) is a coalgebra map, we have
\[
\Delta(F_l(b \times 1_H))
\]
\[
= F_l(((b \times 1_H)_{(1)}) \otimes (F_l(b \times 1_H)_{(2)}))
\]
\[
= F_l(b_{(1)} \times \beta(b_{(2)[1]}) \otimes (\alpha(b_{(2)[0]})) \times 1_H)
\]
\[
= [F_l(\alpha^{-1}(b_{(1)})) F_r(\beta(b_{(2)[1]})) \times \beta^2(b_{(2)[1]})] \otimes (F_l(\alpha(b_{(2)[0]})) \times 1_H).
\]

It follows that
\[
(F_l(b_{(1)}) \times \beta(F_l(b_{(2)[1]})) \otimes (\alpha(F_l(b_{(2)[0]})) \times 1_H)
\]
\[
= [F_l(\alpha^{-1}(b_{(1)})) F_r(\beta(b_{(2)[1]})) \times \beta^2(b_{(2)[1]})] \otimes (F_l(\alpha(b_{(2)[0]})) \times 1_H)
\]

Applying \( id_B \otimes \varepsilon_H \otimes id_B \otimes \varepsilon_H \) to both sides of (3.11) yields (3.8). It follows easily that \( \varepsilon_B \circ F_r = \varepsilon_B \) from (3.7). Applying \( \varepsilon_B \otimes \varepsilon_H \otimes id_B \otimes \varepsilon_H \) to both sides of (3.11) again, we can gain (3.9).

Finally, it is left to us to check part (5). Indeed, for \( b \in B \) and \( h \in H \), we have
\[
F((1_B \times h)(b \times 1_H)) = F((\beta(h_{(1)})) \times \beta^2(h_{(2)})) = F_l(h_{(1)} \alpha^{-1}(b)) F_r(\beta^2(h_{(2)[1]})) \times \beta^3(h_{(2)[2]}).
\]
On the other hand, Since $F$ preserves the multiplication, we compute:

$$F((1_B \times h)(b \times 1_H)) = F(1_B \times h)F(b \times 1_H)$$
$$= (F_r(\beta(h(1))) \times \beta(h(2)))(F_i(b) \times 1_H)$$
$$= F_r(\beta(h(1)))(\beta(h(2)(1)) \cdot F_i(\alpha^{-1}(b))) \times \beta^2(h(2)(2)).$$

Applying $id_B \otimes \varepsilon_H$ to both expressions for $F((1_B \times h)(b \times 1_H))$, we obtain (3.10). □

As the reader might suspect, whether or not $F_l$ is a coalgebra map is explained in terms of $F_R$.

**Corollary 3.4.** Let $F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then, $F_l$ is a monoidal Hom-coalgebra if and only if $F_r(c_{[-1]}(c_0) = 1_B \otimes \alpha^{-1}(c)$, for all $c \in \text{Im}(F_l)$.

**Proof.** Suppose $F_r(c_{[-1]}(c_0) = 1_B \otimes \alpha^{-1}(c)$, for all $c \in \text{Im}(F_l)$. Then we have

$$\Delta(F_l(b)) = F_l(\alpha^{-1}(b(1)))F_r(b(2)_{[-1]}) \otimes F_i(\alpha(b_2(0)))$$
$$= F_l(\alpha^{-1}(b(1)))F_r(F_l(b(2))_{[-1]}) \otimes \alpha(F_i(\alpha(b_2(2))))$$
$$= F_l(b(1)) \otimes F_i(b_2(2)).$$

Conversely, suppose that $F_l$ is a monoidal Hom-coalgebra map. For all $b \in B$, we compute

$$F_r(F_l(b)_{[-1]}) \otimes F_i(b_2(0))$$
$$= F_l(\varepsilon_B(\alpha^{-1}(b(1))))1_B \otimes F_i(\alpha(b_2(0)))$$
$$= F_l(S_B(\alpha^{-1}(b(1)))) \alpha^{-1}(b(2)) F_r((b_2)_{[-1]}) \otimes F_i(\alpha(b_2(0)))$$
$$= [F_l(S_B(\alpha^{-1}(b(1))))] F_i(\alpha^{-1}(b(2))) F_r((b_2)_{[-1]}) \otimes F_i(\alpha(b_2(0)))$$
$$= F_l(S_B(\alpha^{-1}(b(1)))) F_i(\alpha^{-1}(b(2))) F_r((b_2)_{[-1]}) \otimes \alpha(F_i(\alpha(b_2(2))))$$
$$= F_l(S_B(\alpha^{-1}(b(1)))) F_i(\alpha^{-1}(b(2))) F_i((b_2)_{[-1]}) \otimes \alpha(F_i(\alpha(b_2(2))))$$
$$= F_l(S_B(\alpha^{-1}(b(1)))) F_i(\alpha^{-1}(b(2))) F_i((b_2)_{[-1]}) \otimes \alpha(F_i(\alpha(b_2(2))))$$
$$= F_l(b(1)) \otimes F_i(b_2(2)).$$

as desired. □

From Lemma 3.3, we have characterize the conditions that $F_l$ satisfies. It is left to us to characterize $F_r$ as follows.

**Lemma 3.5.** Let $F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then,

(1) $F_r(1_H) = 1_B$.
(2) for all $h, g \in H$,

$$F_r(hg) = F_r(\beta(h(1)))h(2) \cdot F_r(\beta^{-1}(g))),$$

(3) $F_r : H \rightarrow B$ is a monoidal Hom-coalgebra map,
(4) for all $h \in H$,

$$\rho(F_r(h)) = h_{(1)(1)}S(\beta^{-1}(h(2))) \otimes F_r(\beta(h_{(1)(2)}))$$

**Proof.** It is easy to check part (1). We shall check that part (2) holds, for all $h, g \in H$, we calculate on one hand,

$$F(1_B \times hg) = F_r(\beta(h(1)))\beta(g(1)) \times \beta(h(2)) \beta(g(2)),

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and on the other hand,
\[ F(1_B \times hg) = F(1_B \times h)F(1_B \times g) \]
\[ = (F_r(\beta(h(1))) \times \beta(h(2)))(F_r(\beta(g(1))) \times \beta(g(2))) \]
\[ = F_r(\beta(h(1))))(\beta(h(2)\cdot F_r(g(1))) \times \beta^2(h(2))\beta(g(2)). \]

Applying \( id_B \otimes \varepsilon_H \) to both expressions for \( F(1_B \times hg) \), it follows that (3.12) holds.

Let \( h \in H \). To show parts (3) and (4), we compute \( \Delta(F(1_B \times h)) \) in two ways as follows.

\[ \Delta(F(1_B \times h)) = F(1_B \times h(1)) \otimes F(1_B \times h(2)) \]
\[ = (F_r(\beta(h(1))) \times \beta(h(2))) \otimes (F_r(\beta(h(1))) \times \beta(h(2))). \]

On the other hand,

\[ \Delta(F(1_B \times h)) = \Delta(F_r(\beta(h(1))) \times \beta(h(2))) \]
\[ = (F_r(\beta(h(1))) \times F_r(\beta(h(1)))) \otimes (\alpha(\beta(h(1)))) \times \beta(h(2))). \]

Applying \( id_B \otimes \varepsilon_H \otimes id_B \otimes \varepsilon_H \) to the expressions for \( \Delta(F(1_B \times h)) \) gives part (2). Applying \( \varepsilon_B \otimes id_H \otimes id_B \otimes \varepsilon_H \) to the expressions for \( \Delta(F(1_B \times h)) \) again yields

\[ \beta(h(1)) \otimes F_r(h(2)) = F_r(\beta(h(1))) \otimes \beta(h(2))) \otimes F_r(\beta(h(1))) \).

(3.14)

Therefore,

\[ \rho(F_r(h)) = \beta^{-1}(F_r(\beta(h(1))) \otimes \beta^{-1}(h(2)))) \beta^{-1}(h(2))) \otimes F_r(\beta(h(1)))]0 \]
\[ \beta^{-1}(F_r(\beta(h(1))) \otimes \beta^{-1}(h(2))) \otimes F_r(\beta(h(1)))]0 \]

(2.1) \[ \beta^{-2}(F_r(\beta(h(1))) \otimes \beta^{-1}(h(2))) \otimes F_r(\beta(h(1)))]0 \]

(2.5) \[ \beta^{-2}(F_r(\beta^2(h(1))) \otimes \beta^{-1}(h(2))) \otimes F_r(\beta^2(h(1)))]0 \]

(3.14) \[ h(1)_1 S(\beta^{-1}(h(2))) \otimes F_r(\beta(h(1)))]0 \]

which shows that (3.13) holds.

\[ \square \]

**Corollary 3.6.** Let \( F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi) \). Then \( F_1 \) is a left \((H,B)\)-Hom-module map if and only if the condition \( F_1(h_1 \cdot b)F_r(\beta(h(2))) = F_r(\beta(h(1)))F_1(h_2 \cdot b) \). holds.

**Proof.** The necessary condition can be followed easily from (3.10) of Lemma 3.3. Now, we will prove the sufficient part. Suppose that the condition holds. Note that \( F_r \) is a Hom-coalgebra map by (3) of Lemma 3.5. Using this fact and (5) of Lemma 3.3, for all \( h \in H \) and \( b \in B \), we have

\[ h \cdot F_1(b) = \varepsilon_B(F_r(\beta(h(1))))1_B \cdot F_1(\alpha^{-1}(b))) \]
\[ = (S_B(F_r(\beta(h(1))))F_r(\beta(h(2))))(h \cdot F_1(\alpha^{-1}(b))) \]
\[ = S_B(F_r(\beta^2(h(1))))F_r(\beta(h(2))(h \cdot F_1(\alpha^{-2}(b)))) \]

(2.5) \[ S_B(F_r(\beta(h(1))))[F_r(\beta(h(2)))F_r(\beta^2(h(1))) \alpha^{-2}(b)]) \]

(3.10) \[ S_B(F_r(\beta(h(1))))[F_r(\beta(h(2)))F_r(\beta^2(h(1))) \alpha^{-2}(b)]) \]

(2.5) \[ S_B(F_r(\beta(h(1))))[F_r(\beta(h(2)))F_r(\beta^2(h(1))) \alpha^{-2}(b)]) \]

\[ = \alpha_B(F_r(\beta(h(1))))F_r(\beta(h(2))) \cdot \alpha^{-2}(b) \]

\[ = F_1(h \cdot b), \]
which shows that $F_1$ is a left $(H, \beta)$-Hom-module map.

**Lemma 3.7.** Let $F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then, $F_\pi$ is a monoidal Hom-algebra map if and only if $h \cdot F_\pi(g) = \varepsilon_H(h) F_\pi(\beta(g))$, for all $h, g \in H$.

**Proof.** Suppose that $F_\pi$ is a monoidal Hom-algebra map. Using (2) and (3) of Lemma 3.5, for $h, g \in H$, we have:

$$h \cdot F_\pi(g) = [S(F_\pi(\beta(h_{(1)(1)})))F_\pi(\beta(h_{(1)(2)}))] (h_{(2)} \cdot F_\pi(\beta^{-1}(g))) = S(F_\pi(\beta^2(h_{(1)(1)}))) [F_\pi(\beta(h_{(1)(2)})) (\beta^{-1}(h_{(2)} \cdot F_\pi(\beta^{-2}(g)))]$$

$$(2.5) = S(F_\pi(\beta(h_{(1)}))) [F_\pi(\beta(h_{(2)})) (h_{(2)} \cdot F_\pi(\beta^{-2}(g)))]$$

$$(3.12) = S(F_\pi(\beta(h_{(1)}))) F_\pi(h_{(2)} \beta^{-1}(g)) = S(F_\pi(\beta(h_{(1)}))) F_\pi(h_{(2)} \beta^{-1}(g)) = [S(F_\pi(h_{(1)})) F_\pi(h_{(2)}))] F_\pi(g) = \varepsilon_H(h) F_\pi(\beta(g)).$$

If $h \cdot F_\pi(g) = \varepsilon_H(h) F_\pi(\beta(g))$ holds, by using (2) of Lemma 3.5, we have

$$F_\pi(hg) = F_\pi(\beta(h_{(1)}))(h_{(2)} \cdot F_\pi(\beta^{-1}(g))) = F_\pi(\beta(h_{(1)})) \varepsilon_H(h_{(2)}) F_\pi(g) = F_\pi(h) F_\pi(g).$$

□

**Corollary 3.8.** Let $F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. Then $F_\pi$ has a convolution inverse $J_\pi$ defined by $J_\pi(h) = h_{(1)} \cdot F_\pi(S_H(h_{(2)}))$

**Proof.** Let $h \in H$. Then by parts (1) and (2) of Lemma 3.5, we have

$$F_\pi J_\pi(h) = F_\pi(h_{(1)}) J_\pi(h_{(2)}) = \varepsilon_H(h_{(1)(1)} \cdot F_\pi(S_H(h_{(2)})))$$

$$(2.5) = \varepsilon_H(h_{(1)(1)} \cdot F_\pi(S_H(h_{(2)})))$$

and using the fact that $(B, \alpha)$ is a left $(H, \beta)$-Hom-module algebra, we have

$$J_\pi F_\pi(h) = J_\pi(h_{(1)} \cdot F_\pi(S_H(h_{(2)}))) F_\pi(h_{(2)})$$

$$(2.5) = (\beta^2(h_{(1)(1)(1)} \cdot F_\pi(\beta(h_{(1)(1)(2)}))) \cdot F_\pi(\beta^{-2}(h_{(2)})))$$

$$= (\beta^2(h_{(1)(1)(1)} \cdot F_\pi(\beta(h_{(1)(1)(2)}))) \cdot F_\pi(\beta^{-2}(h_{(2)})))$$

$$(2.5) = \varepsilon_H(h_{(1)(1)} \cdot F_\pi(S_H(h_{(1)(2)}))) \cdot F_\pi(h_{(2)}).$$
The proof is completed. □

Using the above lemmas and corollaries what we have got, we can gain the main result.

**Theorem 3.9.** Let $B \times H$ be a Hom-biproduct, let $\pi : B \times H \rightarrow H$ be the projection from $B \times H$ onto $H$, and let $\mathcal{F}_{B,H}$ be the set of pairs $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L} : B \rightarrow B$, $\mathcal{R} : H \rightarrow B$ is a closed map which satisfy the conclusions of Lemma 3.3 and Lemma 3.5 for $F_1$ and $F_r$, respectively. Then

1. The function $\Phi : \mathcal{F}_{B,H} \rightarrow \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$, described by $(\mathcal{L}, \mathcal{R}) \mapsto F$, where $F(b \times h) = \mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h(1)) \times \beta(h(2))$, for all $b \in B$ and $h \in H$, is a bijection. Furthermore, $F_1 = \mathcal{L}$ and $F_r = \mathcal{R}$.

2. Suppose $(\mathcal{L}, \mathcal{R}) \in \mathcal{F}_{B,H}$, then $F \in \text{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)$ if and only if $\mathcal{L}$ is a bijection.

**Proof.** In order to prove (1), we define $\Psi : \text{End}_{\text{Hom-Hopf}}(B \times H, \pi) \rightarrow \mathcal{F}_{B,H}$ by $\Psi(F) = (\Pi \circ F \circ J, \Pi \circ F \circ j)$. It is easily proved that $\Phi$ and $\Psi$ are mutually inverse.

According the definition of $F$, we shall check that $F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$. It is easy to see that $\pi \circ F = \pi$. Note that $F(1_B \times 1_H) = 1_B \times 1_H$ and

$$
\varepsilon(F(b \times h)) = \varepsilon(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h(1)) \times \beta(h(2)))
$$

$$
= \varepsilon_{B}(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h(1))) \varepsilon_{H}(\beta(h(2)))
$$

$$
= \varepsilon_{B}(\mathcal{L}(\alpha^{-1}(b))) \varepsilon_{B}(\mathcal{R}(h(1))) \varepsilon_{H}(\beta(h(2)))
$$

$$
= \varepsilon_{B}(b) \varepsilon_{H}(h),
$$

for $b \in B$ and $h \in H$ which means $\varepsilon \circ F = \varepsilon$.

Let $b, b' \in B$ and $h, h' \in H$. Then,

$$
\begin{align*}
F((b \times h)(b' \times h')) &= F(b(h(1) \cdot \alpha^{-1}(b')) \times \beta(h(2))h') \\
&= \mathcal{L}(\alpha^{-1}(b))(\beta^{-1}(h(1)) \cdot \alpha^{-2}(b'))\mathcal{R}(\beta(h(2))h(1)) \times \beta^2(h(2))h(2) \beta(h(2)') \\
&= \mathcal{L}(\alpha^{-1}(b))\mathcal{L}(\beta^{-1}(h(1)) \cdot \alpha^{-2}(b'))\mathcal{R}(\beta(h(2))h(1)) \times \beta^2(h(2))h(2) \beta(h(2)') \\
&= \mathcal{L}(\alpha^{-1}(b))\mathcal{L}(\beta^{-1}(h(1)) \cdot \alpha^{-2}(b'))\mathcal{R}(h(1)) \times \beta^2(h(2))h(2) \beta(h(2)') \\
&= \mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h(1)) \times \beta^2(h(2))h(2) \beta(h(2)') \\
&= F(b \times h)F(b' \times h').
\end{align*}
$$

Therefore, $F$ is a monoidal Hom-algebra morphism. Next, we shall check that $\Delta \circ F = (F \otimes F) \circ \Delta$ holds. Indeed, for all $b \in B, h \in H$,

$$
\begin{align*}
\Delta(F(b \times h)) &= \Delta(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h(1)) \times \beta(h(2))) \\
&= \Delta(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h(1))) \times \beta(h(2)).
\end{align*}
$$
\[(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)(1)}))_{(1)} \times (\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)(1)}))_{(2)[−1]}h_{(2)(1)}\]
\[\otimes (\alpha(\mathcal{L}(\alpha^{-1}(b))\mathcal{R}(h_{(1)(1)}))_{(2)[0]} \times h_{(2)(2)})\]

\[(17.21)\]
\[\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[−1]} \cdot \beta^{-1}(\mathcal{R}(h_{(1)(1)})) \times (\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]}\mathcal{R}(h_{(1)(2)}))_{[−1]}h_{(2)(1)}\]
\[\otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]}\mathcal{R}(h_{(1)(2)}))_{(1)} \times \beta(h_{(2)(2)}])\]

\[\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[−1]} \cdot \mathcal{R}(\beta^{-1}(h_{(1)(1)})) \times (\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]}\mathcal{R}(h_{(1)(2)}))_{[−1]}h_{(2)(1)}\]
\[\otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]}\mathcal{R}(h_{(1)(2)}))_{(1)} \times \beta(h_{(2)(2)}])\]

\[(3.13)\]
\[\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[−1]} \cdot \mathcal{R}(\beta^{-1}(h_{(1)(1)}))\]
\[\times (\beta(\mathcal{R}(\alpha^{-1}(b))_{(2)[0]}[−1]h_{(1)(2)(1)}S_{H}(\beta^{-1}(h_{(1)(2)(2)})))h_{(2)(1)}\]
\[\otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]}\mathcal{R}(h_{(1)(2)(1)}))_{(2)} \times \beta(h_{(2)(2)}])\]

\[(2.12)\]
\[\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[−1]} \cdot \mathcal{R}(\beta^{-1}(h_{(1)(1)}))\]
\[\times (\beta(\mathcal{R}(\alpha^{-1}(b))_{(2)[0]}[−1]h_{(1)(2)(1)}S_{H}(\beta^{-1}(h_{(1)(2)(2)})))h_{(2)(1)}\]
\[\otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]}\mathcal{R}(h_{(1)(2)(1)}))_{(2)} \times \beta(h_{(2)(2)}])\]

\[(3.8)\]
\[\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[−1]} \cdot \mathcal{R}(\beta^{-1}(h_{(1)(1)}))\]
\[\times (\beta(\mathcal{R}(\alpha^{-1}(b))_{(2)[0]}[−1]h_{(1)(2)(1)}S_{H}(\beta^{-1}(h_{(1)(2)(2)})))h_{(2)(1)}\]
\[\otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]}\mathcal{R}(h_{(1)(2)(1)}))_{(2)} \times \beta(h_{(2)(2)}])\]

\[(2.7)\]
\[\mathcal{L}(\alpha^{-1}(b))_{(1)}(\mathcal{L}(\alpha^{-1}(b))_{(2)[−1]} \cdot \mathcal{R}(\beta^{-1}(h_{(1)(1)}))\]
\[\times (\beta(\mathcal{R}(\alpha^{-1}(b))_{(2)[0]}[−1]h_{(1)(2)(1)}S_{H}(\beta^{-1}(h_{(1)(2)(2)})))h_{(2)(1)}\]
\[\otimes [\alpha(\alpha(\mathcal{L}(\alpha^{-1}(b))_{(2)[0]}\mathcal{R}(h_{(1)(2)(1)}))_{(2)} \times \beta(h_{(2)(2)}])\]
The other conditions which make $F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)$ can be checked easily. Thus the proof of part (1) is completed.

As for (2), suppose $F \in \text{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)$. It is easily showed that $F_l$ and $(F^{-1})_l$ are inverses. Thus $F_l$ is bijective and $(F^{-1})_l = (F^{-1})_l$.

Conversely, suppose that $F_l$ is bijective. Set $G_l = (F_l)^{-1}$. From Corollary 3.8, $F_l$ has a convolution inverse $J_r$. $G_r = G_l \circ J_r = (F_l)^{-1} \circ J_r$. Define $G(b \times h) = G_l(\alpha^{-1}(b))G_r(h(1)_1) \times \beta(h(2))$. For all $b \in B$ and $h \in H$, we compute

\[
G(F(b \times h)) = G_l(F_l(\alpha^{-1}(b))F_r(h(1)_1) \times \beta(h(2)))
\]

\[
(3.5) = G_l(F_l(F_l^{-1}(b))F_l(h(1)_1))(J_l \circ J_r)(\beta(h(2))) \times \beta^2(h(2))
\]

\[
= \alpha^{-1}(b)G_l(F_lF_r(h(1)_1)J_r(h(2))) \times \beta^2(h(2))
\]

\[
= \alpha^{-1}(b)(1_B \varepsilon_H)(h(1)_1) \times \beta(h(2)) \quad \text{(By Corollary 3.8)}
\]

\[
= b \times h.
\]

\[
F(G(b \times h)) = F(F_l(G_l(\alpha^{-1}(b))G_r(h(1)_1) \times \beta(h(2)))
\]

\[
(3.5) = F_l(G_l(F_l^{-1}(b))G_r(h(1)_1))F_r(h(2)) \times \beta^2(h(2))
\]

\[
= (\alpha^{-1}(b)F_l \circ G_r)(\beta^2(h(2))) \times \beta^2(h(2))
\]

\[
= (\alpha^{-1}(b)J_r(\beta^{-1}(h(1)_1))F_r(h(2)) \times \beta^2(h(2))
\]

\[
= \alpha^{-1}(b)(b \times h)(h(1)_1)F_r(h(2)) \times \beta(h(2))
\]

\[
= b \times h. \quad \text{(By Corollary 3.8)}
\]

Thus we have shown that $G \circ F = id_{B \times H} = F \circ G$. 

Let $\mathcal{F}_{B,H}^*$ denote the set of $(\mathcal{L}, \mathcal{R}) \in \mathcal{F}_{B,H}$ such that $\mathcal{L}$ is bijective. Then, the corresponding of part induces a bijection $\mathcal{F}_{B,H}^* \rightarrow \text{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)$.

Let $(B, \alpha)$ be a monoidal Hom-algebra and $(C, \beta)$ be a monoidal Hom-coalgebra over $k$. Let Hom$(C, B)$ be the set of linear maps $f : C \rightarrow B$ satisfying $f \circ \alpha \circ f = f \circ \beta$. Then Hom$(C, B)$ is an ordinary associative algebra with the unit $\eta_B \circ \varepsilon_C$ under the convolution $\ast$. Indeed, for $f, \phi, \varphi \in \text{Hom}(C, B)$ and $c \in C$,

\[
((f \ast \phi) \ast \varphi)(c) = (f(c(1)_1) \phi(c(2)_1)) \varphi(c(2)_2)
\]

\[
= (f(\beta^{-1}(c(1)_1)) \phi(c(2)_1)) \varphi(c(2)_2)
\]

\[
= \alpha(f(\beta^{-1}(c(1)_1)))(\phi(c(2)_1)) \varphi(\beta(c(2)_2))
\]

\[
= f(c(1)_1)(\phi(c(2)_1)) \varphi(c(2)_2)
\]

\[
= (f \ast (\phi \ast \varphi))(c).
\]
and
\[(f \ast (\eta_B \circ \varepsilon_C))(c) = f(c_{(1)})\varepsilon_C(c_{(2)})1_B = \alpha(f(\beta^{-1}(c))) = f(c).\]
Thus it follows that \(f \ast (\eta_B \circ \varepsilon_C) = f\). That \((\eta_B \circ \varepsilon_C) \ast f = f\) can be checked similarly.

The group \(\mathcal{G}(B) = \text{Aut}_{\text{Hom-algebra}}(B)\) acts on the convolution algebra \(\text{Hom}(C, B)\) by
\[f \triangleright g = f \circ g\]
for all \(f \in \mathcal{G}(B)\) and \(g \in \text{Hom}(C, B)\). This action satisfies:
\[f \triangleright (\eta \circ \varepsilon_C) = \eta \circ \varepsilon_C\quad \text{and} \quad f \triangleright (\phi \ast \varphi) = (f \triangleright \phi) \ast (f \triangleright \varphi),\]
for all \(f \in \mathcal{G}(B)\) and \(\phi, \varphi \in \text{Hom}(C, B)\). Let \(\mathcal{U}(C, B)\) be the group of units of the algebra \(\text{Hom}(C, B)\). Then, \(\mathcal{G}(B) \triangleright \mathcal{U}(C, B) \subseteq \mathcal{U}(C, B)\); thus there is a group homomorphism,

\[\Gamma : \mathcal{G}(B) \to \text{Aut}_{\text{Group}}(\mathcal{U}(C, B)) \quad (3.15)\]
given by \(\Gamma(f)(\phi) = f \triangleright \phi\), for all \(f \in \mathcal{G}(B)\) and \(\phi \in \text{Hom}(C, B)\). The resulting group \(\mathcal{U}(C, B) \rtimes \mathcal{G}(B)\) has product given by
\[(\phi, f)(\varphi, f') = (\phi \ast (f \circ \varphi), f \circ f').\]

**Theorem 3.10.** Suppose that \(B \times H\) is a Hom-biproduct. Then, there is a one-to-one group homomorphism \(\text{Aut}_{\text{Hom-Hopf}}(B \times H, \pi) \to \mathcal{U}(C, B)^{op} \times \mathcal{G}(B)\), with which is given by \(F \to (F_r, F_l)\), for all \(F \in \text{End}_{\text{Hom-Hopf}}(B \times H, \pi)\).

As the end of this paper, we consider an example in [13]. Let \(B = k < 1_B, x >\) and the automorphism \(\alpha : B \to B, 1_B \mapsto 1_B\) and \(x \mapsto -x\). \((B, \alpha)\) is both a monoidal Hom-algebra and a monoidal Hom-coalgebra with multiplication, comultiplication and counit given by
\[1_B1_B = 1_B, 1_Bx = x1_B = -x, x^2 = 0,\]
\[\Delta_B(1_B) = 1_B \otimes 1_B, \Delta_B(x) = (-x) \otimes 1_B + 1_B \otimes (-x),\]
\[\varepsilon_B(1_B) = 1, \varepsilon_B(x) = 0.\]
We define \(S_B : B \to B, S_B(1_B) = 1_B, S_B(x) = -x\), which is the convolution inverse of \(1_B\).

Let \(H = k < 1_H, g >\) be the group Hopf algebra with \(g^2 = 1_H\) and \(\Delta_H(g) = g \otimes g, S_H(g) = g = g^{-1}\). Then \((H, id_H)\) is a monoidal Hom-Hopf algebra. \((B, \alpha)\) is \((H, id_H)\)-module algebra and module coalgebra with the action \(\cdot : H \otimes B \to B\) given by
\[1_H \cdot 1_B = 1_B, 1_H \cdot x = -x, g \cdot 1_B = 1_B \quad \text{and} \quad g \cdot x = x.\]
Also, \((B, \alpha)\) is a left \((H, id_H)\)-comodule algebra and comodule coalgebra with the coaction \(\rho_B : B \to H \otimes B\) given by
\[\rho_B(1_B) = 1_H \otimes 1_B, \quad \rho_B(x) = g \otimes (-x).\]
Then \((B \times H = \{1_B \otimes 1_H, 1_B \otimes g, x \otimes 1_H, x \otimes g, \alpha \otimes id_H\})\) is a Radford’s Hom-biproduct with multiplication, comultiplication, counit and antipode defined as follows:

- **Multiplication**

<table>
<thead>
<tr>
<th>(m)</th>
<th>(1_B \times 1_H)</th>
<th>(1_B \times g)</th>
<th>(x \times 1_H)</th>
<th>(x \times g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1_B \times 1_H)</td>
<td>(1_B \times 1_H)</td>
<td>(1_B \times g)</td>
<td>(-x \times 1_H)</td>
<td>(-x \times g)</td>
</tr>
<tr>
<td>(1_B \times g)</td>
<td>(1_B \times 1_H)</td>
<td>(1_B \times g)</td>
<td>(x \times g)</td>
<td>(x \times 1_H)</td>
</tr>
<tr>
<td>(x \times 1_H)</td>
<td>(-x \times 1_H)</td>
<td>(-x \times g)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>(x \times g)</td>
<td>(-x \times g)</td>
<td>(-x \times 1_H)</td>
<td>(0)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

- **Comultiplication**

\[
\Delta(1_B \times 1_H) = (1_B \times 1_H) \otimes (1_B \times 1_H), \quad \Delta(1_B \times g) = (1_B \times g) \otimes (1_B \times g),
\]
\[
\Delta(x \times 1_H) = (-x \times 1_H) \otimes (1_B \times 1_H) + (1_B \times g) \otimes (-x \times 1_H),
\]
\[
\Delta(x \times g) = (-x \times g) \otimes (1_B \times g) + (1_B \times 1_H) \otimes (-x \times g)
\]

- **Counit**

\[
\varepsilon(1_B \times 1_H) = 1 = \varepsilon(1_B \times g), \quad \varepsilon(x \times 1_H) = 0 = \varepsilon(x \times g).
\]
- Hom-antipode

\(S(1_B \times 1_H) = 1_B \times 1_H, \quad S(1_B \times g) = 1_B \times g, \quad S(x \times 1_H) = x \times g, \quad S(x \times g) = -x \times 1_H.\)

Now, we compute the morphisms \(\mathcal{L} \in \text{End}(B)\) satisfying the conclusions of lemma 3.1. Taking a base of \(\text{End}(B)\) \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\) and \(\mathcal{L}_4\) given respectively by

\[
\begin{align*}
\mathcal{L}_1 &: 1_B \mapsto 1_B, \; x \mapsto 0, \\
\mathcal{L}_2 &: 1_B \mapsto 0, \; x \mapsto 1_B, \\
\mathcal{L}_3 &: 1_B \mapsto x, \; x \mapsto 0, \\
\mathcal{L}_4 &: 1_B \mapsto 0, \; x \mapsto x.
\end{align*}
\]

Let \(\mathcal{L} = t_1\mathcal{L}_1 + t_2\mathcal{L}_2 + t_3\mathcal{L}_3 + t_4\mathcal{L}_4\). If \(\mathcal{L}\) satisfies part (2) of, we can get \(t_1 = 1\) and \(t_2 = 0\). Thus \(\mathcal{L} = \mathcal{L}_1 + t_3\mathcal{L}_3 + t_4\mathcal{L}_4\). By part (4) of Lemma 3.1, it follows that \(t_3 = 0\). So \(\mathcal{L} = \mathcal{L}_1 + t_4\mathcal{L}_4\). So there is a bijection between the set of the morphisms \(\mathcal{L} \in \text{End}(B)\) satisfying the conclusions of lemma 2.1 and the set \(\left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \right\} \mid t \in k\}\).

Now, we will discuss the morphisms of \(\text{Hom}(H, B)\) which satisfy Lemma 3.3 in similar way as above. Taking a base of \(\text{Hom}(H, B)\) \(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\) and \(\mathcal{R}_4\) given respectively by

\[
\begin{align*}
\mathcal{R}_1 &: 1_H \mapsto 1_B, \; g \mapsto 0, \\
\mathcal{R}_2 &: 1_H \mapsto 0, \; g \mapsto 1_B, \\
\mathcal{R}_3 &: 1_H \mapsto x, \; g \mapsto 0, \\
\mathcal{R}_4 &: 1_B \mapsto 0, \; g \mapsto x.
\end{align*}
\]

Let \(\mathcal{R} = k_1\mathcal{R}_1 + k_2\mathcal{R}_2 + k_3\mathcal{R}_3 + k_4\mathcal{R}_4\). If \(\mathcal{R}\) satisfies part (1) of Lemma 3.3, it follows that \(k_1 = 1\) and \(k_3 = 0\). Thus \(\mathcal{R} = \mathcal{R}_1 + k_2\mathcal{R}_2 + k_4\mathcal{R}_4\). By (3.13) of Lemma 3.3, we can get \(k_4 = 0\) and furthermore \(\mathcal{R} = \mathcal{R}_1 + k_2\mathcal{R}_2\). Using part (4), we can obtain \(k_2 = 1\). Thus \(\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2\), i.e., \(\mathcal{R}(1_H) = 1_B\) and \(\mathcal{R}(g) = 1_B\). Hence \(\mathcal{F}_{B,H}^* \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid 0 \neq t \in k \right\} \cong k^*\). We can give the concrete characterization of \(\text{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)\). Let \(\mathcal{F} \in \text{Aut}_{\text{Hom-Hopf}}(B \times H, \pi)\). By Theorem 3.9, we have

\[
\begin{align*}
\mathcal{F}(1_B \times 1_H) &= 1_B \times 1_H, \\
\mathcal{F}(1_B \times g) &= 1_B \otimes g, \\
\mathcal{F}(x \times 1_H) &= tx \times 1_H, \\
\mathcal{F}(x \times g) &= tx \otimes g.
\end{align*}
\]

where \(t \in k^*\).

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References

Characterization of automorphisms of Hom-bipродucts


