SOME CHARACTERIZATIONS FOR SPACELIKE INCLINED CURVES

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Abstract. In this paper by establishing the Frenet frame \( \{T, N, B_1, B_2\} \) for a spacelike curve we give some characterizations for the spacelike inclined curves and \( B_2 \)-slant helices in \( \mathbb{R}^4_2 \).

1. Introduction

In the classical differential geometry inclined curves and slant helices are well known. A general helix or an inclined curve in \( E^4_3 \) defined as a curve whose tangent lines make a constant angle with a fixed direction called the axis of the helix. A helix curve is characterized by the fact that the ratio \( \frac{k_1}{k_2} \) is constant along the curve, where \( k_1 \) and \( k_2 \) denote the first curvature and the second curvature (torsion), respectively. Analogue to that A. Magden has given a characterization for a curve \( x(s) \) to be a helix in Euclidean 4-space \( E^4_3 \). He characterizes a helix iff the function

\[
\left( \frac{k_1(s)}{k_2(s)} \right)^2 + \frac{1}{k_3^2(s)} \left( \frac{d}{ds} \left( \frac{k_1(s)}{k_2(s)} \right) \right)^2
\]

is constant where \( k_1, k_2 \) and \( k_3 \) are first, second and third curvatures of Euclidean curve \( x(s) \), respectively and they are not zero anywhere \([2]\). Similar characterizations of timelike helices in Minkowski 4-space \( E^4_1 \) were given by H. Kocayigit and M. Onder \([6]\).

S. Yilmaz and M. Turgut presented necessary and sufficient conditions to be inclined for spacelike and timelike curves in terms of Frenet equations in Minkowski spacetime \( E^4_3 \) \([12]\). A. T. Ali and R. Lopez studied the generalized timelike helices in Minkowski 4-space and gave some characterizations for these curves \([3]\).

M. Onder, H. Kocayigit and M. Kazaz gave the differential equations characterizing the spacelike helices and also gave the integral characterizations for these curves in \( E^4_1 \) \([7]\).

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Izumiya and Takeuchi have introduced the concept of slant helix by considering that the normal lines make a constant angle with a fixed direction. They characterized a slant helix if and only if the function

\[ \frac{\kappa^2}{(\kappa^2 + \tau^2)^{3/2}} \left( \frac{r}{\kappa} \right)'' \]

is constant [10].

A. T. Ali and R. Lopez gave different characterizations of slant helices in terms of their curvature functions [4]. Kula and Yayli investigated spherical images, the tangent indicatrix and the binormal indicatrix of a slant helix and they obtained that the spherical images are spherical helices [9].

M. Onder, H. Kocayigit and M. Kazaz gave the characterizations of spacelike \( B_2 \)-slant helix by means of curvatures of the spacelike curve in Minkowski 4-space. Moreover they gave the integral characterizations of the spacelike \( B_2 \)-slant helix [8].

In this study we investigate the conditions for spacelike curves to be inclined or \( B_2 \)-slant helix in \( \mathbb{R}^4 \) and we give some characterizations and theorems for these curves.

2. Preliminaries

The Semi-Euclidean space \( \mathbb{R}^4_2 \) is the standard vector space equipped with an indefinite flat metric \( (\cdot, \cdot) \) given by

\[ (\cdot, \cdot) = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2 \]  

where \((x_1, x_2, x_3, x_4)\) is a rectangular coordinate system of \( \mathbb{R}^4_2 \). A vector \( v \) in \( \mathbb{R}^4_2 \) is called a spacelike, timelike or null (lightlike) if respectively hold \( (v, v) > 0 \), \( (v, v) < 0 \) or \( (v, v) = 0 \) and \( v \neq 0 = (0, 0, 0, 0) \). The norm of a vector \( v \) is given by \( \|v\| = \sqrt{(v, v)} \). Two vectors \( v \) and \( w \) are said to be orthogonal if \( (v, w) = 0 \).

An arbitrary curve \( \alpha : I \to \mathbb{R}^4_2 \) can locally be spacelike, timelike or null if respectively all of its velocity vectors \( \alpha'(s) \) are spacelike, timelike or null.

Let \( a \) and \( b \) be two spacelike vectors in \( \mathbb{R}^4_2 \). Then there is unique real number \( 0 < \delta < \Pi \), called angel between \( a \) and \( b \), such that \( (a, b) = \|a\| \|b\| \cos \delta \).

Let \( \{T(s), N(s), B_1(s), B_2(s)\} \) be the moving Frenet frame along the curve \( \alpha(s) \) in \( \mathbb{R}^4_2 \). Then \( T, N, B_1, B_2 \) are the tangent, the principal normal, the first binormal and the second binormal fields respectively and let \( \nabla_T T \) is spacelike.

Let \( \alpha \) be a spacelike curve in \( \mathbb{R}^4_2 \), parametrized by arclength function of \( s \). The following cases occur for the spacelike curve \( \alpha \). Let the vector \( N \) is spacelike, \( B_1 \) and \( B_2 \) be timelike. In this case there exists only one Frenet frame \( \{T, N, B_1, B_2\} \) for which \( \alpha(s) \) is a spacelike curve with Frenet equations

\[
\begin{align*}
\nabla_T T &= k_1 N \\
\nabla_T N &= -k_1 T + k_2 B_1 \\
\nabla_T B_1 &= k_2 N + k_3 B_2
\end{align*}
\]
where \( T, N, B_1 \) and \( B_2 \) are mutually orthogonal vectors satisfying the equations

\[
\langle N, N \rangle = \langle T, T \rangle = 1, \quad \langle B_1, B_1 \rangle = \langle B_2, B_2 \rangle = -1 \tag{3}
\]

Recall that the functions \( k_1 = k_1(s) \), \( k_2 = k_2(s) \) and \( k_3 = k_3(s) \) are called the first, the second and the third curvature of the spacelike curve \( \alpha(s) \), respectively and we will assume throughout this work that all the three curvatures satisfy \( k_i(s) \neq 0, 1 \leq i \leq 3 \).

3. Some Characterizations for Spacelike Inclined Curves and \( B_2 \)-Slant Helices in \( R^4_2 \)

Let \( \alpha(s) \) be a non-geodesic spacelike curve in \( R^4_2 \) and let \( \{T, N, B_1, B_2\} \) denotes the Frenet frame of the curve \( \alpha(s) \). A spacelike curve in \( R^4_2 \) is said to be an inclined curve if its tangent vector forms a constant angle with a constant vector \( U \). From the definition of the inclined curve we can write

\[
T.U = \cos \theta \tag{4}
\]

where \( U \) is a spacelike constant vector. Differentiating both sides of this equations we have

\[
k_1 N.U = 0 \tag{5}
\]

Thus we arrive \( N \perp U \). Considering this we can compose \( U \) as

\[
U = u_1 T + u_2 B_1 + u_3 B_2 \tag{6}
\]

where \( u_i, 1 \leq i \leq 3 \) are arbitrary functions. Differentiating (6) and considering Frenet equations, we have

\[
0 = u_1' T + (u_1 k_1(s) + u_2 k_2(s)) N + (u_2' - u_3 k_3(s)) B_1 + (u_3' + u_2 k_3(s)) B_2 \tag{7}
\]

From (7) we find the equations

\[
\begin{cases}
  u_1' = 0 \\
  u_1 k_1(s) + u_2 k_2(s) = 0 \\
  u_2' - u_3 k_3(s) = 0 \\
  u_3' + u_2 k_3(s) = 0
\end{cases} \tag{8}
\]

By using the equations above we have \( u_1 = c = \text{cons}, \)

\[
u_2 = -c k_1(s) = -\frac{1}{k_2(s)} \frac{du_3}{ds} \tag{9}
\]

and

\[
u_3 = -c k_1(s) \frac{d}{ds} k_3(s) \tag{10}
\]

From the equation \( u_2' - u_3 k_3(s) = 0 \) we have

\[
\frac{du_2}{ds} = k_3(s) u_3 \tag{11}
\]
Differentiating $u_2$ we have

$$\frac{d}{ds}\left(-\frac{1}{k_3(s)} \frac{du_3}{ds}\right) = k_3(s)u_3. \quad (12)$$

By a direct computation we have the differential equation

$$\frac{d}{ds}\left(-\frac{1}{k_3(s)} \frac{du_3}{ds}\right) + k_3(s)u_3 = 0 \quad (13)$$

By using exchange variable $t = \int_0^s k_3(s)ds$ in (13) we find

$$\frac{d^2u_3}{dt^2} + u_3 = 0 \quad (14)$$

The general solution of (14) is

$$u_3 = m_1\cos t + m_2\sin t \quad (15)$$

where $m_1, m_2 \in R$. Replacing variable $t = \int_0^s k_3(s)ds$ in (15) we have

$$u_3 = -\frac{c}{k_3(s)} \frac{d}{ds}(\frac{k_1(s)}{k_2(s)}) = m_1\cos(\int_0^s k_3(s)ds) + m_2\sin(\int_0^s k_3(s)ds) \quad (16)$$

Considering equation (16) and (9) we have

$$u_2 = -c\frac{k_1(s)}{k_2(s)} = m_1\sin(\int_0^s k_3(s)ds) - m_2\cos(\int_0^s k_3(s)ds) \quad (17)$$

From the equations above we find

$$m_1 = -\frac{c}{k_3(s)} \frac{d}{ds}(\frac{k_1(s)}{k_2(s)})\cos(\int_0^s k_3(s)ds) - c\frac{k_1(s)}{k_2(s)}\sin(\int_0^s k_3(s)ds) \quad (18)$$

and

$$m_2 = c\frac{k_1(s)}{k_2(s)}\cos(\int_0^s k_3(s)ds) - \frac{c}{k_3(s)} \frac{d}{ds}(\frac{k_1(s)}{k_2(s)})\sin(\int_0^s k_3(s)ds) \quad (19)$$

By taking $A_1 = m_1 + m_2$ and $A_2 = m_1 - m_2$, if we calculate $A_1^2 + A_2^2$ we find

$$c^2\left(\frac{k_1(s)}{k_2(s)}\right)^2 + \frac{c^2}{k_3^2(s)} \left[\frac{d}{ds}(\frac{k_1(s)}{k_2(s)})\right]^2 = constant \quad (20)$$

or

$$\left(\frac{k_1(s)}{k_2(s)}\right)^2 + \frac{1}{k_3^2(s)} \left[\frac{d}{ds}(\frac{k_1(s)}{k_2(s)})\right]^2 = constant. \quad (21)$$

Conversely, let us consider vector given by

$$U = \left\{T - \frac{k_1(s)}{k_2(s)} B_1 - \frac{1}{k_3(s)} \frac{d}{ds}(\frac{k_1(s)}{k_2(s)}) B_2\right\}\cos\theta \quad (22)$$

Differentiating vector $U$ and considering differential equation of (21) we obtain

$$\frac{dU}{ds} = 0 \quad (23)$$
Thus $U$ is a constant vector and so the curve $\alpha(s)$ is an inclined curve in $R^4_2$. Thus we have the following theorem.

**Theorem 1.** Let $\alpha = \alpha(s)$ be a spacelike curve in $R^4_2$. $\alpha$ is an inclined curve if and only if

$$\left(\frac{k_1(s)}{k_2(s)}\right)^2 + \frac{1}{k_3^2(s)} \left\{ \frac{d}{ds} \left( \frac{k_1(s)}{k_2(s)} \right) \right\}^2 = \text{constant.} \quad (24)$$

**Proof.** It is obvious from the computations above. \qed

**Corollary 2.** Let $\alpha = \alpha(s)$ be a spacelike curve in $R^4_2$. $\alpha$ is an inclined curve if and only if

$$k_3(s) \frac{k_1(s)}{k_2(s)} + \frac{d}{ds} \left( \frac{1}{k_3(s)} \frac{d}{ds} \left( \frac{k_1(s)}{k_2(s)} \right) \right) = 0. \quad (25)$$

**Proof.** If we differentiate the equation (24) respect to $s$ we find the equation (25). \qed

Now let us solve the equation (25) respect to $\frac{k_1}{k_2}$. If we use exchange variable $t = \int_0^s k_3(s)ds$ in (25) we have

$$\frac{d^2}{dt^2} \left( \frac{k_1}{k_2} \right) + \left( \frac{k_1}{k_2} \right) = 0. \quad (26)$$

So we arrive

$$\frac{k_1}{k_2} = W_1 \cos \int_0^s k_3(s)ds + W_2 \sin \int_0^s k_3(s)ds. \quad (27)$$

where $W_1$ and $W_2$ are real numbers.

Now we will give a different characterization for inclined curves. Let $\alpha$ be an inclined curve in $R^4_2$. By differentiating (24) with respect to $s$ we get

$$\left( \frac{k_1}{k_2} \right)' + \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \left[ \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \right]' = 0 \quad (28)$$

and hence

$$\frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' = \frac{\left( \frac{k_1}{k_2} \right)'}{\left[ \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \right]'} \quad (29)$$

If we define a function $f(s)$ as

$$f(s) = \frac{\left( \frac{k_1}{k_2} \right)'}{\left[ \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \right]'} \quad (30)$$

then

$$f(s) = -\frac{1}{k_3(s)} \frac{k_1}{k_2} = W_1 \sin \int_0^s k_3(s)ds - W_2 \cos \int_0^s k_3(s)ds. \quad (31)$$

By using (28) and (31) we have

$$f'(s) = -\frac{k_1k_3}{k_2}. \quad (32)$$
Conversely, consider the function
\[ f(s) = -\frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' = W_1 \sin \int_0^s k_3(s)ds - W_2 \cos \int_0^s k_3(s)ds \]
and assume that \( f'(s) = -\frac{k_1k_3}{k_2^2} \). We compute
\[ \frac{d}{ds} \left( \frac{k_1(s)}{k_2(s)} \right)^2 + \frac{1}{k_3^2(s)} \left( \left( \frac{k_1(s)}{k_2(s)} \right)' \right)^2 = \frac{d}{ds} \left[ \frac{1}{k_3^2} (f'^2 + f^2(s)) \right] := \varphi(s) \quad (33) \]
As \( f(s)f'(s) = -\left( \frac{k_1}{k_2} \right)' \) and \( f''(s) = -k_3' \left( \frac{k_1}{k_2} \right) - k_3 \left( \frac{k_1}{k_2} \right)' \) we obtain
\[ f'(s)f''(s) = k_3k_3' \left( \frac{k_1}{k_2} \right)^2 + k_2^2 \left( \frac{k_1}{k_2} \right)^3 \left( \frac{k_1}{k_2} \right)' = 0 \quad (34) \]
As consequence of above computations
\[ \varphi(s) = 2(f'f'' + f''f'^2) - \frac{(f'^2k_3')k_3}{k_3^2} = 0 \quad (35) \]
that is the function \( \left( \frac{k_1(s)}{k_2(s)} \right)^2 + \frac{1}{k_3^2(s)} \left( \left( \frac{k_1(s)}{k_2(s)} \right)' \right)^2 \) is constant. Therefore we have the following theorem.

**Theorem 3.** Let \( \alpha \) be a unit speed spacelike curve in \( \mathbb{R}^3 \). Then \( \alpha \) is an inclined curve if and only if the function \( f(s) = -\frac{1}{k_3(s)} \left( \frac{k_1}{k_2} \right)' = W_1 \sin \int_0^s k_3(s)ds - W_2 \cos \int_0^s k_3(s)ds \) satisfies \( f'(s) = -\frac{k_1k_3}{k_2^2} \) where \( k_1, k_2 \) and \( k_3 \) are the curvatures of \( \alpha \).

**Proof.** The proof can be completed from the computations above. \( \square \)

Now let \( \alpha(s) \) be a spacelike curve in \( \mathbb{R}^3 \) and let \( \{T, N, B_1, B_2\} \) denotes the Frenet frame of the curve \( \alpha(s) \). We call \( \alpha(s) \) as spacelike \( B_2 \)-slant helix if its second binormal vector makes a constant angle with a fixed direction in a vector \( U \). From the definition of the \( B_2 \)-slant helix we can write
\[ B_2 U = \cos \theta \quad (36) \]
where \( U \) is a spacelike constant vector. Differentiating both sides of this equations we have
\[ -k_3B_1 U = 0 \quad (37) \]
Since \( k_3 \neq 0 \) we arrive \( B_1 \perp U \). Considering this we can compose \( U \) as
\[ U = u_1T + u_2N + u_3B_2 \quad (38) \]
where \( u_i, \ 1 \leq i \leq 3 \) are arbitrary functions. Differentiating (38) and considering Frenet equations, we have
\[ 0 = (u_1' - u_2k_1)T + (u_1k_1(s) + u_2')N + (u_2k_2(s) - u_3k_3(s))B_1 + u_3' B_2 \quad (39) \]
From (39) we find the equations
\[
\begin{align*}
\{ & u'_1 - u_2 k_1 = 0 \\
& u_1 k_1(s) + u'_2 = 0 \\
& u_2 k_2(s) - u_3 k_3(s) = 0 \\
& u'_3 = 0
\end{align*}
\] (40)

By using the equations above we have \( u_3 = c = \text{cons} \),
\[
\begin{align*}
u_2 &= \frac{k_3(s)}{k_2(s)} = \frac{1}{k_1(s)} \frac{du_1}{ds}
\end{align*}
\] (41)

and
\[
\begin{align*}
u_1 &= -\frac{c}{k_1(s)} \frac{d}{ds} k_3(s)
\end{align*}
\] (42)

From the equation \( u'_1 - u_2 k_1(s) = 0 \) we have
\[
\begin{align*}
\frac{du_1}{ds} = k_1(s) u_2
\end{align*}
\] (43)

Differentiating \( u_1 \) we have
\[
\begin{align*}
\frac{d}{ds} \left( -\frac{1}{k_1(s)} \frac{du_2}{ds} \right) = k_1(s) u_2.
\end{align*}
\] (44)

By a direct computation we have the differential equation
\[
\begin{align*}
\frac{d}{ds} \left( \frac{1}{k_1(s)} \frac{du_2}{ds} \right) + k_1(s) u_2 = 0
\end{align*}
\] (45)

By using exchange variable \( t = \int_0^s k_1(s)ds \) in (45) we find
\[
\begin{align*}
\frac{d^2 u_2}{dt^2} + u_2 = 0
\end{align*}
\] (46)

The general solution of (46) is
\[
\begin{align*}
u_2 &= m_1 \cos t + m_2 \sin t
\end{align*}
\] (47)

where \( m_1, m_2 \in R \). Replacing variable \( t = \int_0^s k_1(s)ds \) in (47) we have
\[
\begin{align*}
u_2 &= \frac{k_3(s)}{k_2(s)} = m_1 \cos \left( \int_0^s k_1(s)ds \right) + m_2 \sin \left( \int_0^s k_1(s)ds \right)
\] (48)

Considering equation (48) we have
\[
\begin{align*}
u_1 &= -\frac{c}{k_1(s)} \frac{d}{ds} \left( \frac{k_3(s)}{k_2(s)} \right) = -m_1 \sin \left( \int_0^s k_1(s)ds \right) + m_2 \cos \left( \int_0^s k_1(s)ds \right)
\] (49)

From the equations above we find
\[
\begin{align*}
m_1 &= -\frac{c}{k_1(s)} \frac{d}{ds} \left( \frac{k_3(s)}{k_2(s)} \right) \cos \left( \int_0^s k_1(s)ds \right) + \frac{k_3(s)}{k_2(s)} \sin \left( \int_0^s k_1(s)ds \right)
\] (50)

and
\[
\begin{align*}
m_2 &= \frac{k_3(s)}{k_2(s)} \cos \left( \int_0^s k_1(s)ds \right) - \frac{c}{k_1(s)} \frac{d}{ds} \left( \frac{k_3(s)}{k_2(s)} \right) \sin \left( \int_0^s k_1(s)ds \right)
\] (51)
By taking $B_1 = m_1 + m_2$ and $B_2 = m_1 - m_2$, if we calculate $B_1^2 + B_2^2$ we find

$$c^2 \left( \frac{k_3(s)}{k_2(s)} \right)^2 + c^2 \left( \frac{d}{ds} \frac{k_3(s)}{k_2(s)} \right)^2 = constant \quad (52)$$

or

$$\left( \frac{k_3(s)}{k_2(s)} \right)^2 + \frac{1}{k_1^2(s)} \left( \frac{d}{ds} \frac{k_3(s)}{k_2(s)} \right)^2 = constant. \quad (53)$$

Conversely, let us consider vector given by

$$U = \left\{ -\frac{1}{k_1(s)} \frac{d}{ds} \frac{k_3(s)}{k_2(s)} T + \frac{k_3(s)}{k_2(s)} N + B_2 \cos\theta \right\} \quad (54)$$

Differentiating vector $U$ and considering differential equation of $(53)$ we obtain

$$\frac{dU}{ds} = 0 \quad (55)$$

Thus $U$ is a constant vector and so the curve $\alpha(s)$ is a spacelike $B_2$ slant helix in $R^4_2$. As a result we can give the following theorem.

**Theorem 4.** Let $\alpha = \alpha(s)$ be a spacelike curve in $R^4_2$. $\alpha$ is a spacelike $B_2$ slant helix if and only if

$$\left( \frac{k_3(s)}{k_2(s)} \right)^2 + \frac{1}{k_1^2(s)} \left( \frac{d}{ds} \frac{k_3(s)}{k_2(s)} \right)^2 = constant. \quad (56)$$

**Proof.** The proof can easily seen from the computations above. \qed

**Corollary 5.** Let $\alpha = \alpha(s)$ be a spacelike curve in $R^4_2$. $\alpha$ is a $B_2$-slant helix if and only if

$$k_1(s) \frac{k_3(s)}{k_2(s)} - \frac{d}{ds} \frac{1}{k_1(s)} \frac{d}{ds} \left( \frac{k_3(s)}{k_2(s)} \right) = 0. \quad (57)$$

**Proof.** If we differentiate the equation $(56)$ respect to $s$ we have the equation $(57)$. \qed

Now let us solve the equation $(57)$ respect to $\frac{k_3}{k_2}$. If we use exchange variable $t = \int_0^s k_1(s)ds$ in $(57)$ we have

$$\frac{d^2}{dt^2} \left( \frac{k_3}{k_2} \right) + \left( \frac{k_3}{k_2} \right) = 0. \quad (58)$$

So we arrive

$$\frac{k_3}{k_2} = L_1 \cos \int_0^s k_1(s)ds + L_2 \sin \int_0^s k_1(s)ds. \quad (59)$$

where $L_1$ and $L_2$ are real numbers.

Now we will give a different characterization for $B_2$-slant helices. Let $\alpha$ be a spacelike $B_2$-slant helix in $R^4_2$. By differentiating $(56)$ with respect to $s$ we get

$$\left( \frac{k_3}{k_2} \right)' + \frac{1}{k_1} \left( \frac{k_3}{k_2} \right)' \left[ \frac{1}{k_1} \left( \frac{k_3}{k_2} \right)' \right] = 0 \quad (60)$$
and hence

$$\frac{1}{k_1} \left( \frac{k_3}{k_2} \right)' = - \left( \frac{k_3}{k_2} \right)' \left( \frac{1}{k_1} \right)' $$

(61)

If we define a function $f(s)$ as

$$f(s) = - \left( \frac{k_3}{k_2} \right)' \left( \frac{1}{k_1} \right)' $$

(62)

then

$$f(s) = - \frac{1}{k_1(s)} \left( \frac{k_3}{k_2} \right)' = L_1 \sin \int_0^s k_1(s) ds - L_2 \cos \int_0^s k_1(s) ds.$$  

(63)

By using (60) and (63) we have

$$f'(s) = - \frac{k_1 k_3}{k_2}.$$  

(64)

Conversely, consider the function

$$f(s) = - \frac{1}{k_1(s)} \left( \frac{k_3}{k_2} \right)' = L_1 \sin \int_0^s k_1(s) ds - L_2 \cos \int_0^s k_1(s) ds$$

(65)

and assume that $f'(s) = - \frac{k_1 k_3}{k_2}$. We compute

$$\frac{d}{ds} \left[ \left( \frac{k_3(s)}{k_2(s)} \right)^2 + \frac{1}{k_1(s)} \left( \frac{k_3(s)}{k_2(s)} \right)^2 \right] = \frac{d}{ds} \left[ \frac{1}{k_1^2} (f' + f^2(s)) \right] := \varphi(s)$$

(66)

From $f(s)f'(s) = -(\frac{k_3}{k_2})' \left( \frac{k_3}{k_2} \right)'$ and $f''(s) = -k_1' \left( \frac{k_3}{k_2} \right)' - k_1 \left( \frac{k_3}{k_2} \right)'$ we obtain

$$f'(s)f''(s) = k_1 k_3^2 \left( \frac{k_3}{k_2} \right)^2 + k_2 \left( \frac{k_3}{k_2} \right)^2.$$  

(67)

As a consequence of above computations

$$\varphi(s) = 2 \left( f' + \frac{f' f''}{k_1^2} - \frac{(f'^2 k_1')}{k_1^2} \right) = 0$$

(68)

that is the function $\left( \frac{k_3(s)}{k_2(s)} \right)^2 + \frac{1}{k_1^2(s)} \left( \frac{k_3(s)}{k_2(s)} \right)^2$ is constant. Therefore we have the following theorem.

**Theorem 6.** Let $\alpha$ be a unit speed spacelike curve in $\mathbb{R}^3_2$. Then $\alpha$ is a $B_2$-slant helix if and only if the function $f(s) = - \frac{1}{k_1(s)} \left( \frac{k_3}{k_2} \right)' = L_1 \sin \int_0^s k_1(s) ds - L_2 \cos \int_0^s k_1(s) ds$ satisfies $f'(s) = - \frac{k_1 k_3}{k_2}$, where $k_1$, $k_2$, and $k_3$ are the curvatures of $\alpha$.

**Proof.** It is obvious from the above computations.
References


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