A new family of $k$ -- Gaussian Fibonacci numbers

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Abstract

In this manuscript, a new family of $k$ -- Gaussian Fibonacci numbers has been identified and some relationships between this family and known Gaussian Fibonacci numbers have been found. Also, I the generating functions of this family for $k = 2$ has been obtained.

Key Words: Fibonacci numbers, Gaussian fibonacci numbers, Gaussian numbers.

$k$- Gaussian Fibonacci sayılarının yeni bir ailesi

Özet

Bu yazında, yeni bir $k$ -- Gaussian Fibonacci sayıları ailesi tanımlanmış ve bu aile ile bilinen Gaussian Fibonacci sayıları arasında bazı ilişkiler bulunmaktadır. Ayrıca, $k=2$ için bu ailenin üretic fonksiyonlarını elde edilmiştir.

Anahtar Kelimeler: Fibonacci sayıları, Gaussian Fibonacci sayıları, Gaussian sayıları.

1. Introduction

Horadam [1] in 1963 and Berzsenyi [2] in 1977 defined complex Fibonacci numbers. Horadam introduced the concept the complex Fibonacci numbers as the Gaussian Fibonacci numbers. Moawwad El-Mikkawy and Tomohiro Sogabe [3] in 2015 defined a new family of $k$- Fibonacci numbers and they gave $F_n^{(k)}$ and establish some properties of the relation to the $F_n$. There are many studies on Fibonacci and Gaussian Fibonacci numbers. See, e.g. [4-15].

The Binet’s formula of the Fibonacci numbers are defined as follows:

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\[ F_n = \frac{1}{\sqrt{5}} (\alpha^{n+1} - \beta^{n+1}), \quad n = 0, 1, 2, \ldots \]

where \( \alpha = \frac{1+\sqrt{5}}{2} \) and \( \beta = \frac{1-\sqrt{5}}{2} \). The first few Fibonacci numbers are 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots. For more detailed information on these numbers see [16]. The numbers \( F_n \) with the initial conditions \( F_0 = 0 \) and \( F_1 = 1 \) satisfy

\[ F_{n+2} = F_{n+1} + F_n, \quad n \geq 0. \]

The Gaussian Fibonacci numbers: \( GF_n \) for \( n \geq 0 \) are defined

\[ GF_n = GF_{n-1} + GF_{n-2} \]

where \( GF_0 = i, GF_1 = 1 \). The first few Gaussian Fibonacci numbers are \( i, 1, i + 1, i + 2, 2i + 3, 3i + 5, 5i + 8, \ldots \).

2. A new family of \( k \) – Gaussian Fibonacci numbers

**Definition 2.1.** Let \( n \) and \( k (k \neq 0) \) be natural numbers, then according to the division algorithm, there are \( m \) and \( r \) such that \( n = mk + r, \quad 0 \leq r < k \). According to this, we define a new family of generalized \( k \) – Gaussian Fibonacci numbers \( GF_n^{(k)} \) by

\[ GF_n^{(k)} = \left[ \left( \frac{\sqrt{5}}{5} + \left( \frac{5 - \sqrt{5}}{10} \right)i \right) \alpha^m + \left( -\frac{\sqrt{5}}{5} + \left( \frac{5 + \sqrt{5}}{10} \right)i \right) \beta^m \right]^{k-r} \left[ \left( \frac{\sqrt{5}}{5} + \left( \frac{5 - \sqrt{5}}{10} \right)i \right) \alpha^{m+1} + \left( -\frac{\sqrt{5}}{5} + \left( \frac{5 + \sqrt{5}}{10} \right)i \right) \beta^{m+1} \right]^r. \]

For \( k = 2, 3 \) are as follows:

\[ \{ GF_n^{(2)} \} = \{-1, i, 1, i + 1, 2i, 3i + 1, 4i + 3, 7i + 4, 12i + 5, \ldots \}, \]

\[ \{ GF_n^{(3)} \} = \{-i, -1, i, 1, i + 1, 2i, 2i - 2, 4i - 2, 7i - 1, \ldots \}. \]

From Definition 2.1, \( GF_n^{(k)} \) and \( GF_n \) related by

\[ GF_n^{(k)} = (GF_m)^{k-r}(GF_{m+1})^r. \]

If \( k = 1 \), we see that \( m = n \) and \( r = 0 \). So, \( GF_n^{(1)} \) is well-known Gaussian Fibonacci numbers \( GF_n \).

3. Main results

**Theorem 3.1.** Let \( k, m \in \{1, 2, 3, 4, \ldots \} \). For \( k \) and \( m \), \( GF_n^{(k)} \) and \( GF_n \) numbers satisfy
i) \[ \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} GF_{mk+j}^{(k)} = (-1)^{k-1} GF_m GF_{(m+1)(k-1)} \]

ii) \[ \sum_{j=0}^{k-1} \binom{k-1}{j} GF_{mk+j}^{(k)} = GF_m GF_{(m+2)(k-1)} \]

iii) \[ \sum_{j=0}^{k-1} GF_{mk+j}^{(k)} = \frac{GF_m}{GF_{m-1}} \left[ GF_{(m+1)k} - GF_{mk}^{(k)} \right] \]

**Proof.**

i) I have
\[ \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} GF_{mk+j}^{(k)} = (-1)^{k-1} \sum_{j=0}^{k-1} (-1)^{k-1-j} \binom{k-1}{j} (GF_m)^{k-j} (GF_{m+1})^j \]
\[ = (-1)^{k-1} GF_m \sum_{j=0}^{k-1} \binom{k-1}{j} (GF_{m+1})^j (-GF_m)^{k-j} \]
\[ = (-1)^{k-1} GF_m [((GF_{m+1} - GF_m)^{k-1}] \]
\[ = (-1)^{k-1} GF_m GF_{(m+1)(k-1)} \]

ii) In a similar manner, I have
\[ \sum_{j=0}^{k-1} \binom{k-1}{j} GF_{mk+j}^{(k)} = \sum_{j=0}^{k-1} \binom{k-1}{j} (GF_m)^{k-j} (GF_{m+1})^j \]
\[ = GF_m \sum_{j=0}^{k-1} \binom{k-1}{j} (GF_{m+1})^j (GF_m)^{k-1-j} \]
\[ = GF_m [((GF_{m+1} + GF_m)^{k-1}] \]
\[ = GF_m GF_{(m+2)(k-1)} \]

iii) It follows I have
\[ GF_{mk+j}^{(k)} = (GF_m)^{k-j} (GF_{m+1})^j = \left( \frac{GF_{m+1}}{GF_m} \right)^j (GF_m)^k. \]

Using the above equation and some algebraic operations, the desired result is obtained.

**Theorem 3.2.** For the \( GF_n^{(2)} \), I have the following relations:

i. \( GF_{2(m-1)}^{(2)} - GF_m GF_{m-2} = (-1)^{m-1}(i - 2), \quad m \geq 1 \)

ii. \( GF_n^{(2)} = GF_{n-1}^{(2)} + GF_{n-3}^{(2)} + GF_{n-4}^{(2)}, \quad n \geq 4 \)

**Proof.** i) Let \( A \) be the Fibonacci matrix of the form
\[ A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \]

Then, from the matrix \( A \) and definition the Gaussian Fibonacci numbers I have
\[ \begin{bmatrix} GF_m & GF_{m-1} \\ GF_{m-1} & GF_{m-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{m-1} \begin{bmatrix} GF_1 & GF_0 \\ GF_0 & GF_{-1} \end{bmatrix}. \]

The determinants of both sides of the above equation are taken
\[ GF_{m}GF_{m-2} = (-1)^{m-1}(i + 2) \]
\[ GF_{m-1} = GF_{m}GF_{m-2} = (-1)^{m-1}(i - 2) \]
\[ GF_{2(m-1)}^{(2)} - GF_{m}GF_{2m-2} = (-1)^{m-1}(i - 2), \quad \text{from } GF_{m-1}^{2} = GF_{2(m-1)}^{(2)}. \]

ii) If \( n \) is even, i.e.,
\[ GF_{2m}^{(2)} = GF_{2m-1}^{(2)} + GF_{2m-3}^{(2)} + GF_{2m-4}^{(2)}. \]

To illustrate the above equations, I will use the following relations:
\[ GF_{2m}^{(2)} = (GF_{m})^{2} \]
\[ GF_{2m+1} = GF_{m}GF_{m+1}. \]

The relations are readily obtained from Definition 2.1. Now, it can be written as
\[
GF_{2m}^{(2)} = (GF_{m})^{2}
= GF_{m}GF_{m}
= GF_{m}(GF_{m-1} + GF_{m-2})
= GF_{m-1}GF_{m} + GF_{m-2}GF_{m}
= GF_{m-1}GF_{m} + GF_{m-2}(GF_{m-1} + GF_{m-2})
= GF_{m-1}GF_{m} + GF_{m-2}GF_{m-1} + (GF_{m-2})^{2}
= GF_{2m-1}^{(2)} + GF_{2m-3}^{(2)} + GF_{2m-4}^{(2)}
\]

Similarly, if \( n \) is odd, i.e.,
\[ GF_{2m+1} = GF_{2m}^{(2)} + GF_{2m-2}^{(2)} + GF_{2m-3}^{(2)}, \]
the desired result is obtained.

**Theorem 3.3.** The generating function of \( GF_{n}^{(2)} \) are given by
\[
C_{n}^{(2)}(x) = \frac{-1+(-1+i)x+(1-i)x^{2}+(1+i)x^{3}}{1-x-x^{3}-x^{4}}.
\]

**Proof.** I have \( C_{n}^{(2)}(x) = \sum_{n=0}^{\infty} GF_{n}^{(2)}x^{n} \). Then
\[
C_{n}^{(2)}(x) = \sum_{n=0}^{\infty} GF_{n}^{(2)}x^{n}
\]
\[
-xC_{n}^{(2)}(x) = \sum_{n=0}^{\infty} GF_{n-1}^{(2)}x^{n}
\]
\[-x^3 C_n^{(2)}(x) = \sum_{n=0}^{\infty} G_{n-3}^{(2)} x^n \]
\[-x^4 C_n^{(2)}(x) = \sum_{n=0}^{\infty} G_{n-4}^{(2)} x^n, \]

equations can be written. In this case

\[(1 - x - x^3 - x^4) C_n^{(2)}(x) = (1 - x - x^3 - x^4) C_n^{(2)}(x) \]
\[- \left( G_0^{(2)} x + G_1^{(2)} x^2 + G_2^{(2)} x^3 \right) - G_0^{(2)} x^3 \]
\[+ \sum_{n=4}^{\infty} \left( G_n^{(2)} - G_{n-1}^{(2)} - G_{n-2}^{(2)} - G_{n-3}^{(2)} \right) x^n \]
\[= G_0^{(2)} + \left( G_1^{(2)} - G_0^{(2)} \right) x + \left( G_2^{(2)} - G_1^{(2)} \right) x^2 \]
\[+ \left( G_3^{(2)} - G_2^{(2)} - G_0^{(2)} \right) x^3 + 0 \]
\[= -1 + (i + 1)x + (1 - i)x^2 + (1 + i)x^3. \]

Hence, \( C_n^{(2)}(x) \) of \( G_n^{(2)} \) is

\[ C_n^{(2)}(x) = \frac{-1 + (i + 1)x + (1 - i)x^2 + (1 + i)x^3}{1 - x - x^3 - x^4}. \]

Finally, I give two identities without proofs:

- \[ \sum_{j=0}^{n} G_{2j-1}^{(1)} = G_{2n}^{(1)} + (1 - 2i), \]
- \[ \sum_{j=0}^{n} G_{2j-1}^{(2)} = \begin{cases} G_{2n}^{(2)} + (i + 2) & \text{if } n \text{ is even} \\ G_{2n}^{(2)} + (2i) & \text{if } n \text{ is odd} \end{cases}. \]

References


